Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

From the Director

Mathematics is useful in just about everything we study. This spring, Cambridge was host to a family of red-tailed hawks that set up nest along the Alewife Brook Parkway. What does math have to do with red-tailed hawks?

Biologist Kara Donohue gives us an answer, and, I might add, on very short notice! It was only weeks ago when she received an out-of-the-blue email from Girls' Angle asking her if she'd be willing to explain her research with red-tailed hawks in this issue. Thank you, Kara! Also, special thanks to Roger Brissenden and Nimesh Patel for their photos of the Alewife hawks.

Elsewhere, in addition to our regular columns, you can get to know Carleton College assistant professor Helen Wong and read the last installment of Bjorn Poonen's special series on the meaning of addition.

This summer, we've made a diverse collection of Summer Fun problems too. All members and subscribers are invited to submit their solutions.

All my best,
Ken Fan
Founder and Director

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Girls' Angle welcomes submissions that pertain to mathematics.

Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Red-tailed hawks over Cambridge. Soaring hawk by Roger Brissenden. Inset of hawk nestlings by Nimesh Patel.

An Interview with Helen Wong

Helen Wong is an assistant professor at Carleton College. She holds a doctoral degree in mathematics from Yale University. In her Girls' Angle Women In Mathematics video, which can be viewed on our website, she explained a method for summing up consecutive numbers.

Ken: Hi Helen, thank you for agreeing to do this interview for Girls' Angle! Also, thanks for making a WIM video. I'm curious to know, why did you choose the topic of summing the numbers from 1 to 100?

Helen: I remembered learning about Gauss' trick in school and thinking how cleverly simple the answer became once you see it the right way. Up until then, I'd always thought of math as an endless series of tests about the multiplication table…

Ken: (laughter)

Helen: …so it was a nice change to see a formula that was so pretty. And instead of memorizing the formula, all I had to do was to think about how the trick worked.

Ken: Do you use this technique for summing consecutive numbers in your work?

Helen: I was originally going to say "No", because the problems I usually work on don't use numbers all that much. (In topology, we think more about shapes and how they're the same or different.) But then I realized that in my PhD work, I had to work out some examples where Gauss sums were used to calculate the Witten-Reshetikhin-Turaev quantum invariants, which are used to distinguish topological spaces from one another. Gauss sums add up powers of complex numbers, and variations of Gauss' trick appear all over the place.

Ken: Neat! Did you realize that you wanted to become a mathematician when you learned about Gauss' trick?

Helen: I didn't think I wanted to be a mathematician until my sophomore or junior year of college. For a long time, I used to think that there wasn't any math beyond calculus. Then in college, I took a class called Linear Algebra because all my friends were taking it, too. It was then that I finally realized that math was really awesome, that it was totally different than I had thought. We had to think about functions and numbers and structures in a fundamental and foundational way, and at some point, it was almost like studying philosophy. It was difficult, and I loved the challenge and the reward of getting problems right. It wasn't just about rote calculations anymore. I had to think hard!

Don't be afraid to ask questions or to ask for help. But at the same time, don't be afraid when it takes ages and ages to understand a problem - it'll just make figuring it out later that much better!

But even with more math classes after Linear Algebra, I wasn't sure that I wanted to be a mathematician. Mostly it was because I didn't know what a mathematician did all day long. And besides, hadn't they figured out all there is to know already? After I realized that mathematicians were just normal human beings who were lucky enough to have jobs that allowed them to think about new and relevant problems all the time, I was sold!

Ken: I understand that you went to Hungary on a Fulbright scholarship. What was that year like? Do mathematicians often travel overseas?

Helen: It's hard to explain how incredibly fantastic it was to spend a year in Hungary. I took classes at a program called Budapest Semesters in Mathematics and also worked on an independent project on the side. The year between college and graduate school gave me a little bit of a breather and a chance to see the world. It also allowed me to take a few extra math classes I didn't get to fit in during college because I decided to be a math major relatively late. For instance, in Set Theory, we discussed the different kinds of "infinity" and about the Axiom of Choice, which asks whether it is possible to do infinitely many things at the same time. In between classes, we would go sit at old European cafes and think about math and the universe while sipping thick coffee and eating pastries and cake. Because I was on the Fulbright program, I got to visit a lot of the countryside and see more of aspects of Hungary than a usual study abroad student. It was interesting time to be there historically, as Hungary was transitioning from their communist past towards a free-market economy and membership in the European Union.

Talking is a great way for me to sort out the parts that I know for sure are true from the parts where I am more shaky on.

One of the biggest perks of being a mathematician is the opportunity to travel and attend conferences all over the world. Last summer, I spent a week in Trieste, Italy and also a week in New York City. This year, I'm hoping to go to southern France and to Poland and maybe even to Tokyo. Every four years, there is an International Congress of Mathematicians, where all the mathematicians in the world get together and talk about math. This year, it's being held in India, and some of my friends are going. Sadly I won't be able to attend. So I get rather jealous whenever they talk about their plans to hear about math in India!

Ken: Do you have any advice for how best to learn mathematics?

Helen: Everyone has their own favorite way of learning math. For me, I'm not great at reading math textbooks. Instead, I learn math by thinking about and working on small problems, using the books as guides or references. Of course, this isn't true for everyone - I have friends who can read math books like novels! I also spend a lot of time going to math talks and asking questions. I love it when I can get a friend (or stranger) to explain their favorite math to me and to teach me new things. Mathematicians are friendly people, and they are always excited to talk about the math they're thinking about at the moment.

Ken: Will you explain to us what your first published theorem is about?

Helen: One of the big problems in topology is to find out ways of distinguishing three-dimensional spaces from one another. We know that there are (countably[1]) infinitely many three-dimensional spaces, but there's no easy classification that we know of. So instead

[1] A set is "countably infinite" if its elements can be matched up with all the integers.

topologists turn to invariants, which are methods to assign in a consistent way a number to each three-dimensional space. If two spaces have different numbers assigned to them, then they would have to be different spaces to begin with. There are both "good" invariants and "bad" invariants. For example, one invariant attaches the number 1 to every single three-dimensional space. This would be a rather poor invariant, because it wouldn't help us distinguish between any spaces at all! In some sense, the more numbers the invariant uses in the assignment, the better the invariant.

My first theorem was about an invariant called the Witten-Reshetikhin-Turaev quantum invariant, which came from mathematical physics. The theorem said that this invariant was reasonably good, in that the system of assigning numbers to three-dimensional spaces uses (countably) infinitely many numbers. It uses almost all of the possible complex numbers and didn't leave gaps, that is, the values of the quantum invariant formed a dense subset of the complex numbers. It is still possible for the invariant to not distinguish between two different three-dimensional spaces, but it is as good as we could hope for.

Ken: When you get stuck on a problem, what kinds of things do you do to try to get unstuck?

Helen: When I get really stuck on a problem, I try to rope in a friend to help me. I explain the problem to them, and usually they'll ask me questions about the way I'm trying to answer the problem. Sometimes they'll point out flaws in my reasoning, or they'll make me think about the problem in a different way, or they might show me a book or theorem that I didn't know about before. Even if they're not interested in solving the problem with me, I benefit from talking through my thought process. Talking is a great way for me to sort out the parts that I know for sure are true from the parts where I am more shaky on. And it's always nice to have someone other than myself to talk to!

…instead of memorizing the formula, all I had to do was to think about how the trick worked.

Ken: I always ask this question: do you think there is gender bias in mathematics today?

Helen: I do think there is a gender bias in mathematics, but a lot of it is because there aren't as many women as men interested in mathematics now. As with any conversation, the dynamics change depending on the gender balance at the table. I don't want to make generalizations, as they often don't hold up, but I often feel that men can be more assertive in their statements and women more self-deprecatory. It usually doesn't matter whether they're right or not, but it's about the way in which it is said and about how willing they are to argue their point until the sun comes up. So I find it can be hard to do math in a setting where I'm one of very few women, just because I'm not as used to the way everyone else interacts. This is not so much about how the men treat me as much as about how we respond to each other. I think this will surely change as more women become mathematicians.

Ken: I heard that you like to play the harpsichord. Is that true? That's interesting. Most people I know who play a keyboard instrument play either piano or a synthesizer. What do you like about the harpsichord?

Helen: I've played the piano since I was five years old and was taking lessons up until the end of graduate school. But I always wanted to learn the harpsichord, too, and it seemed like now was as good a time to do so as ever. It's a completely different sound that comes out of a harpsichord, and so it makes me think about the music differently. But I must admit that I still am confused a lot at the harpsichord. The keys are smaller than on the piano so my fingers don't always end up where I think they should, and whenever I want a more "connected" sound, my foot inadvertently will tap on an imaginary pedal under the keys. And as always, I wish I had more time to practice.

… I'm not great at reading math textbooks. Instead, I learn math by thinking about and working on small problems, using the books as guides or references.

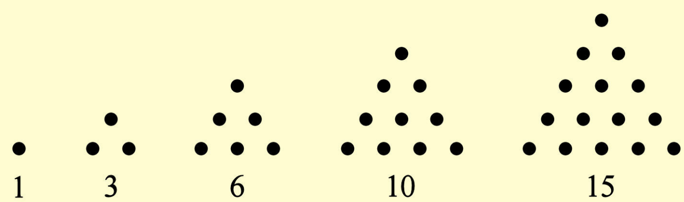
Ken: Do you have any advice for the girls that come to Girls' Angle?

Helen: Don't be afraid to ask questions or to ask for help. But at the same time, don't be afraid when it takes ages and ages to understand a problem - it'll just make figuring it out later that much better!

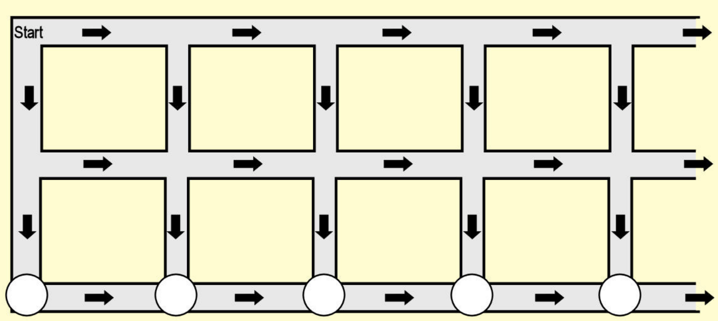
Ken: Thank you so much for this interview!

Triangular Numbers

In Helen's Women In Mathematics video, she describes a nifty way to find the sum of the numbers from 1 to 100. Let T_n be the sum of the numbers 1, 2, 3, …, n. Because this sequence counts the number of dots in triangular arrangements, the T_n are known as **triangular numbers**:



Here's another way that triangular numbers arise. Count the number of paths from the start intersection to each of the circled intersections in the network of one-way streets below.



Finally, let's connect triangular numbers to the topic of this issue's Notation Station: factorials (see page 10). Observe that $T_n = n(n + 1)/2$. This can be shown by using induction or derived by using the technique that Helen describes in her video. Rearranging terms, we see that $2T_n = n(n + 1)$. Therefore, $2T_1 \cdot 2T_3 \cdot 2T_5 \cdots 2T_{2m-1} = (2m)!$. Simplifying slightly, we get this amusing expression for $(2m)!$:

$$(2m)! = 2^m \cdot T_1 T_3 T_5 \cdots T_{2m-1}.$$

Red-Tailed Hawks

by Kara Donohue

This spring, a family of red-tailed hawks took up residence in Cambridge. Locals were treated to a first hand lesson in bird behavior as mother and father raised their hatchlings. Early in June, the nestlings took their first flights and today, the nest is abandoned. Believe it or not, it's very difficult to tell whether a red-tailed hawk is male or female. In this article, Kara Donohue, a biologist at Southern California Edison, explains how she figured out a practical way to do this using mathematics. –Editor

The question. Ever since I saw a captured red-tailed hawk up close and found out that you could not tell males from females, I wanted to come up with a method to figure it out. Male and female red-tailed hawks overlap in size and do not appear to have different plumage characteristics. Previously, it was only possible to tell males from females during the breeding season based on behavioral differences. Is it possible to use commonly taken in-hand measurements of red-tailed hawks to distinguish between males and females? Thanks to a great biostatistics course I took in graduate school, I knew how find an answer to this question. Using DNA analysis and statistics, I was able to give an affirmative answer by providing a practical method to determine gender.

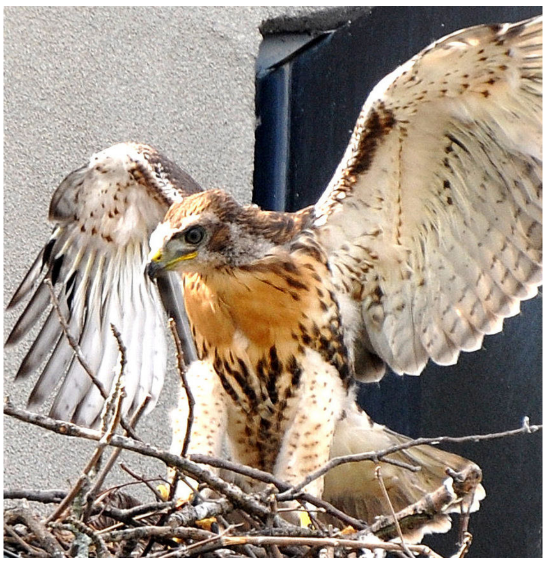


Photo by Roger Brissenden

A red-tailed hawk in Cambridge.

In 2006, my graduate advisor, Dr. Al Dufty, and I published a paper[1] describing the method in the Journal of Field Ornithology. The following explains the steps taken to answer the question.

Statistics. What we wanted to do is come up with some practical way of determining whether a red-tailed hawk is male or female. In other words, we have two groups of hawks that we wanted to discriminate between: male and female. We wanted to be able to discriminate between these two groups using measurements that a field biologist could easily measure, such as hawk mass or tail length.

Of course, today, there is a way to determine a red-tailed hawk's gender that is nearly full proof. DNA analysis can tell males from females. But DNA analysis can be expensive and does not provide immediate results for a biologist working in a remote location.

Instead, what we wanted to do is find some simple formula whose value a field biologist could easily calculate and give a simple criterion using this value to determine whether a red-tailed hawk is male or female. For example, one might propose that the field biologist measure the mass of the hawk and suggest that if this mass is greater than 1 kilogram, then the hawk is

[1] Kara C. Donohue, Alfred M. Dufty Jr. 2006. Sex determination of Red-tailed Hawks (*Buteo jamaicensis calurus*) using DNA analysis and morphometrics. Journal of Field Ornithology 77(1):74-79.

female, otherwise it is male. However, this formula does not work. Many birds would be misclassified with this oversimplified scheme.

In biology, there is a lot of variation. Clear cut answers to classification problems are rare. That is why statistics is used. By studying large samples of birds, we can get a sense of how likely something is or isn't. Our basic strategy then, is to propose some formula whose value a field biologist can compute relatively easily and a simple way to use this formula to decide if a hawk is male or female. We then look at a large number of hawks and apply this formula. We compare how our formula classifies hawks to their actual genders (determined using DNA analysis). Statistical methods tell us how effective the formula is by telling us with what probability it works and how confident we can be that the classification is correct.

To find this formula, we proceed in steps.

The first step is to answer the question: Is there a relationship between the measurements we're interested in and a particular group? If there is no relation, it means that one would not be able to predict with any accuracy what the values of the measurements would be based on the gender of the hawk and vice versa. We use statistics to determine if the measurements have similar values for both male and female hawks or if there are fundamental differences. We start with what is called a null hypothesis. In this case, the null hypothesis would be: There is no difference between male and female red-tailed hawks based on the measurements. If we disprove the null hypothesis, then there is a difference between males and females and we can move on to the next step.

Next, if there is a relationship, how accurately can we assign a gender to a red-tailed hawk? Once we've built a model, it might be that individual birds could be misclassified into the wrong group frequently. We used a statistical technique that tests the model with data that was not used in building the model to see how robust the model is.

Finally, we answered: what is the relationship between the groups and the measurements? Also, what measurements are most useful for classifying into a group? We took several different types of measurements for each red-tailed hawk, but it is important to ask whether all of these measurements are actually useful for distinguishing between males and females. Using statistical methods, we can compute the effectiveness of each measurement type and then find the formula which is most effective.

Data Collection. With a strategy in place, we set out to collect data. For me, this meant spending the fall camped out on top of a mountain in Nevada for two and a half months with other field biologists in a location known for its large raptor (eagles, hawks, falcons, etc.) migration. We captured red-tailed hawks on their southbound journey for

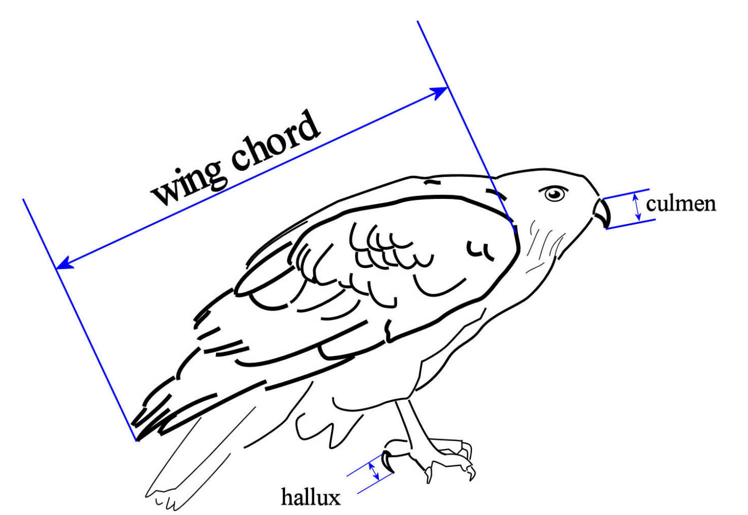


Figure 1. Red-tailed hawk measurements used to determine gender. Not shown but also used is the mass of the hawk.

Photo by Roger Brissenden

A red-tailed hawk chick in Cambridge testing out its wings.

the winter months, plucked a few breast feathers, took several measurements of these birds using rulers and calipers, which are simple tools often used by field biologists, and then released them to continue on their way. This is the best part of field biology, spending time in beautiful and remote places and getting up close to the animals you love. We wrote these measurements on the envelopes containing the individual's feathers. Back in the lab at school, I used the feather tips to extract DNA and determine the sex of each bird. In statistics, the more data one collects the more confidence one can have in the results. In all, 172 birds were analyzed.

The results we got. Our statistical analysis showed that not only was there a difference between males and females based on measurements, but there was a difference between birds born the same year of capture (the "hatch-year") and adult birds. So adults and hatch-year birds were separated for the rest of the tests.

For adult red-tailed hawks, we found that wing chord (see Figure 1 for an illustration of these bird measurements) and body mass are significant morphological measurements for distinguishing between male and female. Here's the formula:

$$Z = 0.166 \times (\text{wing chord}) + 0.026 \times (\text{mass})$$

Measure wing chord in millimeters and bird mass in grams. If $Z > 94.902$, then the bird is female; if $Z \leq 94.902$, then it is male. This equation accurately assigned gender to 98% of the adult red-tailed hawks in our data sample.

Hatch-year birds had more significant measurements than the adults. For hatch-year birds, body mass, wing chord, hallux, and culmen are significant morphological measurements for discriminating between males and females. Here's the formula for hatch-year birds:

$$Z = 0.2 \times (\text{wing chord}) + 0.011 \times (\text{mass}) + 1.302 \times (\text{hallux}) + 1.356 \times (\text{culmen})$$

In this formula, hallux and culmen, like wing chord, are measured in millimeters and mass is again measured in grams. If $Z > 160.933$, then the bird is female; if $Z \leq 160.933$ then it is male. This equation accurately assigned gender to 97% of the hatch-year red-tailed hawks in our data sample.

With these formulae, a field biologist no longer has to perform a DNA analysis to figure out the gender of a red-tailed hawk. With a few simple measurements and a calculator, that gender can now be determined with very high accuracy.

Notation

by Ken Fan

Through time, some ways to notate mathematical ideas have become standard. Learning the notational conventions is important in order to be able to communicate. The best way to learn new notation is the same way we learned the alphabet: repetition. Practice makes perfect!

! (The Factorial)

The following questions all have the same answer:

1. Eight people are standing in a line. How many different ways can they be put in order?

2. How many ways are there to place 8 rooks on an 8 by 8 chessboard so that no two rooks are attacking each other?

3. What is the 8^{th} derivative of x^8?

For all three of these problems, the answer is $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, the product of the integers from 1 to 8, inclusive.

If you replace every eight in each of these three problems with n, then the answer becomes the product of the integers from 1 to n, inclusive (even if chessboards don't really come in other sizes!).

Such products of consecutive integers arise so often in mathematics that a special notation was invented for them. It is called "factorial" and is denoted with an exclamation point like this:

Factorial!

$$8! \equiv 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.$$

In general, if n is a positive integer, then $n!$ is the product of the numbers from 1 to n. In words, $n!$ is pronounced "n factorial".

The factorial $n!$ increases rapidly with n. You may have learned about the rapid growth of exponential functions. Well, $n!$ grows even faster! In other words, fix a constant b and look at the sequence $\frac{b^n}{n!}$. This ratio compares the exponential b^n with the factorial $n!$. After a certain point, this ratio gets closer and closer to zero with increasing n. Can you see why?

For more properties of factorial, see the Summer Fun problem set on page 26.

One last thing: It turns out to be rather convenient to extend the definition of factorial to include 0! by declaring that 0! be 1.

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Emmy Newton had to figure out which n satisfies $n! = 24$ (see volume 3, number 2 of this *Bulletin*). A friend of mine asked me how to figure out n given m such that $n! = m$ in general. For example, she asked, what n satisfies $n! = 2432902008176640000$.

Well, I'll start by writing down what the problem is. Also, factorial really stands for a big product, so I'll replace $n!$ with the product.

4/2/10

Given m, find n such that $n! = m$.

$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = m$

I actually have no idea of any way. I'll set this aside for now and come back later. Hmmmm.

I had an idea while taking a walk! I don't know why I didn't think of this before. I can just divide by 2, 3, 4, and so on until I get 1!

4/10/10 Start with 2 and go up:

Compute $\frac{m}{2}$. If result is 1, $m = 2!$

Otherwise compute $\frac{(\frac{m}{2})}{3}$. If result is 1, $m = 3!$.

Otherwise compute $((^m/_2)/_3)/_4$. If result is 1, $m = 4!$.

etc.

I'll write this as an algorithm that can be readily turned into a computer program.

Algorithm: (assuming $m > 1$ and m is equal to $n!$ for some pos. integer n)

1. Let $n = 2$.
2. Replace m with m/n.
3. If m is 1, the answer is n; stop. Otherwise, increase n by 1 and go back to step 2.

I guess I don't have to assume m is equal to $n!$ for some positive integer n. Instead, in the algorithm if m is ever less than 1, I can report that the original number is not a perfect factorial.

Replace step 3 with this:

3. If $m = 1$, the answer is n; stop. Otherwise, if $m < 1$, report that m is not a "perfect" factorial; stop. Otherwise, if $m > 1$, increase n by 1 and go back to step 2.

Also, to avoid the assumption that $m > 1$ I can let $n = 1$ in step 1.

Let's try this procedure for my friend's specific number.

Try this with $m = 2432902008176640000$.

/2 : 1216451004088320000

/3 : 405483668029440000

/4 : 101370917007360000

/5 : 20274183401472000

/6 : 3379030566912000

/7 : 482718652416000

/8 : 60339831552000

/90 : 670442572800

It's clear that the number is bigger than 9 times 10, so I can do the division by 9 and 10 in a single step by dividing by 90.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Every time I divide by a multiple of 5, I lose a trailing zero.

That makes sense, because there are more factors of 2 than 5 in a factorial, and each trailing zero represents one factor of 2 and one factor of 5.

/11 : 60949324800

/20 : 3047466240

/15 : 203164416

/12 : 16930368

/13 : 1302336

/14 : ~~9445~~ 93024

5814

/16 : 342

17 : 19

18 : 1

19 :

Answer: 20!

So the two trailing zeros mean there are two more factors of 5. Of the numbers after eleven, 15 and 20 are the first with factors of 5, so this must be divisible by both 15 and 20 and the original number is at least 20!. Also, I find it slightly easier to divide by 20 and 15 so I'll go ahead and divide by those now.

This is laborious... and error prone!

Finally, a 1! So the answer is 20!.

Maybe I can use the number of trailing zeros to solve the problem faster.

4/11/10 Number of trailing zeros = number of factors of 5.

Number of factors of 5 in $n!$ is

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \dots$$

$$< \frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \dots = n\left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots\right)$$

$$= n\,\frac{1}{5}\,\frac{1}{1-\frac{1}{5}} = \frac{n}{4}$$

This is actually one of the problems in the Summer Fun problem sets at the end of this issue. See the problem set on factorials.

This is a geometric series which I can sum using the formula for the sum of a geometric series.

$n/4$ is a strict upper bound on the number of trailing zeros. So if there are 4 trailing zeros, the number is at least 17!

For 20!, 4 trailing zeros. Solve $\frac{n}{4} = 4 \Rightarrow n = 16$.

So $n \geq 17$. 17! has $\left\lfloor \frac{17}{5} \right\rfloor + \left\lfloor \frac{17}{25} \right\rfloor = 3$ factors of 5.

So must be 20!, 21!, 22!, 23!, 24!.

Each of these have different numbers of digits.

Whenever you multiply a number by 10 or more, the number of digits will increase by at least 1. So 20!, 21!, 22!, etc. all have different numbers of digits. I wonder if this can be exploited to find the answer even faster! I'll note this and think about it later.

$\boxed{\text{Use number of digits to find answer?}}$

Try $m = \underbrace{263308 \dots\dots 2160000000}_{36 \text{ digits.}}$

I asked my friend for another number to try to see if I can find the answer faster. She produced this huge 36 digit number!

7 trailing zeros $\Rightarrow n > 28$

From what I observed earlier, n must be bigger than 4 times the number of trailing zeros.

29! has 5+1=6 trailing zeros.

Answer is 30!, 31!, 32!, 33!, or 34!

Using calculator: $\frac{m}{30!} \approx 1000$

$\therefore$ answer is 32!

Since 31 is nowhere near a thousand, but 31 times 32 is, and 31 times 32 times 33 is more like 30000, the answer must be 32!

Wow...even though this number is huge, it's not even close to the number of ways a standard deck of 52 cards can be shuffled!

AB 6.17.10

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

The Adventures of Emmy Newton

Episode 5. *The Blue Tiles*

by Maria Monks

Last time: Emmy and Melissa opened yet another door on their quest to reach the Principal's office, and this time were surprised to be greeted by a figure in the doorway.

"Emmy, Melissa! There you two are!" It was Amy, the sixth grade student worker who helped out in the Principal's office during lunch time.

"Amy! What are you doing here? Do you know what this place is?" asked Emmy. Melissa just frowned and crossed her arms. Amy was one of those girls who made fun of her because she liked to do math problems at recess.

"I had no idea there even was a basement in this school," replied Amy. "The Principal sent me to deliver this package to Mr. Wheel's room," she continued. She was holding a small packaged item in her hand. "I went into the staircase to go upstairs, and I found that it led down to a basement. I was bored and decided to change my route, but now I'm locked in here!"

"So are we... Mr. Wheel sent us to get that package from the Principal's office," said Emmy. "So you don't have any idea who set all of this up?"

Amy shook her head and shrugged, trying to retain her usual cool demeanor to mask her fright. Melissa uncrossed her arms and peered past Amy into the next hallway. She blinked. It was rather... blue.

Emmy saw too. "What the heck?" said Emmy, who, despite having been trapped in a cold basement the entire morning, couldn't help but enjoy the journey as they made their way past all of these mathematical obstacles. She stepped passed Amy and took a good look at the hallway. The walls and ceiling were uniformly painted with a deep blue paint and the long fluorescent hallway lights were untidily wrapped in blue cellophane. The metal wall on the far end was a silvery blue, with giant dark glowing blue letters reading, "**I LIKE BLUE.**"

But the floor was black.

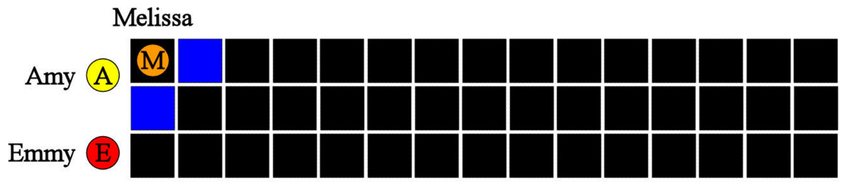
"It's so weird, the floor tiles were blue when I first walked in," said Amy. "When I got to this door and opened it, they all turned black, and that big metal wall on the far side just fell into place behind me, out of nowhere! It looks pretty solid." She sighed. "And I haven't even gotten to eat lunch yet."

"Oh, like we have," snapped Melissa. "We've only been trapped in here for the last three hours, solving riddles to unlock padlocked doors, using the Pythagorean Theorem to construct a ramp up to a passageway, and discovering deep mathematical secrets of permutation generators just to get out of here."

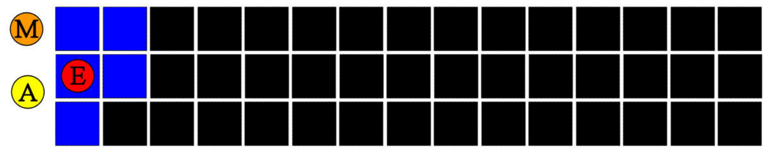
Emmy ignored them and looked more closely at the floor of the long hallway. They were standing on a small patch of grey stone, but the rest of the floor was made up of black square tiles, forming a grid.

"3 by 15," said Emmy, after squinting into the distance and counting the number of tiles that stretched across the hallway.

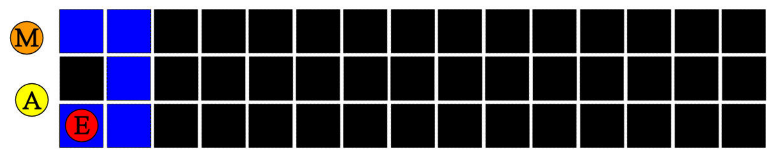
"Well, let's see just how solid that wall is. Maybe we can break through it," suggested Melissa. She stepped forward onto the corner black tile, and its two neighboring tiles instantly lit up, glowing brightly blue.



Melissa gasped and jumped back off the tile, but the blue tiles remained illuminated. Emmy looked in fascination, and stepped onto the tile next to Melissa's, in the middle of the first row. The three neighboring tiles lit up.



"Aha!" She walked on to the third tile in the first row, which currently had one neighbor black and one neighbor blue. The black neighbor lit up, and the blue neighbor turned back to black. "It's just toggling the neighbors of the tiles you stand on!"



"Cool!" said Melissa. "Let's turn all of them blue! Maybe that will get us out of here."

Amy was looking perplexed, but not because of the tiles. "Do you guys smell something burning?" she asked.

The other girls paused. "Fire," Emmy muttered, hoping that she was wrong. She cautiously walked back through the door into the previous room with the monitors. It was much warmer than before, and through the little door on the ceiling she saw the flickering light of flames. She stared for a few moments before the reality of the situation sunk in, then she darted back out to Amy and Melissa and opened her notebook.

"We have to get out of here, fast," said Emmy. "Remember the wooden poles that crashed behind us as we were leaving the south hallway? I think they caught fire... and that's a lot of wood. It could burn the school down!"

"Let's get out of here!" said Amy, and she started to run towards the wall on the other side when Emmy grabbed her.

"Don't! If we're right about this, we need to turn these tiles blue, and it's just going to mess up our strategy if you start running across the hall and creating a mess of blue and black! Let me think, just let me think..."

She stepped on and off the three front tiles again to set all the tiles back to black.

"Okay," said Emmy, "Let's figure this out. We have to make it across the hall so that they're all blue by the time we get over there. We can probably step from one tile to another as long as they share at least one corner." She started drawing dots on a 3 by 15 grid in her notebook. "Let's see, if an odd number of us step on this corner then we need these other two..." she muttered to herself as she devised a strategy for getting the three of them across. Melissa stood dumbstruck, too frightened to think straight. Amy peeked nervously into the previous room to see if the fire was spreading.

"Aha!" Emmy finally exclaimed. "Ok, I think I have a strategy." She told her general strategy to Amy and Melissa, and the three girls quickly figured out the best way to implement it.

Can you see how to get the girls to the other side so that all tiles are blue when they reach the far end? Turn to page 21 for the continuation and their solution.

Addition
Part 3

by Bjorn Poonen

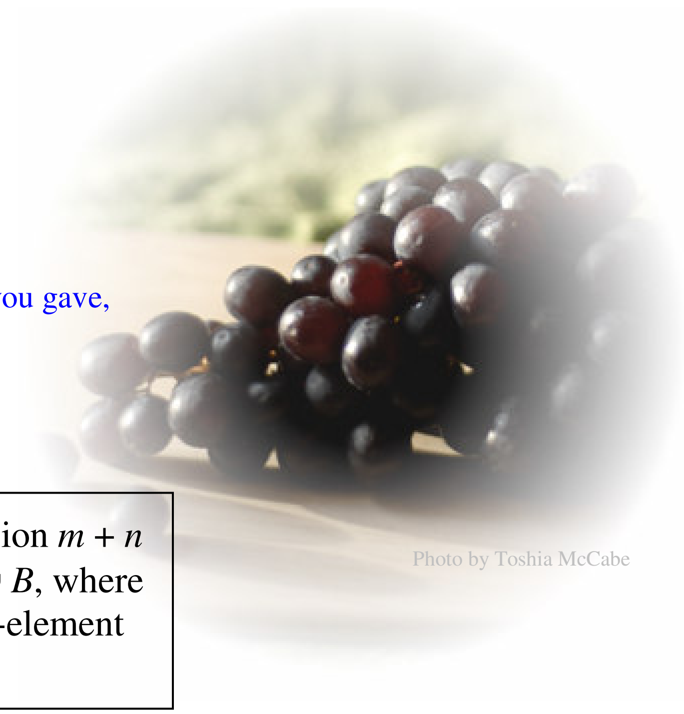


Photo by Toshia McCabe

Do you remember that "definition" of addition you gave, the one that you were so proud of?

The one involving grapes?

Yes, and also the set-theoretic version:

> For any numbers m and n, the expression $m + n$ means the number of elements of $A \cup B$, where A is any m-element set and B is any n-element set such that A and B are disjoint.

Oh yeah, that one is even better since it allows you to use any kind of objects.

Well, these so-called "definitions" don't work.

What do you mean? Sure they do.

I just tried computing 5 + (-2). So I put 5 grapes in my left hand, and then tried putting -2 grapes in my right hand, but that didn't work. Next I tried the set-theoretic definition. But I couldn't find a set with -2 elements.

There isn't any such set.

Aha! So you admit that your definition is wrong!

No, it's not *wrong*. It just has limited applicability. You're applying it in a situation for which it was not intended.

Well, at least you should admit that you were being vague when you said "For any numbers m and n…".

OK, I should have said that m and n were assumed to be nonnegative integers (or more generally, cardinal numbers).

That means that I'm free to define 5 + (-2) any way I want! I'm going to choose a definition that makes it really easy to compute. My definition is: throw away all the minus signs and then add. So 5 + (-2) = 7 and (-4) + (-6) = 10. Easy!

I wouldn't do that if I were you.

Why not? Let's see what FluffyFur thinks. Hey! FluffyFur, are you rolling your eyes at me?

There are good reasons why your definition isn't the standard one.

What makes one definition any better than any other?

A good definition should continue as many patterns as possible, and preserve as many rules as possible. Think about this, for example: the integers form a doubly infinite sequence

…, -3, -2, -1, 0, 1, 2, 3, …

and we can try adding each of these to 5, starting with the positive numbers (where we already have a definition) and working downward:

$$\begin{aligned} 5 + 3 &= 8 \\ 5 + 2 &= 7 \\ 5 + 1 &= 6 \\ 5 + 0 &= 5 \\ 5 + (-1) &= ? \\ 5 + (-2) &= ? \end{aligned}$$

You see, if you want this pattern to continue, it's pretty clear what you have to define 5 + (−2) as.

But what if there is another pattern that suggests a different value?

Well, then you would have to decide which pattern was more important to continue, and go with that one. Or else, if none of the patterns seemed especially compelling compared to the others, you could decide to leave the expression undefined. The latter is what happened with 0/0: in the 7th century, Brahmagupta, the first person to try to define the rules for arithmetic involving 0, said that 0/0 = 0. The pattern

$$\begin{aligned} 0/3 &= 0 \\ 0/2 &= 0 \\ 0/1 &= 0 \\ 0/0 &= ? \end{aligned}$$

does suggest this, but later mathematicians decided that it was better to leave 0/0 undefined because there are equally important patterns like

$$\begin{aligned} 3/3 &= 1 \\ 2/2 &= 1 \\ 1/1 &= 1 \\ 0/0 &= ? \end{aligned}$$

that suggest a different value. So today it is standard to leave 0/0 undefined.

OK, the pattern you showed me suggests that 5 + (-2) = 3, but how can I *make sure* that there is no other pattern suggesting a different value?

All I can say is that a lot of people have written down all the properties of nonnegative number addition they could think of, and they all suggest the same value for 5 + (-2). These properties are important and useful enough that even if you *did* find a new pattern suggesting a different value, you probably wouldn't be able to convince other people to change their mind about 5 + (-2).

What about (-5) + (-2)?

You can define the sum of any two integers by extending the addition table

+	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
9										9	10	11	12	13	14	15	16	17	18
8										8	9	10	11	12	13	14	15	16	17
7										7	8	9	10	11	12	13	14	15	16
6										6	7	8	9	10	11	12	13	14	15
5										5	6	7	8	9	10	11	12	13	14
4										4	5	6	7	8	9	10	11	12	13
3										3	4	5	6	7	8	9	10	11	12
2										2	3	4	5	6	7	8	9	10	11
1										1	2	3	4	5	6	7	8	9	10
0										0	1	2	3	4	5	6	7	8	9
-1																			
-2																			
-3																			
-4																			
-5																			
-6																			
-7																			
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-9																			

so as to continue the pattern that a step to the left or down leads to the number preceding the number at the current position.

That seems like a really time-consuming way to add.

It is. Often the argument that explains *why* something is defined a particular way does not give the best way to compute it. So just continue using whatever method you learned for adding.

I wish I could visualize 5 + (-2) the same way I can visualize 5 + 2 using grapes.

Well, there is a geometric description of addition that is better for this, a description that works not only for integers, but also for rational numbers (fractions like $-3/5$) and even real numbers, which correspond to arbitrary points on the number line.

What is this magical description of addition?

Here it is:

> If x and y are real numbers, then $x + y$ is the position you end up at if you start at x and move y units to the right.

In this description, y is being thought of as an amount of change.

OK, FluffyFur, let's test this, to understand 5 + (−2). I'm putting you at 5, and now let me start pushing you to the right.

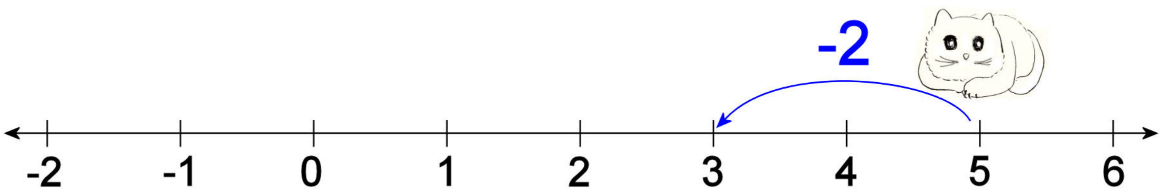
Cat: MEOW!

Wait; −2 units to the right means 2 units to the left.

OK, FluffyFur, let me start pushing you to the left.

Cat: MEOW!

What's wrong this time?



Maybe your cat just doesn't like being pushed around.

Anyway, FluffyFur, you ended up at 3, which is the right answer!

The nice thing about this description of addition is that it arises in all sorts of situations where quantities are being measured: not just position on a line, but also money, volume, and so on.

So if FluffyFur weighs 6 kilograms today, and gains 3 ½ kilograms, …

Then you would have a *very* heavy cat.

One last question: why did you keep calling this a *description* of addition instead of a definition of addition?

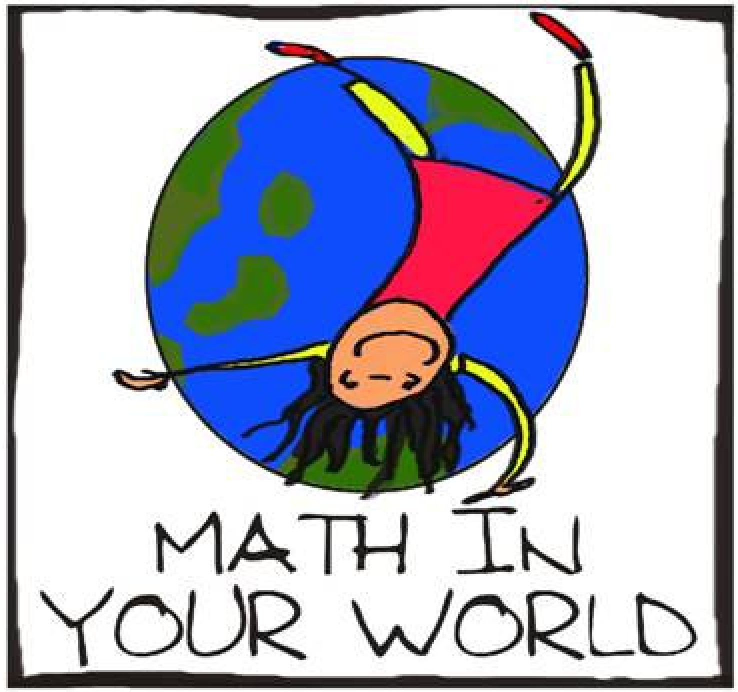
You could also call it a definition, as long as you realize that it depends upon certain axioms (assumptions) about geometry of a line. In fact, all the arithmetic operations on numbers can be defined in terms of geometry. On the other hand, some people prefer to turn things around, to start with some axioms about numbers and then to define geometry in terms of numbers by using coordinates. Either way is fine.

Cat: Purrr…

Editor: Grace Lyo

Finite *yet seemingly infinite!* Lines

By Katy Bold



Logo Design by Hana Kitasei

It is Friday night, and you get to the movie theater at 6:50 to see a show at 7 pm. There are already 20 people in the line to get tickets. Is it likely that you will get your ticket in time to see the movie? Should you leave immediately and find something else to do for the evening?

Let's assume that if you buy the ticket by 7 pm, you are guaranteed a good seat (not one of the seats upfront that require cricking the neck to see the action).

While mathematics is grounded in thorough and detailed computations, sometimes a "back of the envelope" calculation is all you need. A "back of the envelope" computation is something you can do with pencil and paper (or in your head), and usually it involves making estimates. Even though your answer will not be exact, the approximate answer will tell you the relative size of the true answer. For the current problem of determining the waiting time to get a movie ticket, it would be helpful to find out if the expected wait is 5 minutes or 35 minutes.

To estimate the waiting time, we can use the formula:

$$\text{Wait time} = (\text{average time for each group to buy tickets}) \times (\text{\# of groups buying tickets})$$

So, let's estimate the time that each group will spend buying tickets and the number of groups.

To buy tickets, the group must ask for tickets to a particular show, pay for the tickets, and then receive the tickets. Some people pay with cash, some with credit or debit, and hopefully no one pays with dimes and quarters. I assume that most groups will take more than one minute and definitely less than five minutes, and I estimate that the average time to buy tickets is two minutes.

As it is Friday night, I'll guess that most people are at the movie theater in a group (two or more people). Some people may buy their tickets separately, and some people may buy their tickets together. I estimate that, on average, each group buying tickets will have two people. That gives an estimated 20/2 = 10 groups total:

$$\begin{aligned}\text{Wait time} &= (2 \text{ minutes per group}) \times (10 \text{ groups})\\ &= 20 \text{ minutes}\end{aligned}$$

Uh oh! It is 6:50 (maybe it is 6:51 since you spent a minute doing all this math in your head!). The movie starts in 10 (or 9!) minutes, and it will take 20 minutes for everyone in front of you to get a ticket. Is all hope lost?

Probably not. If it is a busy night at the movie theater, there are probably multiple employees selling tickets and we should divide the total time to sell all tickets by the number of people selling tickets. Let's assume there are 4 people taking tickets:

Whew! You can expect to get your ticket in about 5 minutes, leaving a few minutes to grab some popcorn!

Let's consider another type of line- the dreaded post office line. If you have not yet experienced the line at a post office, consider yourself lucky. Some people will take one minute with a teller, and some people will take ten. So, let's say each person will average 5.5 minutes at the teller. If you get at the back of the line at noon with 10 people in front of you and only one teller, you will be in line for about 55 minutes. Is the situation really that bad?

Consider changing the assumptions- for every slow person (10 minutes with a teller), there are two fast people (one minute with the teller)- reducing the average length of time per person to four minutes. Then how long is the wait?

Because people could do many different things at the post office, there is more variation in the length of each transaction compared to movie ticket sales. Running a "back of the envelope" computation requires making more estimates, causing the answer to be more uncertain.

In addition to knowing the average wait time, it may be useful to know other quantities, such as the minimum and maximum wait times. To find the minimum wait time, assume that each of the 10 people in front of you will take only one minute. How would you find the maximum wait time?

An added benefit of doing these calculations mentally is that it takes up time and is a distraction from the boringness of lines!

Take it to your world

In what situations do you wait in a line, and how can you estimate the total waiting time?

What are other situations in which you can use "back of the envelope" computations?

If you wait for a table at a restaurant, the host may give you an expected waiting time. How do you think the restaurant comes up with the time? Are the times usually accurate?

If you want to park in a parking lot that is full, you have to drive around in circles until a spot opens up. When is it worth it to keep driving around and when is it better to look for a different parking lot?

The Adventures of Emmy Newton, continued from page 14.

"Let's go." Emmy began directing the three of them across the board, checking her notebook every few steps to make sure she was on track. The tiles flashed from black to blue to black and back again beneath their feet as they stepped and leapt across the board in an artfully mathematical fashion.

The smell of smoke was getting stronger and the room was noticeably warmer than before. They were halfway across the room when Melissa paused and coughed. As if in response, the fire alarm suddenly blared from above. They heard the sound of rumbling footsteps in the hallway above them.

"Amy, step forward," said Emmy, who knew that pausing was the worst thing to do. "Melissa, take two steps forward and jump." The girls made their way over to the wall and Emmy stepped on the last square, turning the last of the black squares to blue. The girls held their breath, partly in anticipation and partly from the smoke.

Something loud and metal clicked behind the wall and the wall began to raise slowly back up into the ceiling. It revealed an empty stone-walled room with nothing but a door.

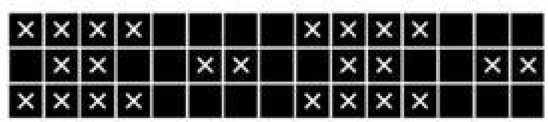
"Cross your fingers!" said Emmy, and she ran to the door and pulled. She couldn't believe it. The door opened, revealing a staircase to the first floor. They had made it!

The three girls ran, coughing, up the stairs and out into the sea of students and teachers escorting them to escape the building. There was the Principal's Office, where they had been trying to get to all along. As they evacuated the building, Emmy just hoped she didn't burn down her school on her way to pick up a package...

TO BE CONTINUED...

Editor: Grace Lyo

Emmy's Solution



"Ok, let's step on these marked tiles an odd number of times, and the rest an even number of times," said Emmy. She showed Amy and Melissa the diagram in her notebook (at right).

"We step on those Xs an odd number of times?" repeated Melissa.

"Yes, see, if we step on a tile two times in a row, it has no effect, because it just switches things on and off! So stepping on something once has the same effect as stepping on it three times, or five times, or seven..."

"Ok, but why is that going to turn everything blue... oh, I see!" said Melissa. "If a tile has an odd number of marked neighbors, then it's going to be blue, for the same kind of reason. And with those as the Xs, every square has an odd number of marked neighbors!"

"Right," said Emmy, "Ok, we don't have much time. Here's the strategy. Amy, come over here." Amy joined them and she and Melissa looked over Emmy's shoulder as she pointed to the squares on her diagram. "We'll start out by each stepping on one of these leftmost tiles." She pointed to the leftmost squares on the diagram, corresponding to the three tiles that were right in front of them. They all turned to face the wall so that left and right corresponded to left and right in the diagram.

"I'll take the top, Melissa takes the middle, and Amy takes the bottom. Then I'll step over and join Melissa so that we hit her square twice, and then I'll leap on to the top square into the next column while each of you move one square to the right."

"Got it," said Melissa. "Then we can all move over another square to the right, and another, and then you join me again to make that one be stepped on an even number of times. We can continue to make moves like that until we're all the way across the hall."

"Why can't we just each walk down one of these long rows and jump an extra time on any tile that needs to be stepped on an even number of times?" asked Amy.

Emmy and Melissa looked at her. "I guess that works too," said Emmy. The strategy seemed to Emmy more interesting if they weren't allowed to just stand on a tile and jump once in place. But certainly, the tiles couldn't know the difference between a new person stepping onto it and the same person jumping on it. Both strategies worked.

Just before setting off across the hall, Emmy wondered what would happen if it was a 3×16 grid of tiles instead. In fact, what happens in the general $m \times n$ case? Can you find a way to get the girls across a 3×16 grid of tiles? An $m \times n$?

Summer Fun!

The best way to learn math is to do math!

Here's this summer's batch of fun problem sets for you.

We invite Girls' Angle members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you may discover on the way there?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems before seeing solutions.

By the way, some of these problems are going to be very unlike those you will find at school. Usually, problems that you get at school are readily solvable. However, some of these problems were designed by the author to require time to solve and cannot be solved immediately.

If you are used to solving problems quickly, it can feel frustrating at first to work on problems that take months to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain or rock climbing. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So there's a meta-problem for those of you who feel frustrated at times doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Set Inclusions

by Julia Zimmerman

In all these problems, we will use capital letters of the English alphabet as they appear below:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

An object is symmetric about a line if half of its image can be reflected about that line to form the whole image. For example, this letter **T** is symmetric about a vertical line. An object has 180° rotational symmetry if half of its image can be rotated 180° about a point to form the whole image. For example, this letter **N** has 180° rotational symmetry about a point halfway along its diagonal.

Some Set Related Terminology

- If an element x is in a set p, then we write "$x \in p$".
- If every element in a set q is also in set p, then we say that "q is a subset of p" and write "$q \subseteq p$".
- If $q \subseteq p$ but $q \neq p$, then we say that "q is a proper subset of p" and write "$q \subset p$".

Note: Some people use the symbol "$\subset$" for subset and put a small "$\neq$" beneath the "$\subset$" symbol to stand for proper subset.

1. What fraction of the alphabet is symmetric about a line? What fraction of the alphabet has 180° rotational symmetry?

Let a be the set of every capital letter of the alphabet.
Let b be the set of capital letters which are symmetric about some line.
Let c be the set of capital letters which are symmetric about a vertical line.
Let d be the set of capital letters which are symmetric about a horizontal line.
Let e be the set of capital letters which have 180° rotational symmetry.
Let f be the set of capital letters which have 180° rotational symmetry about a point that is not part of the image of the letter.

2. Are any of these sets defined above subsets of any of the other sets? For example, is $b \subseteq c$? Is $c \subseteq b$? How do you know?

3. Are any of these sets *proper* subsets of any of the other sets?

4. Is it true that each of the sets a, b, c, d, e, and f is a subset of itself? Why or why not?

For problems 5 and 6, let p, q, and r be sets.

5. If $p \subset q$ and $q \subset r$, then is it true that $p \subset r$? Is it true that $p \subseteq r$? If $p \subseteq q$ and $q \subseteq r$, then is it true that $p \subset r$?

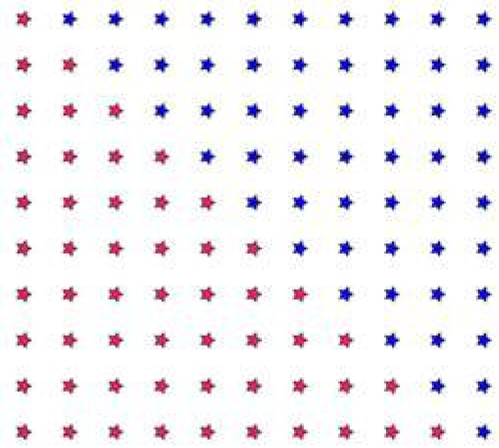
6. Suppose $p \subseteq r$ and $q \subseteq r$, and we know that the $x \in p$. What other sets necessarily contain x?

7. For each integer m, define S_m to be the set of multiples of m. In other words, S_m contains the numbers km for all integers k. When is $S_m \subseteq S_n$? When is $S_m \subset S_n$? When is $S_m = S_n$? What elements are in both S_m and S_n?

Summer Fun!

Adding Up Numbers

by Helen Wong



For this problem set, you might find it helpful to watch my Women In Mathematics video on the Girls' Angle website.

1. The picture at right has red and blue stars arranged in 10 rows and 11 columns. How many red stars are there in each row? Can you see how it can be used to explain Gauss' trick to show that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{1}{2}(10)(11)$?

2. All together, the seven Harry Potter books contain a total of 4100 pages. Amy is an incredibly gifted girl (and extremely quick at reading), but it takes her a while to get hooked onto the books. The first night, she reads only the first page. The second night, she is a bit more interested and manages to read the next two pages. The third night, she starts where she left off and reads the next three pages. She continues this way, so that on the nth night, she reads n new pages. What page will she be on at the end of one week? How long will it take her to finish all seven books?

3. What is the sum of the first five *odd* numbers 1 + 3 + 5 + 7 + 9? The first six odd numbers? The first seven odd numbers? Do you see a pattern?

4. Jo wants to collect Silly Bandz. She and her aunt have struck a deal. For the first week that Jo gets an A in class, her aunt will give her one new bracelet. If Jo keeps up her A for a second week, her aunt will give her two new bracelets at the end of the second week. If Jo keeps it up for three weeks in a row, her aunt will give her four new bracelets at the end of the third week. Because Jo's aunt worries that Jo will forget about their deal, she writes down her promise using a chart:

Week	New bracelets
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128
9	256
10	512

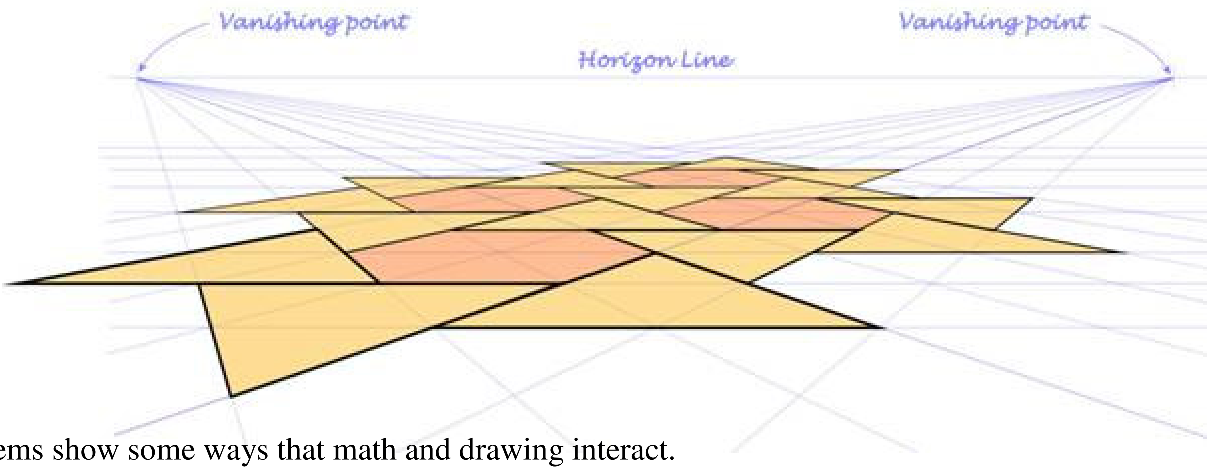
Week	Total bracelets
1	1
2	3
3	7
4	15
5	
6	
7	
8	
9	
10	

Help Jo and her aunt fill in the chart for the total bracelets! Is there a pattern? Can you find a formula?

Summer Fun!

Draw!

by Ken Fan



These problems show some ways that math and drawing interact.

1. Make a scale drawing of the floor plan of your home.

2. A circle in a perspective drawing will always look like some sort of ellipse. Here's a standard way to draw an ellipse: Take a string. Place your drawing on top of something that you can put a thumb tack into, such as a scrap of soft wood. Push two tacks through the paper into the backing. Tie the ends of the string to the tacks leaving slack. Now, take a pencil and use the tip of the pencil to hold the string taut. Draw around the tacks keeping the string taut and the pencil vertical at all times. The result is an ellipse! What kind of ellipse do you get if you tie both ends of the string to the same tack?

It's challenging to draw perfect geometric figures like circles and straight lines with a free hand. But if you have a compass for making circles and a straightedge to make straight lines, many geometric figures can be drawn with great accuracy. The next three problems all utilize a compass and straightedge.

3. Think of a way to draw a perfect equilateral triangle with the aide of a compass and straightedge.

4. Now try drawing a perfect square.

5. How about a perfect regular pentagon? Can you draw a perfect regular pentagon using a compass and straightedge? Here's a handy fact: $\cos 72° = \frac{\sqrt{5}-1}{4}$.

6. In Vermeer's *The Art of Painting* (see page 31), there is a perspective drawing of a checkered floor. Make a perspective drawing of a tiled floor of your own design. If you're not sure which one to try, I'll suggest starting with a brick pattern or a tiling with equilateral triangles. As an example, at the top of this page is a perspective drawing of a tiling that consists of regular hexagons and equilateral triangles. Auxiliary lines used to construct the drawing have been left in place.

Send your drawings to girlsangle@gmail.com!

The Factorial

by Lightning Factorial

This Summer Fun problem set explores properties of the factorial (see page 10).

1. Complete the following table.

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24						

2. Show that the number of trailing zeros in $n!$ is given by

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{5^k} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \ldots$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. This formula is used in this issue's *Anna's Math Journal* on page 11.

3. What is the coefficient of $abcd$ in $(a + b + c + d)^4$?

4. Ten books all by different authors are found shelved in no particular order. What is the probability that they are ordered alphabetically by the authors' last names?

Wilson's theorem states that if p is a prime number, then $(p - 1)! + 1$ is divisible by p. The next few problems outline a way to prove this. Fix a prime number p.

5. Check that Wilson's theorem is true for $p = 2$ by direct computation.

6. Assume from now on that $p > 2$ and let m be a positive integer less than p. Consider the first $p - 1$ multiples of m: m, $2m$, $3m$, …, $(p - 1)m$. Show that no two of these multiples of m share the same remainder when divided by p. In other words, let km and lm be two of these multiples with $1 \le l < k < p$. Show that p does not divide the difference $km - lm$.

7. Use 6 to conclude that there is exactly one positive integer $k < p$ which makes the product km leave a remainder of 1 when divided by p.

8. Show that there are exactly 2 positive integers x less than p that make $x^2 - 1$ divisible by p. (Recall that $p > 2$.) What are they?

9. Use 7 and 8 to show that $(p - 1)!$ can be written as 1 times $p - 1$ times a product of $\frac{p-3}{2}$ numbers that each leave a remainder of 1 when divided by p. From this, show that $(p - 1)! + 1$ is divisible by p.

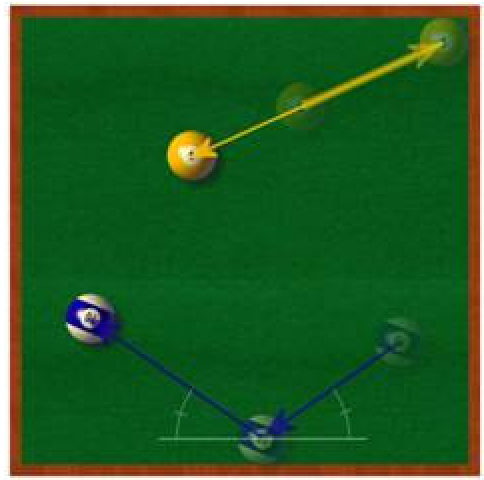
Summer Fun!

Pool

by Grace Lyo

Casi is spending the summer at her parents' cabin in the woods. There aren't very many kids around, nor is there a computer with an internet connection. There is, however, a square pool table, some pool balls, and a cue stick.

What's pool? In case you've never played pool before, a pool ball is a hard ball about the size of a peach. The ball is placed on the pool table, and a cue stick, which is a long, straight stick, is used to hit the ball in a precise direction. This is called "shooting" the ball. The ball then rolls on the table until it reaches the edge. The edge of the table is upraised (the upraised part is called a "curb"), so that when the ball runs into it, it bounces.

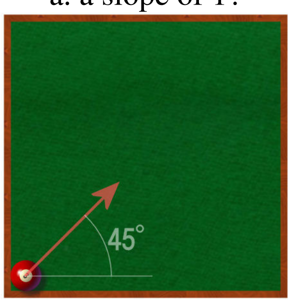


Casi's puzzle. By experimenting, Casi notices that whenever the ball hits the curb, it bounces as in the picture at right. If the ball bounces off a side curb, the two indicated angles are equal. If the ball bounces from the corner, it comes back along the same line that it went in. She also notices that if she hits the ball hard, it bounces around the table and sometimes comes back to where it started.

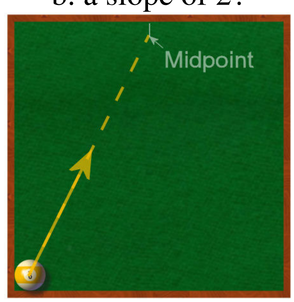
She decides to systematically shoot the ball from the bottom left corner of the table and count how many bounces are required before the ball returns to the starting place (if it ever does). She hopes that if she keeps track of the slope/angle at which she shoots the ball and the corresponding number of bounces, she can find a pattern. Let's figure out what she discovers.

1. How many times will the ball bounce before returning to the start position if she shoots with…

a. a slope of 1?



b. a slope of 2?



c. a slope of 3?

d. a slope of n, where n is an arbitrary positive integer?

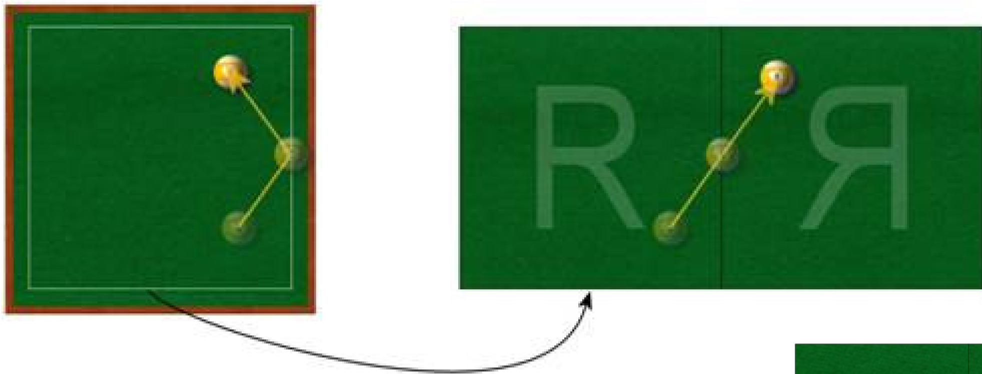
2. Do the same thing as in question 1, but with slopes 1/2, 1/3, ..., $1/n$.

3. Now she tries positive rational numbers (numbers n/m, where n and m are positive integers).

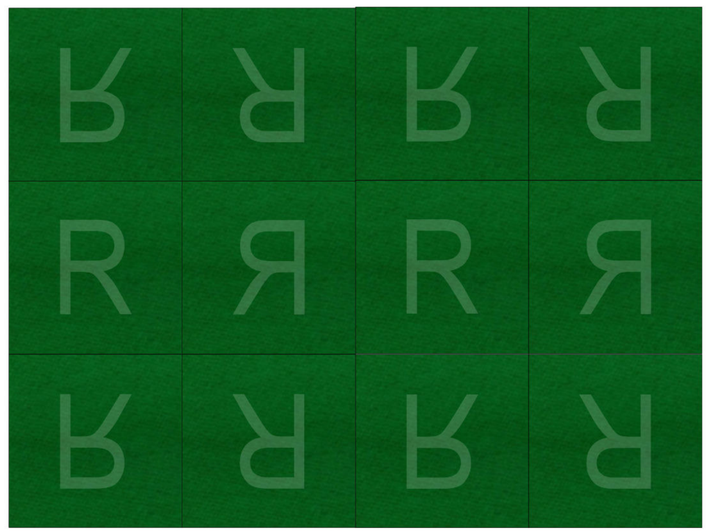
4. What if she shoots the ball at a 60° angle? If you're having trouble, you can make an educated guess by doing your computations with rational numbers that are better and better approximations for $\sqrt{3}$. I.e., first use the approximation 2 for $\sqrt{3}$, then try 1.7, then 1.73, etc.

Summer Fun!

Through the looking glass. (Don’t read this before answering questions 1-4!!) Now Casi gets an interesting idea. She decides to draw a picture of the pool table and the ball rather than shooting with the actual pool table. The interesting part is that she will not only draw a picture of the pool table, but also a picture of its mirror image immediately adjacent to it, as in the picture at left.



When she draws a picture of the ball’s trajectory, she can draw a straight line instead of a bent one. It’s as if the ball passes through a mirror into “mirror image land” rather than bouncing. She can, in fact, fill an entire plane with pictures of the table and its mirror image.

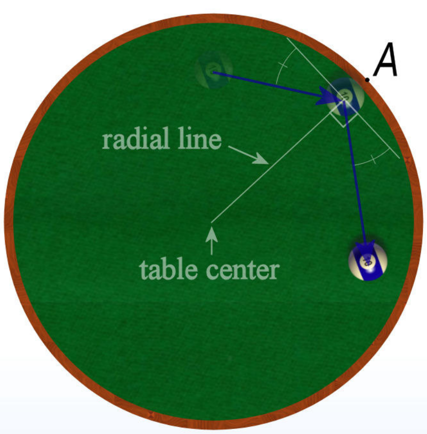


5. On the drawing of mirror images, draw the path of the ball that is shot with a slope of 1.

6. On the drawing of mirror images, draw the path of the ball that is shot with a slope of 2.

7. Revisit questions 1-4 using this trick.

Use your imagination. Now Casi begins to think about ways to change the problem a little bit. (Before reading on, spend a little time thinking about ways to change the problem yourself). For instance, she could try moving the starting point of the ball. She could also try changing the shape of the table. She decides to figure out what happens when the table is a circle.



How the ball bounces. When the ball hits the curb, say at point *A*, it acts just as it would act if the rounded curb were replaced by its tangent at that point. If you don’t know what a tangent is, look at the diagram at left. It depicts what happens when the ball hits the curved curb.

Exercises. Casi decides to place her ball at the very bottom point of the circle.

8. Suppose she shoots the ball at a 45° angle to the tangent. How many times will the ball bounce before returning to the starting position? Try some other angles of your choice.

9. For what angles will the ball eventually return to where it started?

Summer Fun!

Fibonacci Numbers

by Ben Boyajian

The Fibonacci sequence is defined recursively by $F_1 = 1$, $F_2 = 1$, and, for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

1. Complete the following table of Fibonacci numbers:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	1	2	3	5	8						

2. Which of these Fibonacci numbers are even? Which are multiples of 3? Which are multiples of 5? Can you predict in general which Fibonacci numbers will be multiples of 2, 3, or 5? Can you prove your assertions?

3. Prove that if n is a multiple of m, then F_n is a multiple of F_m.

4. Now consider non-Fibonacci numbers. For example, which Fibonacci numbers are multiples of 4? How about multiples of 7? (Hint: $F_6 = 8$ is the first Fibonacci number that is a multiple of 4, and $F_8 = 21$ is the first Fibonacci number that is a multiple of 7).

5. Can you predict in general which Fibonacci numbers are multiples of k, where k is an arbitrary positive integer? Can you prove your assertions?

6. Complete the table of greatest common factors of F_n and F_m at right. What patterns do you notice? Can you prove your assertions?

Table of Greatest Common Factors of F_n and F_m

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
F_1	1	1	1	1	1	1	1	1	1	1	1	1
F_2	1	1	1	1	1	1	1	1	1	1	1	1
F_3	1	1	2	1	1	2	1					
F_4	1											
F_5	1											
F_6	1											
F_7	1											
F_8	1											
F_9	1											
F_{10}	1											
F_{11}	1											
F_{12}	1											

8. Define a sequence by $L_n = F_{2n}/F_n$. Find the first several values of L_n. What do you notice about this sequence? Can you prove your assertions?

9. Consider sums of Fibonacci numbers. Let $S_n = F_1 + \ldots + F_n$. Compute the first few values of S_n. Can you find an explicit formula for S_n? (Hint: Recall the explicit formula for the Fibonacci sequence. See volume 3, number 1 of this *Bulletin*.)

10. Let O_n be the sum of the first n odd Fibonacci numbers and let E_n be the sum of the first n even Fibonacci numbers. Compute the first few values of O_n and E_n. Can you find explicit formulas for O_n and E_n?

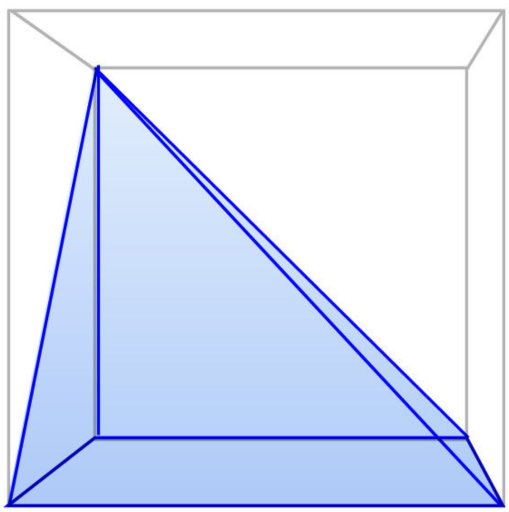
Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

<u>Session 6 – Meet 11 – April 29, 2010</u>

Mentors: Rediet Abebe, Lauren Cipicchio, Grace Lyo, Lauren McGough,
Maria Monks, Charmaine Sia, Liz Simon, Amy Tai, Julia Yu



A square pyramid inside a cube.

Some girls found a very nice way to show that the volume of a certain square pyramid (see figure at left) whose height is equal to the side length s of the base is equal to one third the volume of the cube whose side length is also s.

They observed that you can take 3 congruent copies of this pyramid and fit them together to recover the cube. Do you see how?

This fact is related to the fact that a cube has a 120° rotational symmetry. Any object with such a symmetry can be split into 3 congruent pieces. Looking at the pyramid, notice that its longest edge is a main diagonal of the cube. If you twice rotate the pyramid by 120° around this main diagonal, you will see how the other two pyramids fit together to form the cube.

Although this might seem like a very specific fact having to do with a very specific pyramid, it isn't such an exaggeration to say that it contains the key fact from which one can deduce that the volume of any pyramid with base area B and height h is given by $\frac{1}{3}Bh$.

<u>Session 6 – Meet 12 – May 6, 2010</u>

Mentors: Rediet Abebe, Lauren Cipicchio, Wei Ho, Grace Lyo,
Ariana Mann, Maria Monks, Liz Simon, Julia Yu

We held our traditional end-of-session Treasure Hunt.

This was by far the most challenging Treasure Hunt we've held but the girls did a great job taking up the challenge. Congratulations!

Here is one of the problems from the Treasure Hunt. This is *The Art of Painting* by the Dutch painter Jan Vermeer. The painting is designed to be viewed from a certain distance. What is this ideal viewing distance? Round off your answer to the nearest foot. Scale: 1 inch = 6 inches.

Member’s Thoughts

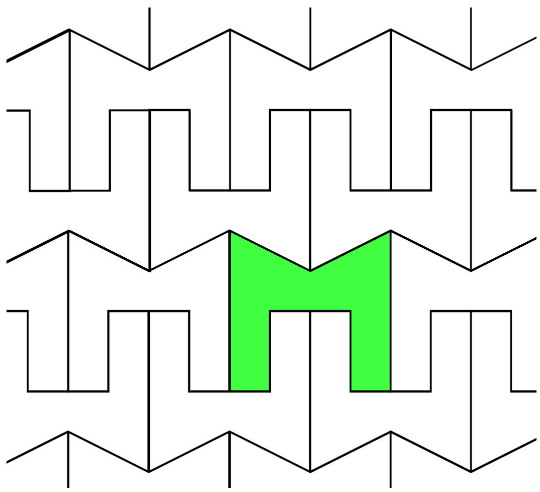
Font Design

Have you ever designed your own font?

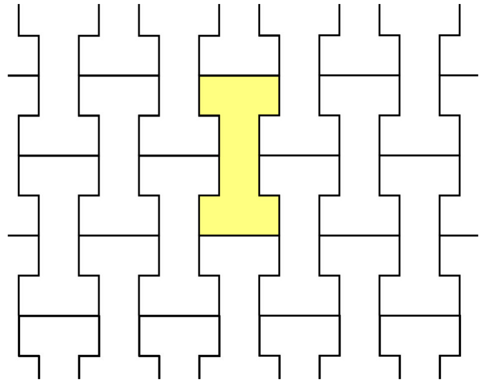
A font is a size and style for writing characters such as the alphabet.

At the club, some girls, working with mentor Charmaine Sia, became engaged with designing a special character font that had the property that every character could be used to tile the plane. Recall that a shape tiles the plane if the plane can be filled entirely with congruent copies of the shape in such a way that the shapes do not overlap.

For example, for the letter M, they came up with . Here’s how this  tiles the plane:



The letter I didn’t need too much modification: . And here’s the  tiling:



Design your own tessellating font! Send in your solutions to girlsangle@gmail.com.

Calendar

Session 6: (all dates in 2010)

January	28	Start of sixth session!
February	4	
	11	
	18	No meet
	25	Jericho Bicknell, Waltham Fields Community Farm
March	4	
	11	
	18	
	25	No meet
April	1	Cynthia Rudin, MIT Sloan School
	8	
	15	Elissa Ozanne, Harvard Medical School and MGH
	22	No meet
	29	
May	6	

Session 7: (all dates in 2010)

September	9	Start of the seventh session!
	16	
	23	
	30	
October	7	
	14	
	21	
	28	
November	4	
	11	Veteran's Day – No meet
	18	
	25	Thanksgiving - No meet
December	2	
	9	

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) is free for members and can be purchased by others. Please contact us if you'd like to purchase printed issues.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

In what ways can a girl participate? There are 2 ways: **membership** and **active subscription** to the Girls' Angle Bulletin. **Membership** is granted per session and includes access to the club and extends the member's subscription to the Girls' Angle Bulletin to one year from the start of the current or upcoming session. You can also pay per session. If you pay per session and come for three meets, you will get a subscription to the Bulletin. **Active subscriptions** to the Girls' Angle Bulletin allow the subscriber to ask and receive answers to math questions through email. Please note that we will not answer email questions if we think that we are doing the asker's homework! We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. Note that you can receive the Girls' Angle Bulletin free of charge. Just send us email with your request.

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, Courant Instructor/PIRE fellow, NYU
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, Moore Instructor, MIT
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, Senior Research Scientist, Harvard Medical School.
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For: ☐ Membership
☐ Active Subscription (interact with mentors through email)

Parents/Guardians: ____

Address: ____ Zip Code: __________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about? ____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________
(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session membership
  - ☐ $50 for a one year active subscription
  - ☐ I am making a tax free charitable donation.

- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________

Girls' Angle

A Math Club for Girls