Girls' Angle Bulletin

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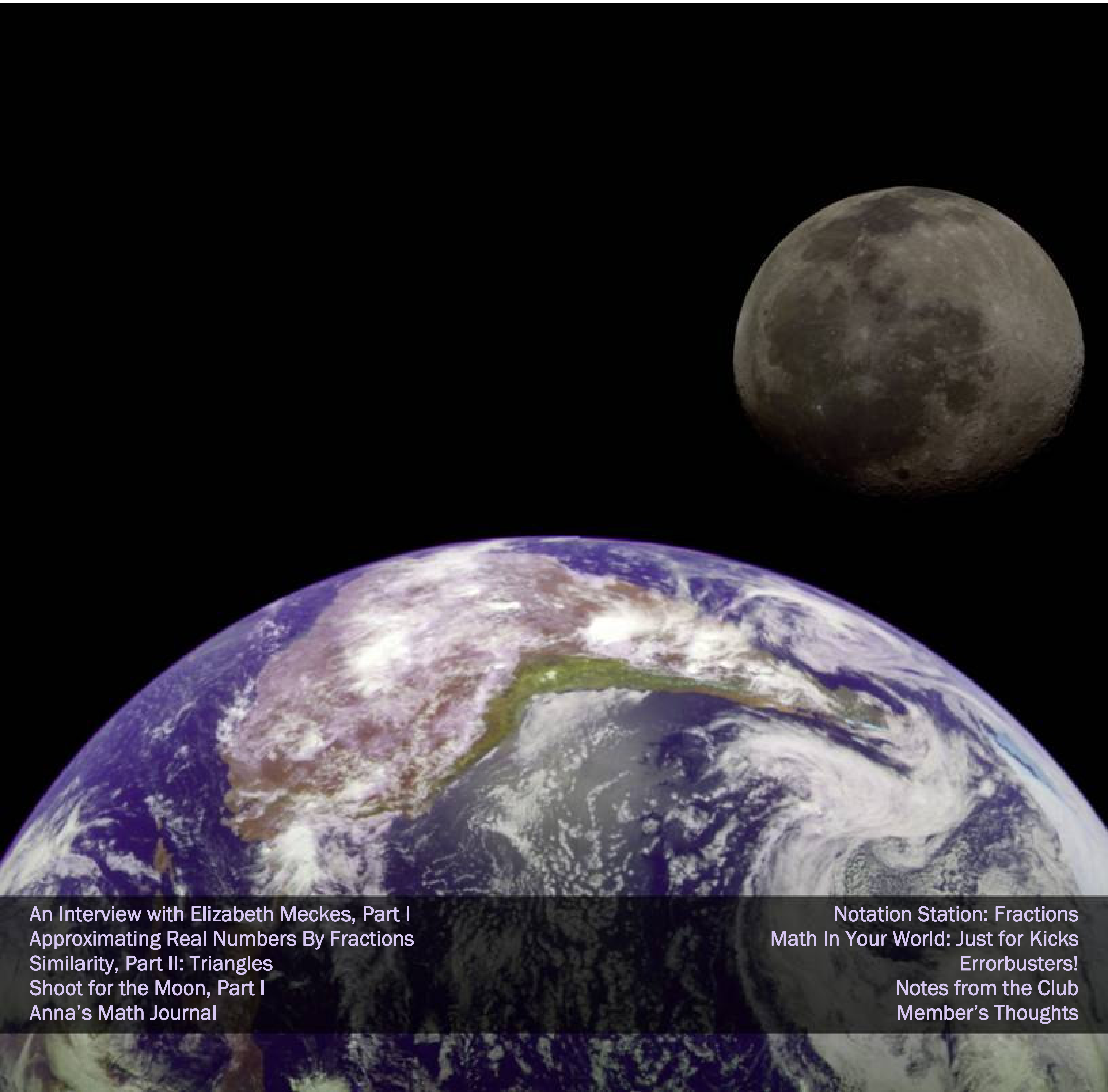
To Foster and Nurture Girls' Interest in Mathematics



From the Director

Math is highly interconnected. Every new math fact you learn can affect your understanding of other math facts you know. Sophisticated mathematical concepts can affect the way one understands even the most basic concepts.

For this reason, at Girls' Angle, we routinely ask mathematicians to explain math. We have a postdoctoral mentor at every meet. And we try to get mathematicians to write articles, like Bjorn Poonen's series on the meaning of addition. In this issue, mathematician David Speyer invites readers to ponder mysterious patterns that appear in a very natural problem, namely, finding rational approximations to π.

Also in this issue, Julia Zimmerman's back with a story that delights while it enlightens, Cammie Smith Barnes keeps us from straying into error, we meet Case Western Reserve assistant professor Elizabeth Meckes, and Katherine Sanden takes over *Math In Your World* from Katy Bold. Katy recently wed and is moving on to new adventures. Katy, thank you so much for all your wonderful contributions to Girls' Angle!

- Ken Fan, Founder and Director

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Girls' Angle Bulletin

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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: The Earth and the Moon, courtesy of NASA.

An Interview with Elizabeth Meckes, Part I

Dr. Elizabeth Meckes is an assistant professor of mathematics at Case Western Reserve University. She is a fellow of the American Institute of Mathematics and received her doctoral degree from Stanford under the direction of Persi Diaconis.

Ken: Hi Dr. Meckes, thank you so much for doing this interview for the Girls' Angle Bulletin. My first question for you is: What is your favorite kind of mathematics?

Elizabeth: This is a hard question to answer, so I'll answer it in a couple of ways. Firstly, if you had to classify me by area, you would call me a probabilist (someone who studies probability theory), and moreover, put me on the more "analytical" end, which means I use a lot of ideas and techniques from the field called "real analysis". One of the things I like about probability theory is that there are a lot of rather counter-intuitive results either about or using probability. I don't just like counter-intuitive results because I'm ornery; they can be a dramatic demonstration of the value of mathematics. Intuition can be wrong, and it can also break down and not tell you anything about a question you care about. Math is a way to move past intuition, and you know it's really done something for you when you come up with a result that you wouldn't have otherwise guessed was true.

To answer a different interpretation of the question, my favorite kind of math at any given moment is usually whatever I'm working on (broadly construed; it's not as though it's interesting every minute). I find that it's hard to be really interested in something until I've delved pretty deeply into it; real appreciation comes after understanding, which comes after a lot of hard work. This is a little unfortunate from one perspective, because it means you can't tell how interesting something will be to you without a lot of initial investment of time and energy, but it also means that you can be surprised by where you find interesting math.

Ken: How did you get interested in probability?

Elizabeth: Through my Ph.D. advisor, Persi Diaconis. I went to graduate school having almost no idea what I wanted to do, although I knew I was more inclined toward analytic (as opposed to algebraic) areas. The usual system to find a thesis advisor is to start by doing reading courses with several professors, sort of like a trial period. I did one with Persi, and enjoyed it. He seemed like someone I'd enjoy talking to once a week for a few years, so I asked him to be my advisor. I strongly advocate this as a way to choose an advisor: you can get interested in almost anything (and you can change areas later, although it's not easy), but it's important to have a good relationship with your advisor. If you dread those weekly meetings, graduate school will feel like a very unpleasant eternity. I could have done a lot of different things with Persi (he has very broad interests); I got into the part of probability that I did mainly because I got interested in a series of lectures he gave about Stein's method (more about that below) during my second year.

Ken: When did you realize that you wanted to become a mathematician? Do you remember a specific experience that got you excited about mathematics?

Elizabeth: I'd always sort of assumed I'd be an academic scientist of some sort, and when I started college I thought I'd probably be a physicist. But I found when taking classes, especially in my sophomore year when I started taking "real math" courses (that is, proof-based courses for math majors like abstract algebra and real analysis, as opposed to calculus, which is taught to a large audience of mostly non-math majors) that the way my mind worked seemed perfectly suited to being a mathematician. Math classes were certainly hard for me, but hard in a different way from other classes; math made sense to me in a way that physics didn't. I felt like I'd found my niche, and never really considered doing anything else.

Ken: What do you do, as a mathematician?

Elizabeth: That's a great question; I think most people really aren't clear on what mathematicians do all day. I once saw a great description of a mathematician's week: "Monday, tried to prove theorem. Tuesday, tried to prove theorem. Wednesday, tried to prove theorem. Thursday, tried to prove theorem. Friday: theorem false." So for one thing, I, like most mathematicians, am wrong a lot. But that doesn't really answer your question: what does it mean to spend your days and years trying to prove theorems? Sometimes it starts with a conjecture: something you read, a conversation with another mathematician, or something that you've been thinking about on your own makes you suspect that a specific statement is true. So you think about how to try to prove it: read other people's papers looking for ideas, scribble thoughts as they come to you, or just sit and think about it. Other times it's more exploratory: "I wonder what happens if you try to apply this argument in this other setting?" I usually try quite a few blind alleys, but can normally tell when I'm actually on to something. Of course, I don't know for sure if I've really proven something until I write it down carefully, as if I'm explaining it to something else. I really believe that you don't understand anything until you can explain it to someone else, and trying to do so is a great exercise to clarify your ideas and figure out which things you still don't understand.

Ken: You mentioned a mathematical tool that you like to use: Stein's method of exchangeable pairs. Can you explain to us what that is, or what it is essentially based on?

Elizabeth: Okay; I think most people are familiar with the bell-shaped curve, what I would call the Gaussian distribution. There's a general class of theorems called "central limit theorems", which are about showing that some specific quantity "looks Gaussian". The classic example is in terms of coin tossing. If you toss a coin 1,000 times, you expect to get about 500 heads and 500 tails. But of course you probably won't get *exactly* 500 of each, just close to that. So you could look a little closer and ask: if you perform the experiment millions of times and make a histogram of the number of heads from each experiment, what will it look like? We expect a big peak at 500, but can we say more about the shape of the histogram? The answer is yes: it's a bell-shaped curve, and that fact is the classical central limit theorem. You can write down a formula for the bell-shaped curve, and it has a lot of special properties; all of the approaches to proving central limit theorems take advantage of at least one of these special properties. Stein's method is probably the newest. It was invented in the late sixties by Charles Stein. That might sound like quite awhile ago, but when you think about the fact that people have known about the bell-shaped curve for more than two hundred years, saying much of anything really new and important about it is pretty amazing. The special property that Stein's method takes advantage of is the fact that the function which describes the bell-shaped curve is the only function there is that satisfies a certain differential equation; if some other curve (like a curve fitted to the coin-tossing histogram described above) "comes close" to satisfying the same differential equation, then it's close to the bell-shaped curve. This was Stein's fundamental idea, and a huge number of papers have been written extending and exploiting this idea. The exchangeable pairs part has to do with a way of actually implementing Stein's insight. It turns out that you can try to detect whether a curve governing a histogram like the coin-tossing one will satisfy this special differential equation by checking that the collection of outcomes of the experiment change in a very specific way when you change the experiment just a little. For example, you could take each of your coin-tossing experiments, but throw away the last toss and do it again, and replot the histogram. It will shift a little, and analyzing the particular way it shifts is a way to detect this differential equation at work. This all probably sounds a little silly, because if you already have the histogram, you can tell whether it looks bell-shaped or not, right? The thing is, often you don't have the histogram (say, because it's too hard to compute), but the amazing thing is that even if you can't plot the histogram, you can sometimes still figure out some things about how it would look if you could plot it, like how it would change if you tweaked the experiment a little.

Ken: Can you explain to us one of your own results?

Elizabeth: One of my favorite results is a central limit theorem in the context of Riemannian geometry. Riemannian geometry is about the geometry of curved spaces—think of the surface of a sphere as opposed to the surface of a table. On such a curved space, there are special functions (called "eigenfunctions of the Laplacian") which encode a lot of information about the geometry of the space—they are related to the famous question "Can you hear the shape of a drum?" In general, if you have a complicated function on a complicated space and you're trying to get a handle on what it's like, one interesting thing to think about is the distribution of the values it takes on; like the distribution of students' grades on an exam, knowing about which values the function takes on and how often might give you meaningful information. What I proved was a result about when the value distribution of one of these special functions looks like the bell-shaped curve. In particular, I showed that if you pick one of these functions on a high-dimensional sphere at random, it will probably have a value-distribution that looks like a bell-shaped curve. I like the mix of fields here, the interaction between geometry and probability.

I once saw a great description of a mathematician's week: "Monday, tried to prove theorem. Tuesday, tried to prove theorem. Wednesday, tried to prove theorem. Thursday, tried to prove theorem. Friday: theorem false." So for one thing, I, like most mathematicians, am wrong a lot.

Another reason I like this result is that it is an example of something I think is really interesting: the study of "high-dimensional phenomena". Here, "high" means tending to infinity, so even a million dimensions isn't considered high. The idea of high dimensions is a little hard to wrap your head around; most of us are used to thinking in three dimensions, and that's it. But one of the great things about math is that once you turn your intuition about the world (in this case, about three-dimensional space) into a mathematical framework, you can abstract and generalize it without needing to follow any kind of rules that you feel hampered by from the physical world; the only rules you need to follow are mathematical ones. So you can start talking about space of arbitrarily many dimensions, and make sense of it.

To be continued…

Approximating Real Numbers By Fractions

by David E. Speyer

As you may have heard, π is irrational. There is no fraction p/q which is equal to π. Nonetheless, there are some popular approximations people often use. For quick and dirty work, $\pi \approx 3$. A much closer approximation is $\pi \approx 22/7 = 3.142857142857\ldots$. An even better one is $355/113 = 3.14159292035398230088\ldots$. (I'm not writing out the full period of this one; it goes 112 digits before repeating!)

In what sense are these the best approximations to π? Why, for example, is 31/10 not an equally good candidate? How could we have found these numbers methodically—and found them for any real number like $\sqrt{13}$ or $\sqrt[3]{2}$, not just for π? These are the questions we will try to answer in this article and its sequel.

First of all, when is p/q a good approximation to π? The first answer one might think of is: When the difference between π and p/q is as small as possible. But, by that standard, we can always do better! The fraction 31/10 is closer to π than 3 is, 314/100 is closer than 31/10, 3141/1000 is closer than 31/10; we can always take more digits and get closer. What we want from an approximation is not just that p/q be close to π, but that p/q be close to π and p and q be small. We'll measure the size of p and q by $p + q$.

This leads us to our first definition:

Definition For any positive integer N, we will say that p/q is the best approximation to π of size N if, among all fractions r/s with $r + s \le N$, the fraction p/q is the closest to π.

It turns out that it will clarify things later if we separate the notion of being a good *upper approximation* to π and a good *lower approximation*.

Definition For any positive integer N, we will say that p/q is the best lower approximation to π of size N if, among all fractions r/s with $r + s \le N$ and $r/s < \pi$, the fraction p/q is the closest to π. We will say that P/Q is the best upper approximation to π of size N if, among all fractions r/s with $r + s \le N$ and $r/s > \pi$, the fraction P/Q is the closest to π.

For example, let's take $N = 30$. There are 496 pairs (p, q) of nonnegative integers such that $p + q \le 30$ (can you see how to compute this?) Of course, we only want fractions that are in lowest terms, and we don't want to consider $p/0$ (which isn't even defined!). Throwing these out leaves 278 fractions p/q with $p + q \le 30$. Here are the ones near π:

$$\ldots \frac{8}{3} < \frac{19}{7} < \frac{11}{4} < \frac{3}{1} < \pi < \frac{22}{7} < \frac{19}{6} < \frac{16}{5} < \frac{13}{4} < \frac{23}{7} < \frac{10}{3} \ldots$$

So 3/1 is a good lower approximation to π, the best one with $p + q \le 30$. And 22/7 is a good upper approximation, again, the best with $p + q \le 30$.

We can try this for any value of N, and the bounds on π will get tighter and tighter as N increases. Here's how it happens. We'll start at $N = 5$:

N	Best Lower Approximation	Best Upper Approximation
5	3/1	4/1
6	3/1	4/1
7	3/1	4/1
8	3/1	4/1
9	3/1	7/2
10	3/1	7/2
11	3/1	7/2
12	3/1	7/2
13	3/1	10/3
14	3/1	10/3
15	3/1	10/3
16	3/1	10/3
17	3/1	13/4
18	3/1	13/4
19	3/1	13/4
20	3/1	13/4

As you can see, there is a lot of repetition, because going to a larger N doesn't always help us find a better approximation. We'll keep the table readable by listing only the rows where one of the approximations changes:

N	Best Lower Approximation	Best Upper Approximation
5	3/1	4/1
9	3/1	7/2
13	3/1	10/3
17	3/1	13/4
21	3/1	16/5
25	3/1	19/6
29	3/1	22/7
33	25/8	22/7
58	47/15	22/7
91	69/22	22/7

So far, we have followed our noses. We made a definition of what a good upper approximation and a good lower approximation are, we figured out how to compute them, and we computed a bunch of them. Now, it is a good time to look back at our data and see what we can observe.

For many values of N, the lower approximation p/q and upper approximation P/Q don't change as we increase N. Then, all of a sudden, a new number a/b pops up, which is between p/q and P/Q. If a/b is less than π, then a/b replaces the fraction p/q and, if a/b is greater than π, then a/b replaces P/Q.

We started with the bounds (3/1, 4/1). Then 7/2 landed in the middle of the interval, and replaces 4/1. Then our bounds were (3/1, 7/2), until 10/3 came along to be a better bound. Let's rewrite our table to emphasize this picture of new fractions squeezing themselves in between the previous bounds:

$$\begin{array}{lll|lll} \frac{10}{3} & \to & \left(\frac{3}{1},\frac{7}{2}\right) & \frac{22}{7} & \to & \left(\frac{3}{1},\frac{19}{6}\right) \\ \frac{13}{4} & \to & \left(\frac{3}{1},\frac{10}{3}\right) & \frac{25}{8} & \to & \left(\frac{3}{1},\frac{22}{7}\right) \\ \frac{13}{4} & \to & \left(\frac{3}{1},\frac{10}{3}\right) & \frac{47}{15} & \to & \left(\frac{25}{8},\frac{22}{7}\right) \\ \frac{16}{5} & \to & \left(\frac{3}{1},\frac{13}{4}\right) & \frac{69}{22} & \to & \left(\frac{47}{15},\frac{22}{7}\right) \\ \frac{19}{6} & \to & \left(\frac{3}{1},\frac{16}{5}\right) & \frac{??}{??} & \to & \left(\frac{69}{22},\frac{22}{7}\right) \end{array}$$

And now, here is my challenge to you.

Challenge What will be next fraction, to insert itself between 69/22 and 22/7?

If you look at the data in this table, you should be able to figure it out without listing all the thousands of fractions with two digit numerator and denominator.

I also want to point out a very nonobvious pattern in this data. You might wonder how tight our bounds for π are. If π is bounded between $\frac{p}{q}$ and $\frac{P}{Q}$, how small is $\frac{P}{Q}-\frac{p}{q}$? Let's try a few examples:

$$\frac{4}{1}-\frac{3}{1} = \frac{1}{1} \qquad \frac{19}{6}-\frac{3}{1} = \frac{1}{6}$$

$$\frac{25}{8}-\frac{22}{7} = \frac{1}{56} \qquad \frac{69}{22}-\frac{22}{7} = \frac{1}{154}$$

Surprise! As far as our data goes, $\frac{P}{Q}-\frac{p}{q}$ always equals $\frac{1}{qQ}$!

Of course, the fact that the denominator is qQ isn't a big surprise; that's just the common denominator you have to put these fractions over in order to subtract them. But the numerator is a surprise. In that last computation,

$$\frac{22}{7}-\frac{69}{22}=\frac{22\times 22-69\times 7}{22\times 7}=\frac{484-483}{22\times 7}=\frac{1}{22\times 7}.$$

why do the 484 and 483 cancel like that?

Next time, we'll resolve the challenge and explain the surprise. Unless you figure them out first!

Similarity, Part II: Triangles

by Ken Fan
edited by Jennifer Silva

Last time, we decided to take a close look at the concept of similarity as applied to triangles. Recall that two figures are similar to each other if ratios of corresponding lengths are equal. Another way of putting it is to say that two figures are similar if either is a scale model of the other.

What might a mathematician who has never thought about similarity and triangles do to begin to get an understanding of the topic? What would you do?

Of course, I can't predict what you or any mathematician would do, so I'm going to write about what a mathematician might do to gain an understanding of similarity and triangles. A mathematician might (in no particular order) do any of the following:

1. Draw many triangles and carefully put them into groups of similar triangles. Then, look for patterns in the groups by asking: What do triangles in the same group have in common? How do triangles in separate groups differ?

2. Think more about the notion of similarity in general and relate each thought to triangles.

3. Decide that there are other shapes worth looking at with respect to similarity that are easier to think about than triangles and study those instead. Perhaps the mathematician is more comfortable pondering circles, rectangles, or angles instead of triangles. After contemplating other shapes, perhaps thinking about triangles will be easier.

4. Draw a triangle and then try to create other triangles that are similar to it. Carefully observe what the other triangles must have in common with the first triangle, as well as in what ways they can differ.

5. Start by assuming something without concern for whether it is true or false and try to prove or disprove it. For example, one might just assert, "All triangles are similar to each other!" Then, one might try to prove that it is true or find a counterexample that shows it is false. If it turns out to be false, one could then try to alter the statement and sculpt it into something that is true.

6. Any combination of the above.

7. Something else that isn't even described above!

Why don't you give it a try? Play with the concept of similarity and triangles and see what you can discover. See if you can get to a level of understanding where you can give different criteria that establish that two triangles are or are not similar to each other. Can you describe a set of triangles no two of which are similar to each other and that contains a triangle that will be similar to any triangle you can think of? When you're ready, continue on to page 20.

Shoot for the Moon, Part I

Written and illustrated by Julia Zimmerman

She could almost see the man's face—he was easier to see than the bunny, anyway, which she could only make out if she kind of tilted her head. And squinted. And if the bunny had one ear longer than the other. And... only three legs.

"Clara, Earth to Clara, do you read me? Over."

Clara looked at Elizabeth, startled out of her reverie. The girls were in Elizabeth's living room, surrounded by an array of notes, pencils, and textbooks; Clara was standing at the window, and Elizabeth was on the floor near her with a book on her lap, but she was looking up at Clara rather than at the pages, and wearing a mischievous grin.

"Clara, do you even need to stare out of the window to see the moon? I mean, your head is *already* lost in outer space."

Clara snorted and tossed one of the sofa pillows at Elizabeth. "Okay, so I was a little distracted—but I think I've already gone over most of the stuff for tomorrow's test. What are you reviewing now?"

Elizabeth shrugged. "Diameter of the earth, names and order of the planets, types of meteors, distance to the moon—"

"Wait," Clara interjected, "I don't remember that last one. What is it, again?"

"About 384,403 kilometers, or 238,857 miles."

"I can remember that," said Clara. "Say... how do you think they measured that in the first place?"

Elizabeth looked at her innocently. "Don't you remember hearing about this in class?" Clara shook her head and waited for Elizabeth to explain. "Well, it was hard—it took years to assemble, then they had to get the thing attached to the spaceship, and *then* they had to figure out how to unroll it as they went... it took eight astronauts just to unspool it!"

Clara stared at her blankly. "... Unspool it? What are you talking about?"

Elizabeth's smile widened and she said, as sincerely as possible, "The giant tape measure."

There was a moment of silence, and then both girls burst into laughter. The image was a ridiculous one—and, of course, an unlikely one.

Clara wondered how the distance from Earth to the moon had actually been measured... Had the process been complicated? Did somebody have to invent special equipment? Was there an experiment? Elizabeth took up her book and returned to studying, but Clara's thoughts continued to wander—could you fly to the moon and back and time how long it took to figure out the distance involved? But then... the equation

$$\textit{distance} = \textit{speed} \times \textit{time}$$

required that the spaceship fly at constant speed, or that the speed be measured accurately at all times. The former was impossible, and the latter extremely challenging and impractical...

Elizabeth looked over at Clara and sighed. She picked up the sofa pillow and put it back on the couch, and then started walking towards the den, where the computer was. "All right, I'm curious, too—let's see if the Internet can tell us how they did it."

"Clara, I found something!" Elizabeth was staring excitedly at the computer screen, gesturing Clara over. "Here's a brief description of an experiment run here in the US to measure the distance from the Earth to the moon: it says here that... an observatory sent out pulses of light from a laser beam, and this light reflected off of an array of mirrors known as corner reflectors... and when this light hit a target at the observatory back on Earth, the scientists timed how long it had taken for the light to get from the laser to the moon and then back to Earth. They used the equation *distance* = *speed* × *time* to calculate it.

Since the light went to the moon and back, and they only wanted half that distance, they divided the end result by two. The equation relies on the relationship between constant speed, time, and distance.

"The time it takes for light to get from Earth to the moon is about 2.5 seconds, and since light has a constant speed, c (about 186,282 miles per second[1]), they could easily compute the distance by plugging in the known quantities—time and speed—and solving for the unknown quantity—distance."

"Well, that doesn't sound too complicated..." said Clara.

Elizabeth nodded. "I'm really curious about that stuff about 'corner reflectors,' though! I've never heard of those before, have you?"

Clara nudged Elizabeth with her elbow. "I guess I'm not the only one who's been distracted from studying, am I, Liz?"

Elizabeth sat down next to Clara, scooting her over so they could both fit in front of the computer. "Fine, fine. But now that we know about some of the experiment we can't just stop reading about it! I mean, it makes sense to finish what we've started, right?"

"Oh, definitely. Absolutely. No other option."

"I know you're making fun of me."

"Would I do that? Me??"

Elizabeth shot Clara a look to let her know she was not fooled. "Corner. Reflectors. Look it up."

"Yes, ma'am! Corner reflectors, here we come..."

"Okay, Elizabeth—here's some information about the mirror arrays placed on the moon, the corner reflectors... A corner reflector consists of three (or sometimes two) mirrors, attached like the corner of a cube, so that each is perpendicular to the others. Oh, and there's a picture…"

Elizabeth looked at the diagram and, reading the article next to it, said, "So, it looks like corner reflectors were used because they have the special property that light beams exiting, or bouncing off, the corner reflector travel in a path parallel to the path on which they came into, or first approached, the corner reflector."

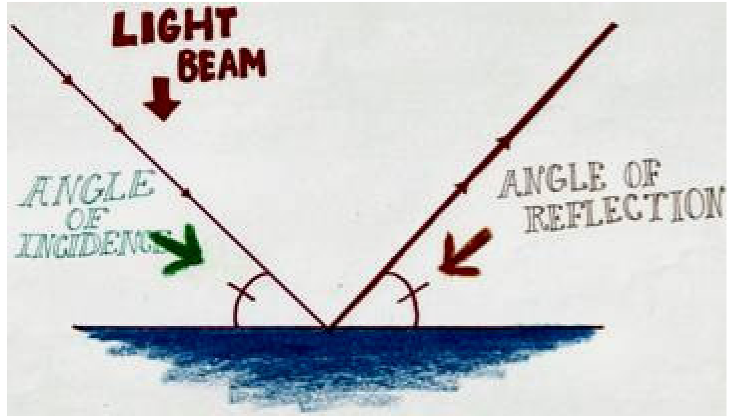
Clara nodded in agreement. "Yes, and this property results from the geometric arrangement of the mirrors in the corner reflectors. It says here the underlying principle has been known since the time of ancient Greece...

"When a light beam reflects off of a surface, the angle that the beam of light makes with the surface before it hits is the same as the angle that the beam will make with the surface after it hits, as light travels away from the surface. This website says that same idea this way: 'The angle of incidence is equal to the angle of reflection.'"

"Hmmm... I'm going to draw a diagram," said Elizabeth.

[1] The speed at which light travels changes when the light beam passes from one medium to another; it is inversely proportional to the **index of refraction** of the medium. The speed v_{medium} at which light travels through a given medium is equal to the fastest speed light can travel, c, divided by the index of refraction of the medium, n_{medium}, that is, $v_{\text{medium}} = c/n_{\text{medium}}$. The index of refraction in a vacuum is 1. Space is very similar to a vacuum, so v_{space} is very close to c. Air has an index of refraction of about 1.0003, so v_{air} is slightly less than v_{space}. The speed of light can only change when traveling from one medium to another.

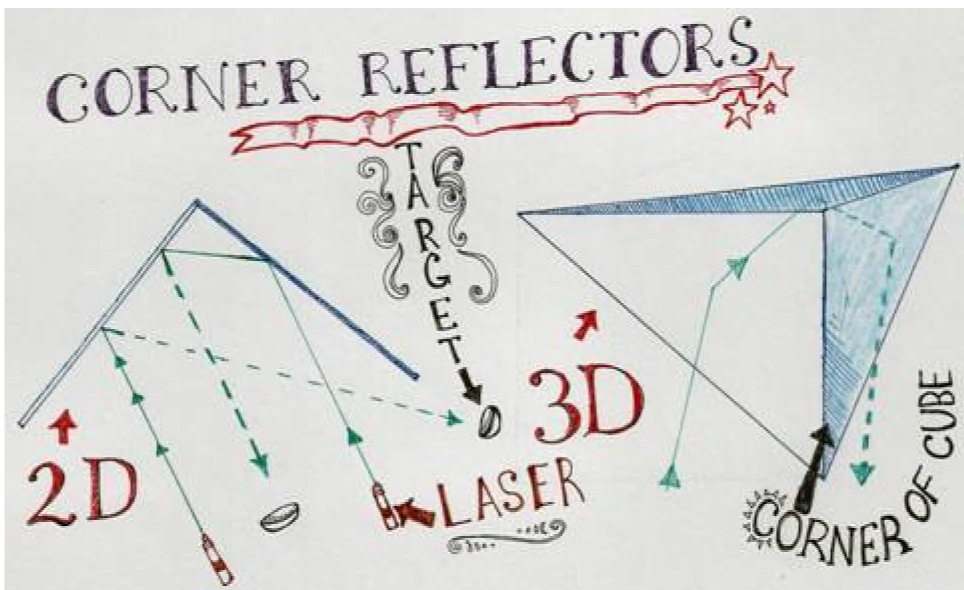
Meanwhile, Clara continued to follow links. She found one that gave a little of the history and read out loud to Elizabeth: "The ancient Greeks knew a lot about optics (the behavior of streams of light) because optics involves a lot of geometry—something at which the ancient Greeks excelled. Aristotle first stated the law of reflection: that the path a stream of light takes will be the same regardless of which "end" you take as the beginning. It then follows that the angle of incidence is equal to the angle of reflection, since if you reverse the direction of the light but keep the same path, the angle of incidence and angle of reflection switch places by definition. Does that match with what you drew in your diagram, Elizabeth?"



"Yep—so far, so good."

"Hmmm. So, why wouldn't they use a regular mirror for the experiment? Why did they choose corner mirrors?"

Elizabeth studied her diagram. "Well, the special thing about corner mirrors seems to be related to the angles of incidence and reflection. The light hits one side of the array and bounces around until it leaves the array along a path parallel to the one by which it came in... What happens to the angles of incidence and reflection in the case of a regular mirror?" Elizabeth pictured several beams of light hitting the two types of mirror.



"Want to see if your diagram reflects what happens if we make our own corner mirror?" asked Clara.

"Sure, Clara, hang on—I think there's a laser pointer in my brother's room..." Elizabeth got up to get the laser, but Clara grabbed her arm.

"Wait! Did you get it?? I said 'reflects'—if the diagram 'reflects' what will happen!"

Elizabeth rolled her eyes. "Your razor-sharp wit is as cutting as ever, Clara."

Clara giggled. "Okay, go get the laser. I'm going to find some mirrors. Where's the screw driver? I'm going to take the little medicine cabinet doors down from the bathroom."

"It's in the left kitchen drawer—wait, what are you going to do?? Wait a minute, Clara!" Elizabeth ran after her friend. "I don't know if that's a good idea!"

The Experiment

All of their materials assembled, Elizabeth and Clara were ready to try out the corner reflector array, using the reflector Clara had taped together from the doors of the medicine cabinet in Elizabeth's bathroom. Luckily, Elizabeth's house had a pretty long hallway...

They turned out all the lights in the hall and they put the corner reflector on a chair at the far end. Elizabeth held a paper plate labeled 'OBSERVATORY TARGET', and Clara held the laser pointer.

Clara shone it at the array—the red beam of light appeared as a dot on the wall, and Elizabeth moved her arm until the dot was on the plate. Whenever Clara moved the laser pointer, as long as she kept it trained on the corner reflector, Elizabeth found that all she had to do to keep the beam on the target was hold the target approximately behind the laser pointer. If Clara moved right, Elizabeth had to move the target right, and if Clara moved left, Elizabeth had to move the target left.

Next, Elizabeth and Clara hauled out a wall-mirror that usually hung on the back of Elizabeth's bedroom door. They moved the corner reflector and placed this mirror sideways on the chair so that its back rested flush against the back of the chair.

When they shone the laser at this new set-up, they encountered something unexpected: they had thought, based on what they'd read, that it would be obviously "harder" or more complicated to get the light to hit the target when they used a regular mirror than it had been when they'd used a corner reflector. However, they found that it was just as easy to catch the light beam with the target. If Clara did move the laser pointer a lot, Elizabeth found that all she had to do was move the target in the opposite direction from the laser pointer: if Clara moved the laser pointer to the left, Elizabeth had to move the target to the right.

They tried the corner reflector again—but found that keeping the beam on the target was no easier than it had been with the regular mirror. They wondered if they were doing something wrong, but the laser seemed to be working, the regular mirror seemed to be reasonably flat, and their corner reflector was at nice 90 degree angles. They couldn't think of anything else to check, so they decided to believe their rather surprising results.

"Even though the experiment turned out a little funny, it is a pretty good corner reflector, isn't it?" Clara asked proudly.

"I can't believe you took apart the cabinet..." groaned Elizabeth. "I sure hope you know how to put those doors back on!"

"Relax, Liz," grinned Clara. "You just have to screw them back on. No big deal."

"No big deal..." repeated Elizabeth, skeptically. "Why am I worried whenever I hear you use that phrase?"

Clara smiled at her. "I have no idea."

The girls had put back most of their materials, and Clara was about to begin removing tape from her corner reflector. She handed the laser pointer to Elizabeth to replace in her brother's room. Elizabeth accepted it absentmindedly; she seemed to be contemplating something.

"You know what *I* have no idea about?" Elizabeth was staring at her reflection in the corner mirror.

Clara waited expectantly.

"...Why did they use a corner reflector? It was just as easy to use the regular mirror as it was to use the corner mirror." Elizabeth's upside-down face looked back at her, puzzled.

Clara picked up Elizabeth's diagram and studied it briefly. "I don't know," she said, "but I'm curious, too." She set the diagram down and reached again for her corner reflector, planning to disassemble it.

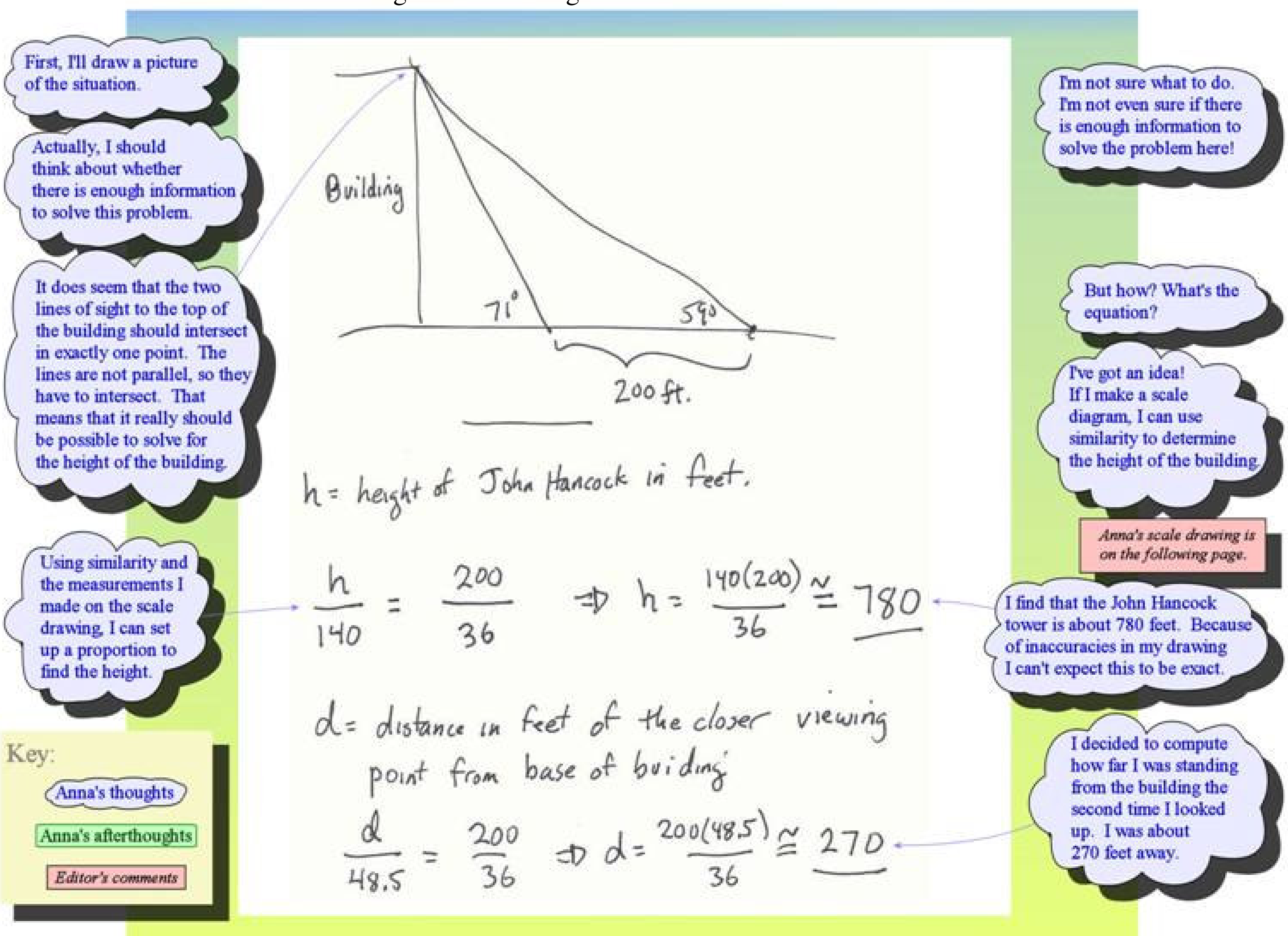
"Wait!" Elizabeth stopped her. "I have a feeling we may need that again..."

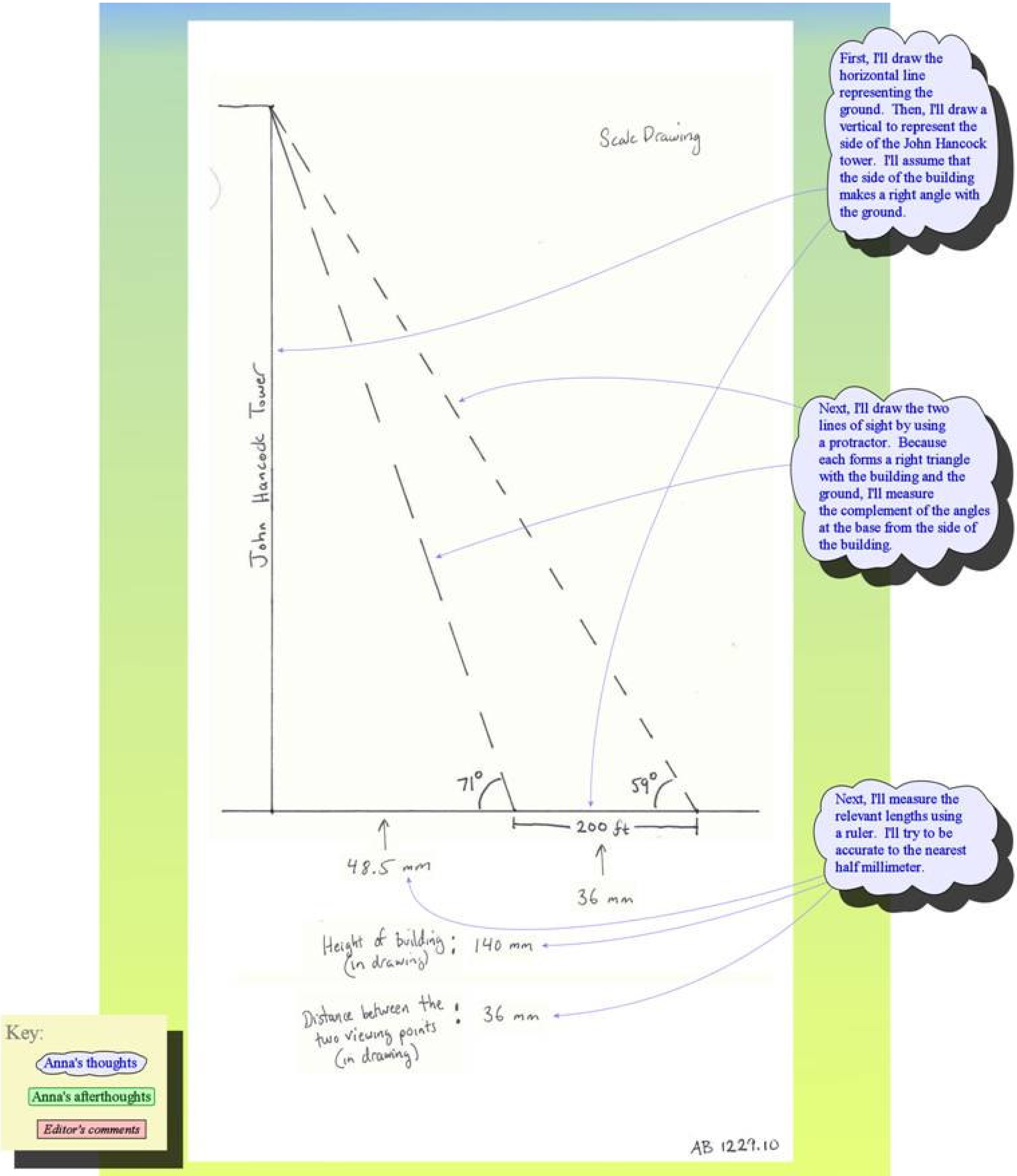
Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

During the holidays, Anna found herself in downtown Boston admiring the John Hancock tower, the tallest building in New England. Near the corner of Boylston and Dartmouth, she noted that her line of sight to the top of the tower made a 59° angle to horizontal. When she walked 200 feet closer toward the building, she found that the angle changed to about 71°. Later, she wondered if she could figure out how high the tower stands from this information alone.




Scale Drawing
John Hancock Tower
71°
59°
200 ft
48.5 mm
36 mm
Height of building (in drawing): 140 mm
Distance between the two viewing points (in drawing): 36 mm
AB 1227.10
First, I'll draw the horizontal line representing the ground. Then, I'll draw a vertical to represent the side of the John Hancock tower. I'll assume that the side of the building makes a right angle with the ground.
Next, I'll draw the two lines of sight by using a protractor. Because each forms a right triangle with the building and the ground, I'll measure the complement of the angles at the base from the side of the building.
Next, I'll measure the relevant lengths using a ruler. I'll try to be accurate to the nearest half millimeter.
Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

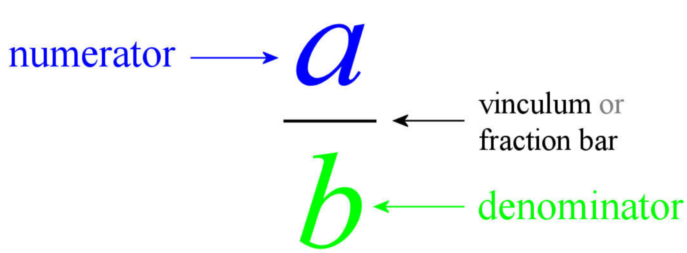
Notation

by Ken Fan

Through time, some ways to notate mathematical ideas have become standard. Learning the notational conventions is important in order to be able to communicate. The best way to learn new notation is the same way we learned the alphabet: repetition. Practice makes perfect!

Fractions

Fraction notation is just another way to express the concept of division.



A fraction.

The fraction above means "*a* divided by *b*." Here are other ways of expressing the same thing:

$$\frac{a}{b} = a \div b = a/b = a : b$$

With fractions, there are implied parentheses around both the numerator and denominator:

$$\frac{21+3}{2\times 6} = (21 + 3) \div (2 \times 6) = 24 \div 12 = 2.$$

But,

$$\frac{21+3}{2\times 6} \neq 21 + 3 \div 2 \times 6 = 21 + 9 = 30.$$

Because division by zero is undefined, no meaning is attached to a fraction with a zero in the denominator.

Sometimes, the word "fraction" specifically refers to a fraction where the numerator and denominator are both integers. Such fractions are also known as **rational numbers**, and here's a quick problem to show how useful rational numbers can be: Three apple pies have to be divided evenly among seven people. How much apple pie does each person get? Answer using a rational number in the form of a fraction, using a percentage, and using a decimal number. Which answer do you find easiest to work with?

MATH IN YOUR WORLD

Logo Design by Hana Kitasei

Just for Kicks

By Katherine Sanden

I recently bought a colorful pair of sneakers– apparently sneakers like these are called “kicks.” I get excited every time I wear them in public. What is it that makes these shoes so much fun for me? It’s partly the bright colors. But even more than that, it’s the feeling of being unique. I like to think I’m the *only* person around who wears these shoes.

I would do a double-take if I were walking down the street and I saw someone wearing the exact same kicks. “What are the odds?!” I would exclaim.

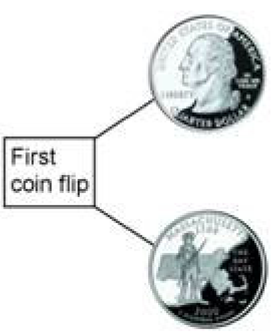
This got me thinking… what *are* the odds?

Let’s consider a simple scenario. Imagine that three friends go to a store that has four choices of shoes (red shoes, blue shoes, green shoes, and black shoes). What is the probability that two (or more) of them pick the same color shoes?

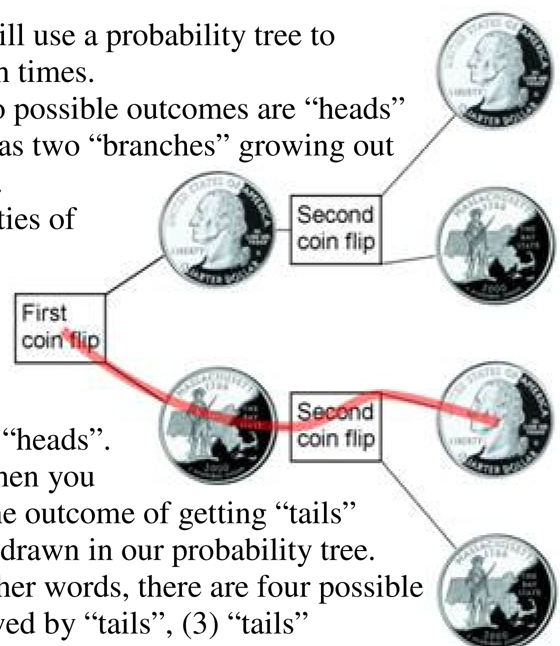
To answer this question, we’ll draw a picture known as a **probability tree**. A probability tree is a picture of all possible outcomes, connected by lines, which we call “branches”. This idea is best illustrated by an example.

Example of a probability tree

Say we have a fair coin and we flip it twice. We will use a probability tree to compute the probability that the coin lands on “heads” both times.



The first time we flip the coin, the two possible outcomes are “heads” and “tails”. So we draw those two outcomes as two “branches” growing out of the “First coin flip” (see the figure[1] at left).



Now we flip again. The two possibilities of “heads” or “tails” branch out of each of our first outcome possibilities (at right).

How do we interpret this tree? You can think of it like a board game. You start at the “First coin flip” and flip a coin. If you get “heads” you move up along the branch to “heads”. If you get “tails” you move along the branch to “tails”. Then you flip the coin again, and move accordingly. For instance, the outcome of getting “tails” followed by “heads” would be represented by the red path drawn in our probability tree.

Notice that there are four “leaves” of the tree; in other words, there are four possible final outcomes: (1) “heads” both times, (2) “heads” followed by “tails”, (3) “tails” followed by “heads”, and (4) “tails” both times. Since we’re dealing with a fair coin, there is an equal likelihood of ending up at any of the four final outcomes.

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[1] Quarter-dollar coin images courtesy of the United States Mint

When each outcome is equally likely, we can compute the probability of a particular outcome by counting the number of such outcomes and dividing by the number of all possible outcomes. From our tree, we can see that there is only one way to get two heads and four total possible outcomes. Thus we can read off of the tree that the probability of getting “heads” both times is $\frac{1}{4}$.

Using this tree, we can also read off other probabilities. For example, what is the probability that at least one of the coins is “tails”? From the tree, we can see that there are 3 final outcomes in which at least one “tail” is flipped. So we can see that the probability is $\frac{3}{4}$.

Now let’s take this idea of a probability tree, and apply it to the shoe situation. We have three friends, let’s call them A, B, and C, going to a store that offers four choices of shoes (red shoes, blue shoes, green shoes, and black shoes). What is the probability that two (or more) of them pick the same color shoes?

Let’s make some simplifying assumptions:

1. They are equally likely to purchase any shoe color (this is similar to our assumption that the coin was a fair coin).

2. No one can see what others are buying (i.e. no one’s choice is influenced by anyone else’s choice).

3. There is an infinite supply of each type of shoe, meaning we’ll never run into the case where someone has bought the only pair left.

Notice that these assumptions may not be true in real life! As we go through the math, see if you can figure out where each assumption is used.

Now, let A pick a pair of shoes first. She can pick the red shoes, blue shoes, green shoes, or black shoes, giving us four different possibilities. Thus begins our tree (see the following page). The tree shows the various possibilities after each friend has picked a shoe. There are 64 leaves on the right side of the diagram representing each of the 64 possible ways 3 friends can pick shoes from a choice of 4 colors. For instance, the red dot at the top represents the outcome in which A, B, and C all choose red shoes. The black dot at the bottom represents the outcome in which A, B, and C all choose black shoes.

I put a star next to each outcome in which two or more friends chose the same colored shoes. Of the 64 total possible outcomes, there are 40 starred ones. Because all the different outcomes are equally likely, the probability that two or more friends choose the same colored shoes is $\frac{40}{64}$ or $\frac{5}{8}$. That means it’s more likely than not that if the friends pick according to our assumptions, at least two of them will end up with the same colored shoes! If they don’t want that to happen, they better coordinate their shopping efforts.

Notice that when I drew the tree, I had A pick first, followed by B, then C. In this situation, does it matter who picks first? Imagine you drew a tree where B picks first, followed by A, then C. Would you get the same result?

Can you use this tree to determine the probability that all three friends pick the same shoes? How about the probability that all three friends pick different shoes? What is the probability that *exactly* two of the friends pick the same colored shoes?

Take it to your world

Suppose we have three different shoe colors instead of four. Draw a tree to represent all the possible outcomes that can occur when 3 friends choose from 3 different colors. What is the probability that two or more of them will end up picking the same colored shoes? What is the probability that they all pick different colored shoes? What is the probability that they all end up picking the same colored shoes?

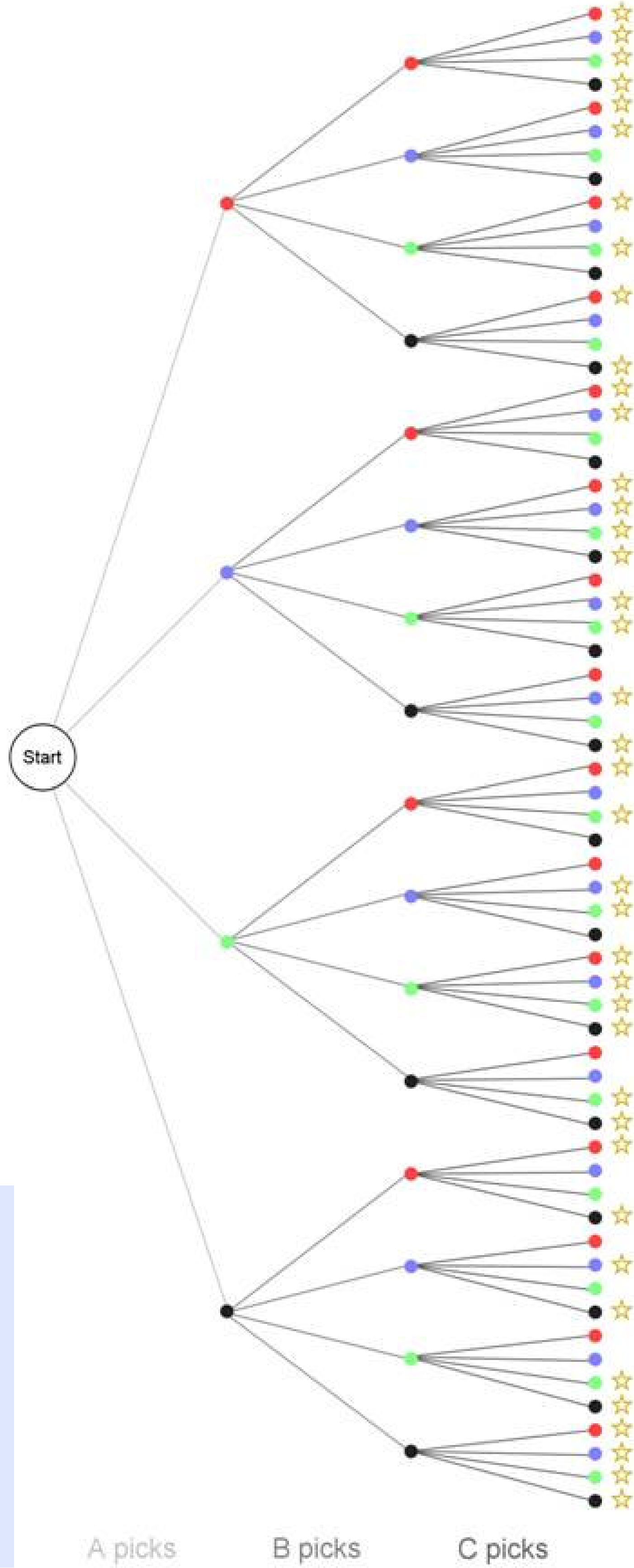
What if we wanted to do this problem for a store with 100 shoes? The tree would be huge! There would be 100 branches in the first generation (the layer where A picks her shoes). There would be a total of 10,000 branches in the second generation (where B picks her shoes). How many branches would there be in the third generation (where C picks her shoes)? It wouldn't really be practical to actually draw out the whole tree. In order to solve the problem with so many shoes, you have to start noticing patterns in these trees and find a less direct way of counting outcomes. Can you figure out what the probability is that three friends all pick different shoes in a store that offers 100 different kinds of shoes?

Can you find a formula which gives the probability that 3 friends all choose different shoes in a store that offers N different shoe types (under the same assumptions that we made in our worked out example)?

A special message from Katy Bold:

It has been a wonderful opportunity to write for the Girls Angle Bulletin. I am excited to let you know that Kat Sanden is taking over the Math In Your World column. Kat was a student in my math class five years ago at Princeton, and she always comes up with fun ways to think about math (she once wrote a comic about the Pythagorean theorem – it made me laugh out loud). I look forward to her columns and also to watching Girls' Angle continue to grow.

- Katy



Similarity, Part II: Triangles

Continued from page 9.

What have you convinced yourself about triangles and similarity?

How does what you found compare to the following facts?

1. Two triangles are similar if and only if they have the same three angles.

2. Two triangles are similar if and only if the sides of one can be matched up with the sides of the other in such a way that each side of one is matched with a different side of the other and the three ratios formed by dividing the length of a side in the first triangle by the length of the matching side in the other triangle are all equal.

Did you also notice the general fact that similar angles have equal measure?

I won't prove these facts here, but if you have proven them to yourself, that's great! The reason I won't prove them now is that I have not laid the foundation to be able to actually prove anything about similarity and triangles yet. In order to do this, I'd first have to provide a definition for triangle, which I haven't done.

If you're curious to see one way the first two facts listed above can be proven, take a look at Euclid's *Elements*. Specifically, read propositions 4 and 5 in Book VI. Before you read them, just remember that an actual proof of these facts will necessarily involve writing that is more detailed and precise. Also, keep in mind that there are other ways of defining the notions of triangle and similarity, so there can be many different proofs of these facts. In fact, a modern mathematician's proof of these facts will likely look quite different from the one Euclid published. After all, Euclid's proof was written over two thousand years ago!

Let's see if we can describe a set of triangles, none of whose members are similar to each other, but which contains a representative of every "similarity class" of triangles. In other words, for any triangle you can dream up, this set will contain one, and exactly one, triangle that is similar to it.

I'll try to use the first fact listed above about angles. Consider a triangle. Let's call the measures of the three angles of the triangle in degrees A, B, and C. Any triangle which has angles that measure A, B, and C degrees, according to the first fact listed above, will be similar to this triangle. So if we can list all angle triples (A, B, C) which can be the angles of a triangle exactly once, that would accomplish our goal.

We have to be careful not to list a triple twice. For example, (30, 60, 90) (which corresponds to a triangle with angles that measure 30°, 60°, and 90°) would represent the same triple as (90, 30, 60). To prevent this kind of double listing for the task at hand, we will insist on writing the angle measures in order from smallest to largest.

Recall that the sum of the angles of a triangle is always equal to 180°, so $A + B + C = 180$. This means that once two angles are known, the other is determined as well. So we really only have the flexibility to specify two angle measures in a triangle. I'll concentrate on specifying the smaller two of the three angles.

The smallest angle of a triangle cannot exceed 60°; if it did, then the sum of the three angles would have to be greater than 3(60°) = 180°, which cannot happen in a triangle. It is possible for the smallest angle to be exactly 60°, as this actually happens in an equilateral triangle. Also, the smallest angle of a triangle must be greater than 0°.

So, if (A, B, C) is a triple of angle measures with $A \le B \le C$, we must have

$$0 < A \le 60.$$

If we fix A, what are the possibilities for B?

Well, we must observe our constraint that $A \le B$. But we must also choose B so that C (which equals $180 - A - B$) is greater than or equal to B. So B must satisfy,

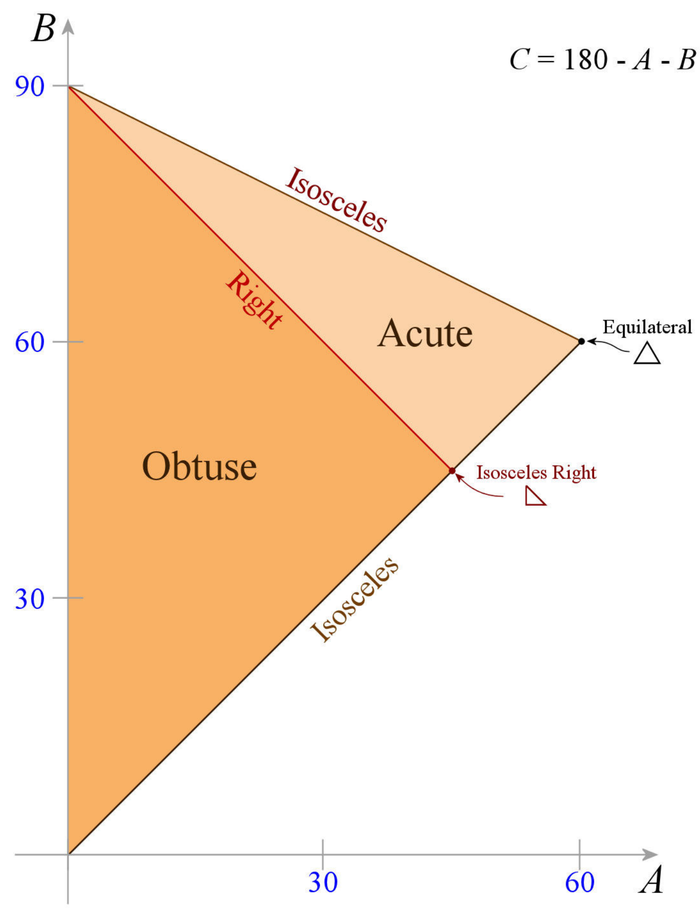
$$A \le B \le 180 - A - B.$$

Can you show that this is the same as $A \le B \le 90 - \dfrac{A}{2}$?

We can summarize all of this information in the picture at right. Because C is determined once A and B are specified, the picture only shows two axes, the A axis and the B axis. Each point in the shaded region (or on the boundary labeled "Isosceles") corresponds to a similarity class of triangles. By the way, why is the left edge of the diagram not part of the classification space?

If you pick a point in the classification space, can you picture in your mind's eye a triangle in the represented similarity class? If you move the point around, can you imagine the representative triangle deforming accordingly?

Next time, we'll focus just on the line that represents right triangles!



An abstract picture of similarity classes of triangles. Every triangle has angle measures corresponding to some point in the reddish triangle. Notice that the isosceles triangles sit along the boundary and the line representing right triangles splits the region in two: the obtuse and acute triangles.

Errorbusters!

by Cammie Smith Barnes

Teaching mathematics at a very small liberal arts college for women is both a challenge and a joy. Over the course of a semester, I get to know my students well as I spend a lot of time working with many of them individually. These students work through exercises with me one-on-one, and I have the opportunity to see how each of them thinks through the solution to a problem. This gives me insight into why certain concepts initially perplex many students, and helps me guide them to a clear understanding of the ideas behind the math.

One particularly common misconception that I have seen made by students in courses ranging from precalculus to abstract algebra is something I will call "confusing exponentiation with multiplication." I've seen this error in two forms. Occasionally a student will actually switch an exponent to a factor, as in the incorrect equation:

$$2^3 = 2 \cdot 3.$$

The above equation cannot hold true, since 2^3 is 8, whereas $2 \cdot 3$ is 6.

The more prevalent and persistent form of the error, however, is slightly more abstract. Exponents—which I will discuss shortly, as an introduction for those who haven't yet seen them, and as a review for those who have—have many rules that make intuitive sense, once one has a chance to grasp them concretely and then to generalize from there. When a student has not yet had the opportunity to truly understand the concepts behind the rules, however, she may confuse taking a number that already has an exponent to yet another exponent with taking the product of a number to one exponent times that same number to another exponent. In other words, I often see the following mistake:

$$(x^n)^m = x^n x^m,$$

also frequently encountered in the equivalent form

$$(x^n)^m = x^{n+m},$$

for some numbers or variables x, n, and m.

To see why these equations are false in general, let us turn to my promised introduction to exponents. First, what does exponentiation mean?

Suppose that Julia receives a birthday present, wrapped neatly in a cubic box, a few days before she is allowed to open the gift. In trying to investigate what the box could hold, she measures the length of one side of the cube. Suppose that this length happens to be 10 inches. Then, since the length of a face of the cube equals its width, the width will also be 10 inches; thus the area of the face is $10 \cdot 10 = 100$ square inches.

We say that $10 \cdot 10$ is "ten squared," and we use a mathematical shorthand called an exponent to record that we're multiplying two factors of 10 with each other. That is, we define

$$10^2 = 10 \cdot 10.$$

The 10 is called the **base** and is the number being multiplied with itself. The small 2 in the upper right corner of the left-hand side of the equation is called the **exponent**.

Now, what if Julia also wants to find the volume of the box, to see whether it might be big enough to hold the gift for which she hopes? She can calculate the volume by multiplying the length times the width times the height of the box, which in the case of a cube are all equal to each other. So for Julia's present, the volume is $10 \cdot 10 \cdot 10 = 1000$ cubic inches. So Julia's present could hold many of her favorite chocolate truffles (since each is a little bigger than a cubic inch in size)!

In exponent form, we would write:

$$10^3 = 10 \cdot 10 \cdot 10 = 1000,$$

because we are multiplying three factors of 10 with each other. The "10^3" is stated as "ten cubed."

We can generalize this idea to other exponents. For instance, we can write

$$10^4 = 10 \cdot 10 \cdot 10 \cdot 10 \text{ or } 10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10,$$

which we can say out loud as "ten to the fourth power" and "ten to the fifth power," respectively.

We'll also define $10^1 = 10$ and $10^0 = 1$. Moreover, we can use any base, not just 10. For instance, consider 2^3, as referred to above. We now know that $2^3 = 2 \cdot 2 \cdot 2$, which is indeed 8 rather than 6. You can multiply by grouping together factors, like this: $(2 \cdot 2) \cdot 2 = 4 \cdot 2 = 8$. We can also see that $3^2 = 3 \cdot 3 = 9$ and that

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = (3 \cdot 3)(3 \cdot 3) = 9 \cdot 9 = 81.$$

More abstractly,

$$x^n = x \cdot x \cdots x,$$

where there are n factors of x on the right side.

Now let us turn to a couple of beautiful rules that exponents obey. If we take the time to learn these rules, it makes working with exponents a lot easier.

1. Product Rule: Multiplication with the same base becomes addition in the exponents.

For example, this means that

$$4^2 \cdot 4^3 = 4^{2+3} = 4^5.$$

Let's see why this is true. We can write 4^2 as $4 \cdot 4$ and 4^3 as $4 \cdot 4 \cdot 4$. Therefore,

$$4^2 \cdot 4^3 = (4 \cdot 4)(4 \cdot 4 \cdot 4) = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5,$$

because we are multiplying two factors of 4 with another three factors of 4, making a total of five factors of 4 in all.

As another example, note that

$$5^4 \cdot 5^6 = (5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5) = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^{10}.$$

The abstract form of this rule is often written as:

$$x^n x^m = x^{n+m},$$

where x, n, and m are some numbers or variables.

2. Power Rule: Repeated exponentiation becomes multiplication in the exponents.

For instance, suppose we want to calculate $(4^2)^3$, or in other words that we want to first square 4, then cube the result. We can again rewrite 4^2 as $4 \cdot 4$, so that

$$(4^2)^3 = (4 \cdot 4)^3.$$

But then, cubing $(4 \cdot 4)$ just means multiplying three factors $(4 \cdot 4)$ with each other. That is,

$$(4^2)^3 = (4 \cdot 4)^3 = (4 \cdot 4)\ (4 \cdot 4)\ (4 \cdot 4) = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6.$$

Here we see that there are six factors of 4 in total. Note that $2 \cdot 3 = 6$. This is not a coincidence! In general, the rule is:

$$(x^n)^m = x^{m \cdot n},$$

where x, n, and m are each numbers or variables.

So now, using these two rules, we see in particular that $(4^2)^3 = 4^{2 \cdot 3} = 4^6$, whereas $4^2 \cdot 4^3 = 4^{2+3} = 4^5$. Thus the equation

$$(4^2)^3 = 4^{2+3}$$

is simply not true! Be careful not to mix up these two rules of exponentiation. Hence, in general,

$$(x^n)^m = x^{n+m}$$

is not true, either!

Now try your hand at simplifying expressions with exponents using the following exercises. Match each expression on the left with its equivalent expression on the right. The answers can be found on page 29.

	Left		**Right**
(1)	2^5	(a)	27
(2)	3^3	(b)	x^8
(3)	4^3	(c)	32
(4)	$3^6 \cdot 3^5$	(d)	x^{15}
(5)	$(3^6)^5$	(e)	64
(6)	x^5x^3	(f)	3^{30}
(7)	$(x^5)^3$	(g)	3^{11}

Edited by Jennifer Silva

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

<u>Session 7 – Meet 9 – November 4, 2010</u>

Mentors: Keren Gu, Samantha Hagerman, Ryan Heffrin,
Jennifer Melot, Rediet Tesfaye, Bianca Viray

Special Visitor: Valerie Gordeski, Raytheon

Valerie talked about the mathematics behind sound. She used a tuning fork and a bowl of water so that the sound waves could be visualized in the water. She then showed them how the frequency and amplitude of the sound wave affects the note that is produced. Then, the girls got to sing into a computer and see the sound waves that they produced with their own voices.

Bianca worked with **Sophia** on the number of divisors of a number. See Member's Thoughts on page 28.

Some members, who were working with fractions, wondered why they were important to learn. See page 16.

Other members solved combinatorics problems whose solutions made heavy use of products of consecutive integers.

<u>Session 7 – Meet 10 – November 18, 2010</u>

Mentors: Jennifer Balakrishnan, Samantha Hagerman, Liz Simon, Rediet Tesfaye, Bianca Viray

Special Visitors: Anita Suhanin and Noam Weinstein

A number of girls worked to earn brownie points by solving problems related to Tigran Ishkhanov and Grace Lyo's article, *Brownie Points*, in the last issue of this Bulletin. Here are two of the games that had to be analyzed:

Game 1. There are 4 boxes. One of the boxes contains 100 brownie points and the others contain nothing. You can open as many boxes as you wish, one at a time, but you must pay P brownie points to open a box. How much should P be? (The amount P is fixed for the whole game.)

Game 2. I roll a 6-sided die. You can take the number rolled or opt to roll a second die. But if you roll a second die, you must accept whatever number you get on the second roll. The number of brownie points you get is 10 times the number you end up with.

(To understand what is supposed to be determined, please see Tigran and Grace's article.)

Here's a solution to the first game. There are four possibilities. The box with the 100 brownie points is either in the first, second, third, or fourth box. Each of these possibilities is equally likely. Let's consider what would happen in each of these four cases.

If the 100 points are in the…	…first box, then the player would pay P to open it, and discover the 100 points and stop, earning 100 – P brownie points.
	…second box, then the player would pay P to open the first box and find it empty. The player would have to decide whether to open a second box. The cost P to open a box is designed so that the game is fair with 4 boxes. However, now, with the first box opened, the player faces only 3 remaining boxes, one of which contains the 100 brownie points. This would be just like the player facing 4 remaining boxes but knowing that the 100 brownie points was in one of the first 3 boxes, which is a better situation for the player than facing 4 remaining boxes and not having any idea where the brownie points could be. Therefore, it favors the player to keep on opening boxes. So the player will pay another P to open the second box, find the 100 brownie points and walk away with 100 – 2P brownie points.
	…third box, then the player, by similar reasoning as above, would continue opening boxes until the third box is opened and walk away with 100 – 3P brownie points.
	…fourth box, then again, the player will end up opening all 4 boxes and walk away with 100 – 4P brownie points.

If the player plays the game N times, where N is very large, then, on average, the player will make

$$\frac{N}{4}((100-P)+(100-2P)+(100-3P)+(100-4P))$$

brownie points. This simplifies to (100 – 2.5P)N brownie points, or, 100 – 2.5P brownie points on average per game. If P is a fair price, the player will neither gain nor lose brownie points by playing this game over and over, so the fair price satisfies 100 – 2.5P = 0. Solving for P, we find that P = 40.

After the break, special guests Anita Suhanin and Noam Weinstein performed a math song that Noam composed. Both are singer/songwriters. Noam is based in New York City and Anita is based in Boston. Listen to more of their music online by searching for their names.

Girls' Angle thanks Petsi Pies for giving us a deep discount on their yummy brownies! In the past, Petsi Pies has given out free or discounted pies on National Pi Day to those who are able to memorize enough of the decimal digits of π. Hopefully they'll do this again on March 14!

Session 7 – Meet 11 – December 2, 2010

Mentors: Jennifer Balakrishnan, Keren Gu, Samantha Hagerman,
Ryan Heffrin, Rediet Tesfaye, Bianca Viray

Special Visitor: Lenore Cowen, Tufts University

Lenore is a professor of computer science. She talked about a problem in graph theory that is related to a problem in yeast genetics. Graph theory is a branch of mathematics that studies dots that are connected by paths, much like a finished "Connect-The-Dots" drawing. The dots are called **nodes** and the paths are called **edges**. In Lenore's problem, she colored the nodes of the graph black or white. She called an edge that connected nodes of the same color a **monochromatic** edge and an edge that connected nodes of opposite color a **dichromatic** edge.

The problem she posed was: Given a graph, can one color the nodes so that there are always at least as many dichromatic edges as monochromatic edges?

After the break, some girls continued work on combinatorial problems. If you had trouble with those problems, this issue's *Math In Your World* column on page 17 discusses a simple tool that can be helpful in organizing and solving such problems.

Session 7 – Meet 12 – December 9, 2010

Mentors: Keren Gu, Ariana Mann, Liz Simon, Rediet Tesfaye

We held our traditional end-of-session Treasure Hunt! In this hunt, every girl made an important contribution. Liz and Keren both played roles as characters in the hunt. Keren kept a secret number which **Molly**, using only yes/no questions, managed to figure out. It turned out to be 2π. Meanwhile, **Charlotte** defeated Liz four times in head-to-head games of Math Tag to earn a clue plus 3 additional "strikes" for the girls.

Here's one of the problems from the Treasure Hunt. It was solved by **Rowena**.

> Ellen's phone rang. It was Ryan. She picked up, "Hey Ryan!"
> "Ellen, let's meet up!"
> "Where?"
> "How about the Pizza Parlor? It's straight ahead of me, 300 meters."
> "Hmmm, the Pizza Parlor? I'd have to walk forward 240 meters, turn ninety degrees right, and then walk another 500 meters. Why don't we meet at the bakery? It's a little closer for me…just 640 meters straight ahead."
> "The bakery? I'd have to walk forward 320 meters, turn ninety degrees to the left, and then walk another 640 meters to get there! That's even farther than you'd have to walk to get to the Pizza Parlor!"
> "Hmmm…actually, it'll be faster if I just walk to where you are now…so just hold tight…be there soon…bye!"
>
> How many meters separate Ellen and Ryan?

Working together, the girls managed to unlock the present with just six minutes to spare! Congratulations to all and we hope to see all of you when Girls' Angle resumes on January 27.

On the Number of Divisors of a Number

How many divisors does a number have?

This session, **Sophia** thought a lot about this question.

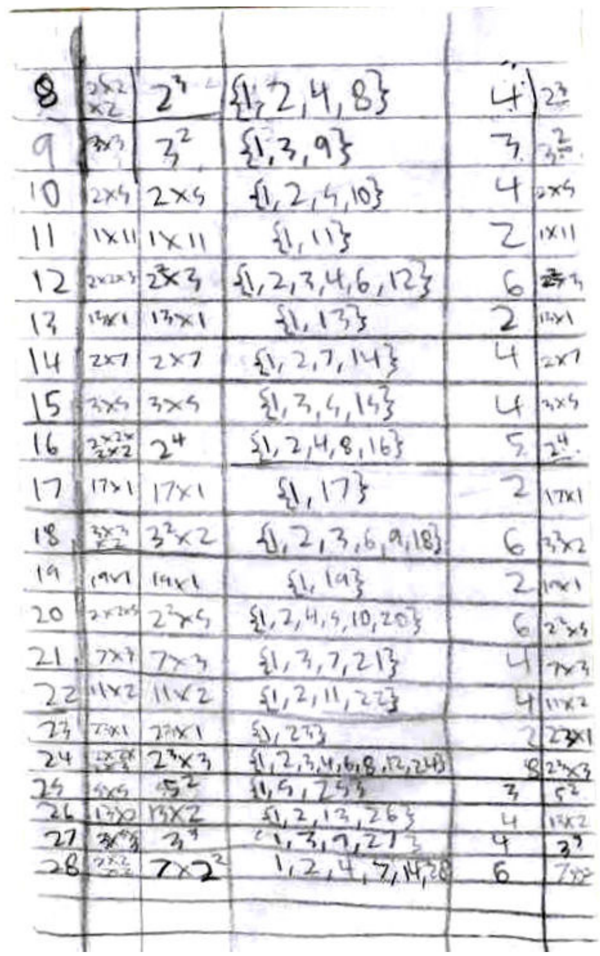
To investigate, the first thing she did was make a nice table of data. Working out lots of examples and looking for patterns is a common problem solving technique that many mathematicians employ frequently.

With such nice data, one can start looking for patterns and start making conjectures.

When are there exactly 2 divisors? When are there exactly 3 divisors? When is the number of divisors odd? Is there a pattern between the number of divisors of n and of $n + 1$? How about between n and $2n$?

Eventually, **Sophia** figured out a formula that expresses the number of divisors in terms of the exponents that appear in the number's prime factorization.

Her knowledge came in really handy at the end-of-session Treasure Hunt which asked: *How many odd numbers less than 1000 are there with exactly 12 divisors?*



8	2×2×2	2^3	{1,2,4,8}	4	2^3
9	3×3	3^2	{1,3,9}	3	3^2
10	2×5	2×5	{1,2,5,10}	4	2×5
11	1×11	1×11	{1,11}	2	1×11
12	2×2×3	2^2×3	{1,2,3,4,6,12}	6	[illegible]
13	13×1	13×1	{1,13}	2	13×1
14	2×7	2×7	{1,2,7,14}	4	2×7
15	3×5	3×5	{1,3,5,15}	4	3×5
16	2×2×2×2	2^4	{1,2,4,8,16}	5	2^4
17	17×1	17×1	{1,17}	2	17×1
18	3×3×2	3^2×2	{1,2,3,6,9,18}	6	3^2×2
19	19×1	19×1	{1,19}	2	19×1
20	2×2×5	2^2×5	{1,2,4,5,10,20}	6	2^2×5
21	7×3	7×3	{1,3,7,21}	4	7×3
22	11×2	11×2	{1,2,11,22}	4	11×2
23	23×1	23×1	{1,23}	2	23×1
24	2×2×2×3	2^3×3	{1,2,3,4,6,8,12,24}	8	2^3×3
25	5×5	5^2	{1,5,25}	3	5^2
26	13×2	13×2	{1,2,13,26}	4	13×2
27	3×3×3	3^3	{1,3,9,27}	4	3^3
28	7×2×2	7×2^2	1,2,4,7,14,28	6	7×2^2

The table **Sophia** made showing the divisors of the numbers from 8 to 28.

Play with this concept. Test your own understanding of it by asking yourself questions and seeing if you can answer them. For example, which number less than 1000 has the most number of divisors? For any d, are there numbers with exactly d divisors? For a given d, what is the smallest number with d divisors? Is there a relationship between the number of divisors of a product nm and the number of divisors of n and m?

Finally, here's a good tip to keep in mind when working out examples like this. Notice that **Sophia** started with the number 8. She probably started with 8 because she considered the numbers 1 through 7 to be too easy. However, it's good practice to always start at the beginning, even if you think the first cases are too easy to consider. *Don't neglect the trivial cases.* Working systematically from the beginning and proceeding step by step is a good habit to get into. You might be amazed what insights you can get even by working the simplest cases! Furthermore, if you have incomplete data, some amazing patterns just might escape your notice.

Calendar

Session 7: (all dates in 2010)

September	9	Start of the seventh session!
	16	
	23	
	30	Tanya Khovanova, Research Affiliate, MIT
October	7	
	14	
	21	Jane Kostick, woodworker
	28	
November	4	Valerie Gordeski, Raytheon
	11	Veteran's Day – No meet
	18	Anita Suhanin and Noam Weinstein
	25	Thanksgiving - No meet
December	2	Lenore Cowen, Tufts University
	9	

Session 8: (all dates in 2011)

January	27	Start of sixth session!
February	3	
	10	
	17	
	24	No meet
March	3	
	10	
	17	
	24	No meet
	31	Susan Barry
April	7	
	14	
	21	No meet
	28	
May	5	

Girls' Angle thanks Microsoft and MITX for naming Girls' Angle as a beneficiary of the 2010 Women's Leadership Forum which was held at the Microsoft New England Research and Development Center in Cambridge, Massachusetts.

Girls' Angle thanks Draper Laboratories for their generous financial support of our stellar mentors…the heart and soul of Girls' Angle.

Here are answers to the *Errorbusters!* problems on page 24.

1-c, 2-a, 3-e, 4-g, 5-f, 6-b, 7-d

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session and includes access to the club and extends the member's subscription to the Girls' Angle Bulletin to one year from the start of the current or upcoming session. You can also pay per session. If you pay per session and come for three meets, you will get a subscription to the Bulletin. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, Courant Instructor/PIRE fellow, NYU
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, Moore Instructor, MIT
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, Senior Research Scientist, Harvard Medical School.
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Bulletin Sponsorship

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: ________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session membership
  - ☐ Bulletin Sponsorship: $60-99 Subscribing. $100-999 Bronze. $1000-2999 Silver. $3K+ Gold.*
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

*Silver and Gold sponsors may specify a recipient for their sponsored copies of the Bulletin.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: __________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

Girls' Angle

A Math Club for Girls