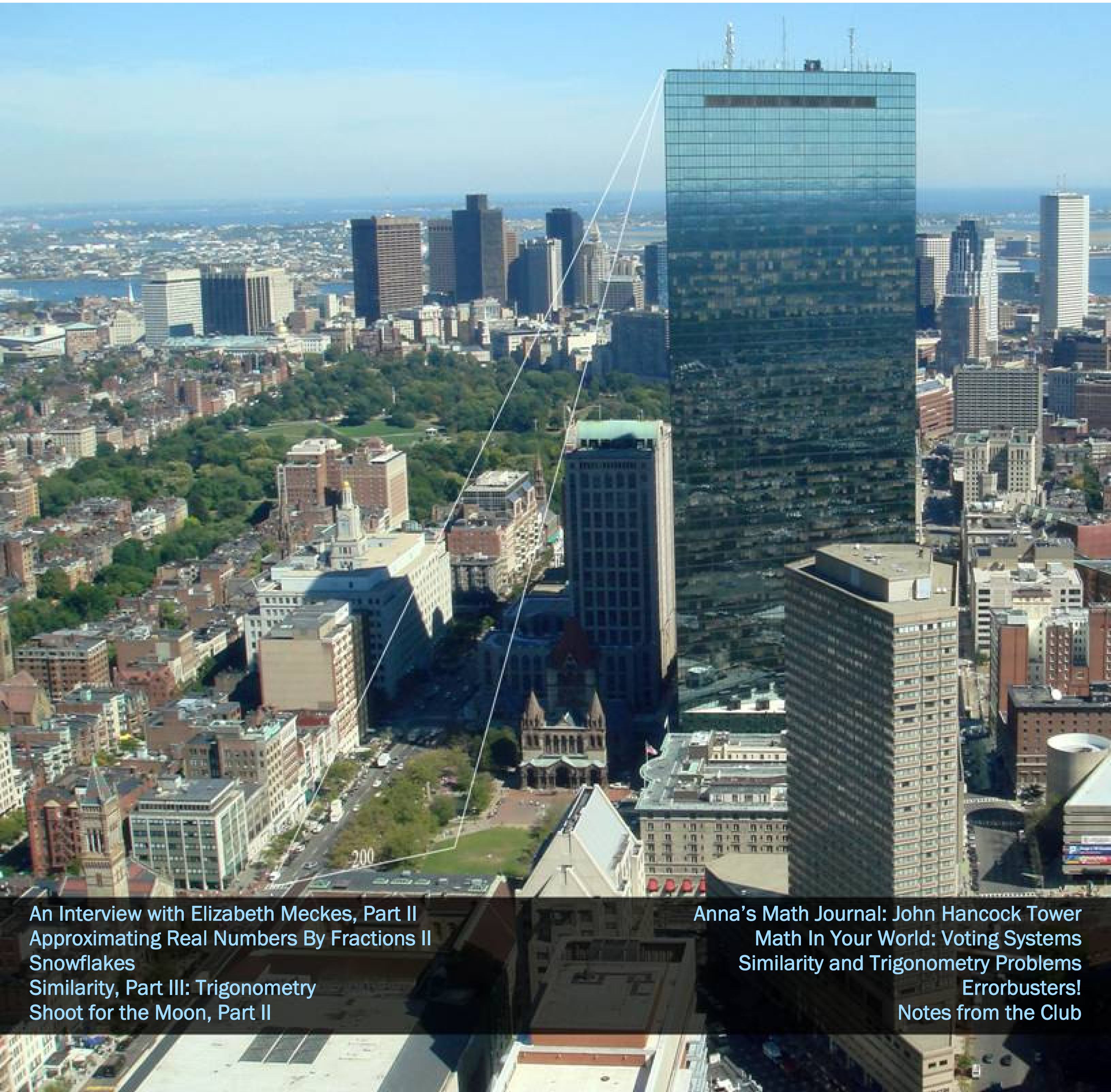
# Girls' Angle Bulletin

February 2011 • Volume 4 • Number 3

*To Foster and Nurture Girls' Interest in Mathematics*



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# From the Director

Girls' Angle welcomes Brown University Tamarkin Assistant Professor Bianca Viray to our Advisory Board. Many members already know her as one of our active postdoctoral mentors. Dr. Viray has a Ph.D. from UC Berkeley.

The content of this Bulletin does vary over a wide range of difficulty levels. We hope that most will find something in here that is challenging. Please don't get frustrated if you find material in here that you feel you can't understand. You can always put the material aside and return to it later. If you don't completely give up, you will eventually understand it! And remember, members and subscribers are always welcome to ask us about Bulletin content or, indeed, anything mathematical!

Think about it…in order to expand human knowledge and understanding, one must tackle problems that nobody has yet understood. So it is in the nature of scientific and mathematical research to be more often in a fog than to have clarity.

- Ken Fan, Founder and Director

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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

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## Girls' Angle:
### A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*



On the cover: The John Hancock Tower in Boston. Image adapted from upload.wikimedia.org/wikipedia/commons/2/22/2007-0925-Boston-JohnHancockTower.jpg by Bobak Ha'Eri.

# An Interview with Elizabeth Meckes, Part II

**Ken**: Last time we were talking about high-dimensional spaces and phenomena in high-dimensions; can you tell us more about how they come up?

**Elizabeth**: For me, phenomena in high-dimensional spaces are interesting from a purely abstract mathematical perspective; that is, I just think they're interesting in their own right. But amazingly, sometimes the theories built up by mathematicians that come from this kind of generalization and abstraction, and are carried out purely because the mathematicians find them interesting, can turn out to be useful again back in the real world. This is what the famous physicist and mathematician Eugene Wigner called "the unreasonable effectiveness of mathematics in the natural sciences". In this case, it turns out that high-dimensional spaces are crucially important in applications all over the place: engineering, computing, physics, biology, chemistry, and more. You might think that as the dimension of a space goes up, things just get more and more complicated, and to some extent that's true. But it's also true that sometimes in high dimensions, patterns emerge from the chaos. The result I mentioned about value distributions of eigenfunctions on the sphere is like that; as the dimension goes up, these special functions are horrendously complicated to write down (you can do it, but trust me, you don't want to see the formulas). Even so, looked at in a certain way, that complication washes out and they all look pretty much the same, and pretty simple.

**Ken**: Fascinating! Do you have any advice for how best to learn mathematics?

**Elizabeth**: Yes, although I won't promise I've always followed my own advice (on the other hand, when I haven't I've usually regretted it).

First of all, learning math and doing math are really the same thing; it's an active process. There's a reason that math teachers assign so much homework: there's a real limit to how much understanding you can gain by watching someone else do math. To really understand something you have to work through it yourself. This is not in any way to suggest that teachers can't help; the insights that math teachers can provide about how to think about things can be incredibly valuable, but only in conjunction with struggling with it on your own. And I do mean struggling; understanding math doesn't come easy to anyone (if it has to you so far, don't worry— if you keep at it long enough you'll get to be as confused as the best of them).

Secondly, talking to other people about math is probably the best tool there is to gain better understanding. This includes people at all levels: those who know a lot more than you, a lot less than you, and everyone in between. Like I said before, explaining something to someone else is a fantastic way to clarify your own thinking and figure out which things you're still unsure about. When I think I've figured out something important (or just something that was hard for me and may not be all that important), I usually run it by someone else. This is frequently how I discover mistakes in my work. It helps the other way too; I can't count the number of times I've been stuck on something, went to ask someone else for help, and in the course of explaining the problem, I managed to figure out the solution.

**Ken**: To what extent is having female role models important for girls getting into math?

**Elizabeth**: This is a very personal issue, and my best advice on the subject is: Don't let anyone answer this question for you. Of course, it's great to have role models you can relate to (female or male), and on the other hand, you can excel without them. The girls who come to Girls' Angle are lucky in this respect, since they already have access to a lot of women who are top mathematics researchers.

The point I want to make is: when confronted with a choice that forces you to weigh how important it is to have a female mathematician as a role model, trust yourself. You are the best judge of how much this matters; if you feel that having a role-model or mentor is important to you, by all means seek one out. On the other hand, if the path you find most attractive doesn't involve such a mentor, then don't let that hold you back.

**Ken**: Do you have any hobbies aside from math?

**Elizabeth**: I must say I find this question a little strange, because I don't consider math a hobby; it's my job. This is actually an important point, and I'm glad to have the opportunity to discuss it. There's an idea out there that mathematicians as a rule are totally in love with mathematics and wouldn't stop to eat or sleep unless someone made them. This is really too bad, because I think that young people just getting started in math (particularly graduate students) can get really discouraged by this, and think that they shouldn't be mathematicians because that's not them. Anecdotally, this is particularly a problem for young women. Probably a lot of the girls that come to Girls' Angle do think of math as a hobby, and that's great. Some of them will continue in math and become mathematicians, which is also great. And the vast majority of those will probably some time, in graduate school if not sooner, stop thinking of math as a hobby and start thinking of it as a job. And that's fine, and probably close to inevitable, no matter what you decide to do with your life. There's a big difference between doing something for fun and doing it as a career, and anything you spend enough of your time on to call a career will be hard or boring or frustrating or all of the above, at least some of the time. It doesn't mean that it's not the right career for you; if you're lucky enough to be doing something for a living that you used to do for fun, it's probably one of the best career choices you could have made. But it will still be a drag sometimes, so it's good to have other things in your life too.

In light of all that, I'll rephrase the question as: "Other than math, how do you spend most of your time?"

Firstly, I have a wonderful family and spend most of my free time with my husband, Mark, and our two-year old daughter, Juliette. Mark is also a mathematician and people are always asking whether Juliette will be a mathematician, too. I always say, "I sure hope not". I want her to feel free to find her own thing (although I certainly wouldn't interfere if she did decide to do math). For now, she can almost count to ten, although for some reason she usually skips three. It's wonderful to be a mother.

One of the main activities I spend time on at home is cooking. There's a great moment that I really relate to in the movie "Julie and Julia", when Julia Child's husband asks her what she likes to do and she says with a laugh, "Eat!" All of my family like to eat well, and I spend a lot of my evenings and weekends in the kitchen. Juliette's favorite food is duck confit ravioli.

**Ken**: Do you have any advice for the girls that come to Girls' Angle?

**Elizabeth**: Just that I'm really impressed with the initiative of the girls that are getting involved in interesting math outside of school— keep it up! Sometimes it's a bumpy curve, but on average it just keeps getting better.

**Ken**: Thank you so much for this interview!

> To really understand something you have to work through it yourself.

# Approximating Real Numbers By Fractions II

by David E. Speyer

Last time, we ended with a challenge and a surprise. Let's review where we were, and continue.

We were discussing rational approximations to real numbers, using the problem of approximating $\pi$ as our example. We defined $p/q$ to be the best lower approximation to $\pi$ of size $N$ if, among all fractions $r/s$ with $r + s \leq N$ and $r/s < \pi$, the fraction $p/q$ is closest to $\pi$. Similarly, we defined $P/Q$ to be the best upper approximation to $\pi$ of size $N$ if, among all fractions $R/S$ with $R + S \leq N$ and $R/S > \pi$, the fraction $P/Q$ is closest to $\pi$.

We started computing best lower and best upper approximations, as $N$ increased. So, for $N \leq 12$, the best lower and best upper approximation were 3/1 and 7/2 respectively. Once $N$ got up to 13, the fraction 10/3 slipped into the middle, forming a better upper approximation than 7/2. When $N$ got up to 17, 13/4 slipped in between 3/1 and 10/3. The next several steps were:

$$\frac{10}{3} \to \left(\frac{3}{1}, \frac{7}{2}\right) \qquad \frac{19}{6} \to \left(\frac{3}{1}, \frac{16}{5}\right) \qquad \frac{47}{15} \to \left(\frac{25}{8}, \frac{22}{7}\right)$$

$$\frac{13}{4} \to \left(\frac{3}{1}, \frac{10}{3}\right) \qquad \frac{22}{7} \to \left(\frac{3}{1}, \frac{19}{6}\right) \qquad \frac{69}{22} \to \left(\frac{47}{15}, \frac{22}{7}\right)$$

$$\frac{16}{5} \to \left(\frac{3}{1}, \frac{13}{4}\right) \qquad \frac{25}{8} \to \left(\frac{3}{1}, \frac{22}{7}\right) \qquad \frac{??}{??} \to \left(\frac{69}{22}, \frac{22}{7}\right)$$

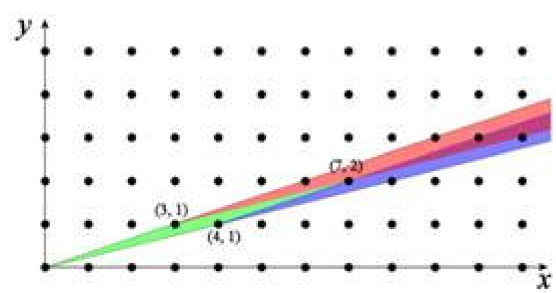
I left you with a challenge: What fraction will be inserted between 69/22 and 22/7?

We'll give a proof later but, for now, let's look at the data. At every step, let $(p/q, P/Q)$ be the pair of fractions we are dealing with. Then, every time, the fraction we insert is $(p + P)/(q + Q)$. For example, we insert $47/15 = (25 + 22)/(8 + 7)$ between 25/8 and 22/7. So, if we believe this pattern will continue to hold then the next fraction inserted will be $(69 + 22)/(22 + 7) = 91/29$. Since $91/29 \approx 3.1379$, which is less than $\pi$, we will insert 91/29 as a lower approximation, replacing 69/22.

Notice that $(p + P)/(q + Q)$ is always between $p/q$ and $P/Q$. So it makes sense that this fraction might be inserted. More specifically, suppose that, at some point, we have the interval $(p/q, P/Q)$. If we don't insert any other fraction before we get to the value of $N$ with $N = (p + P) + (q + Q)$, then $(p + P)/(q + Q)$ will be inserted. So what we need to show is that no fraction is inserted before $(p + P)/(q + Q)$.

Suppose, then, that $r/s$ is the fraction inserted into $(p/q, P/Q)$ before we get to $(p + P)/(q + Q)$. We will obtain a contradiction. The assumption that $r/s$ is inserted before $(p + P)/(q + Q)$ means that $r + s < (p + P) + (q + Q)$.

The figure at right depicts the insertion of the next fraction between $(p, q) = (3, 1)$ and $(P, Q) = (4, 1)$. On the next page, in a similar, but more generic figure, we have plotted $(p, q)$, $(P, Q)$ and $(p + P, q + Q)$. For clarity, that figure is not drawn to scale. We want to show that $(r, s)$ is $(p + P, q + Q)$, meaning that the next point inserted is at the tip of the green parallelogram.

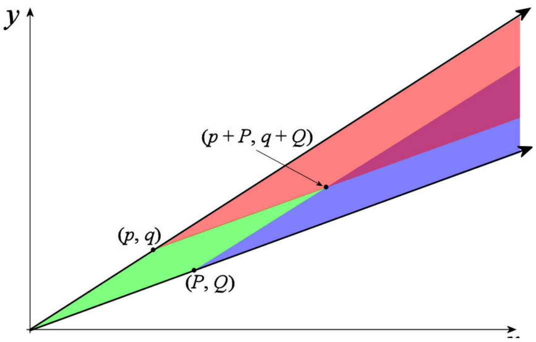


The hypothesis that $r/s$ is between $p/q$ and $P/Q$ means that $(r, s)$ is between the rays from the origin through $(p, q)$ and $(P, Q)$. That means that at least one of the following must be true:

(1) $(r, s)$ is in the red region: the cone with vertex at $(p, q)$ and rays in directions $(p, q)$ and $(P, Q)$.

(2) $(r, s)$ is in the blue region: the cone with vertex at $(P, Q)$ and rays in directions $(p, q)$ and $(P, Q)$.

(3) $(r, s)$ is in the green region: the parallelogram with corners at $(0, 0)$, $(p, q)$, $(p + P, q + Q)$ and $(P, Q)$.



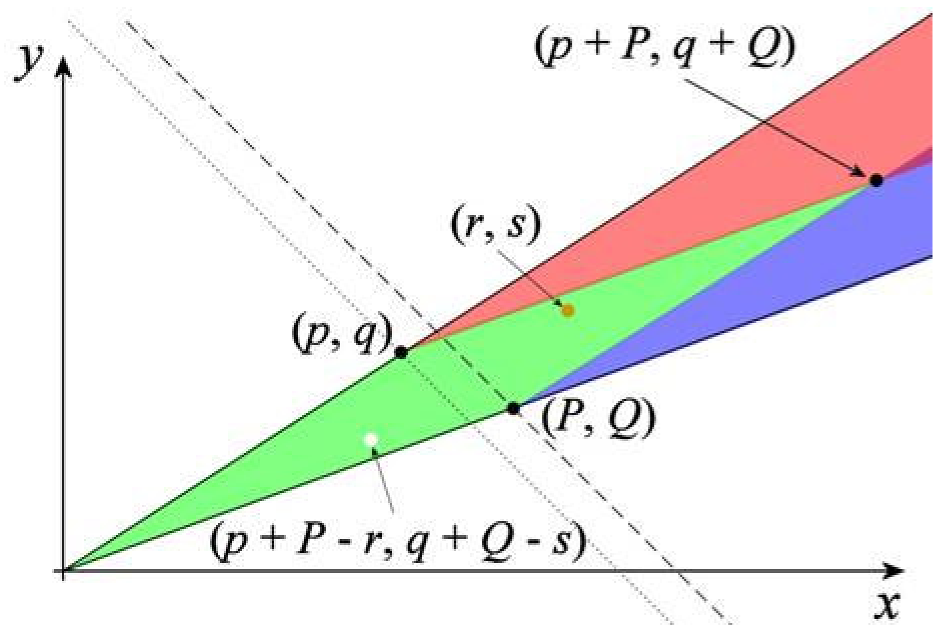

The overlap of the red and the blue regions is colored purple.

If $(r, s)$ is in the red region, then $(r – p, s – q)$ lies in between the two rays from the origin. So $p/q < (r – p)/(s – q) < P/Q$. Then the fraction $(r – p)/(s – q)$ would be inserted into $(p/q, P/Q)$ before $r/s$ was, so $r/s$ is not the fraction to be inserted into this interval.

Similarly, if $(r, s)$ is in the blue region, then $(r – P)/(s – Q)$ would be inserted into $(p/q, P/Q)$ before $r/s$ was. So, a point in the blue region cannot be the next fraction inserted.

Finally, we come to the green region (see image below).

Suppose the next point inserted was $(r, s)$, a point inside the green parallelogram. We have depicted $(r, s)$ in brown. In reality, the next point inserted is $(p + P, q + Q)$, the tip of the parallelogram. We want to consider the possibility that it is some other point.




Let me be clear about what is going on here. We are currently making a proof by contradiction, where we assume the opposite of what we want to show. We will eventually conclude that $(r, s)$ is at the tip of the parallelogram after all. Because our picture is depicting something that in reality doesn’t happen, we cannot draw an honest picture of this. Our supposition, from which we hope to draw a contradiction, is that there is such a point $(r, s)$ inside the green region, but not at the tip $(p + P, q + Q)$.

Since we insert $r/s$ after $p/q$ and $P/Q$, the point $(r, s)$ must be above the line $x + y = \max(p + q, P + Q)$, drawn as a dashed diagonal line. So $(p + P – r, q + Q – s)$ is below the dotted diagonal line, which means that $(p + P – r) + (q + Q – s) < \min(p + q, P + Q)$. I have drawn $(p + P – r, q + Q – s)$ in white. Notice that since $(r, s)$ isn’t at the tip $(p + P, q + Q)$ of the parallelogram, the white point $(p + P – r, q + Q – s)$ is not the origin. (In particular, $q + Q – s$ isn’t zero, so we don’t have to worry about dividing accidentally by zero.)

But then, if $(p + P - r)/(q + Q - s) < \pi$, then $(p + P - r)/(q + Q - s)$ is a better lower approximation than $p/q$ is! (Because $(p + P - r, q + Q - s)$ is below the ray through $(p, q)$.) Since $(p + P - r) + (q + Q - s) < p + q$, that means the fraction $p/q$ was never a best approximation after all, a contradiction! Similarly, if $(p + P - r)/(q + Q - s) > \pi$, then $(p + P - r)/(q + Q - s)$ is a better upper approximation than $P/Q$. In either case, we have reached a contradiction.

So $r/s$ can't be in the red region, can't be in the blue region, and can't be anywhere in the green parallelogram except at the tip.

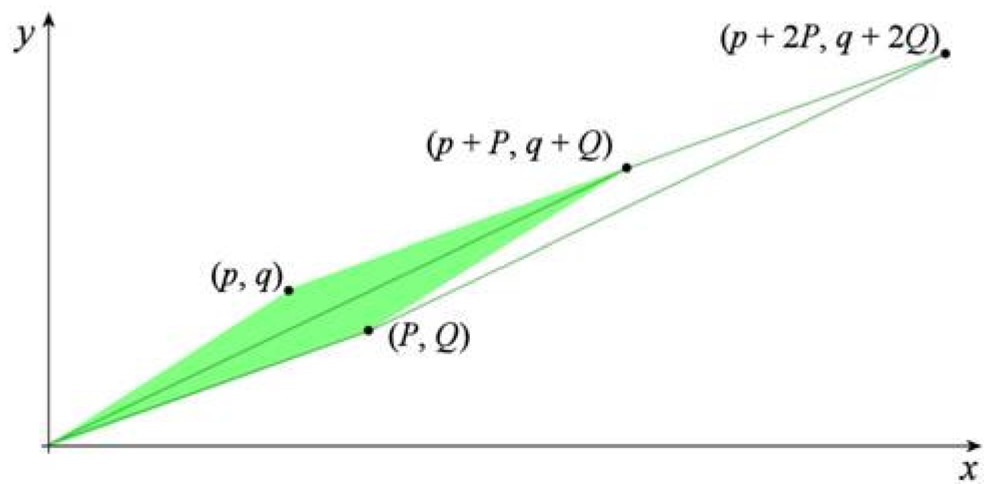
We reach the conclusion that the next fraction inserted will be $(p + P)/(q + Q)$, just as we guessed from our data! To tell whether $(p + P)/(q + Q)$ will be inserted as an upper or a lower bound, one must actually compute this fraction and compare it to $\pi$.

Finally, we come to the surprise from last time. Remember that we observed, but could not explain, that the difference $P/Q - p/q$ was always $1/(qQ)$.

By basic algebra, we compute $P/Q - p/q = (Pq - pQ)/(Qq)$. So our task is to prove that $Pq - pQ = 1$.

There is a geometric way to think about this. The number $Pq - pQ$ is the area of the green parallelogram. (Can you show this?) Back at the beginning, we have the parallelogram with vertices (0, 0), (3, 1), (7, 2) and (4, 1), and this has area 1. What we need to show is that, at every step, the area remains 1.

Look at the figure below.




This figure shows two parallelograms. The first one, shaded in green, has vertices (0, 0), $(p, q)$, $(p + P, q + Q)$ and $(P, Q)$. In the second parallelogram, drawn with a dark green outline only, we have inserted $(p + P)/(q + Q)$ as a new lower bound. So the dashed parallelogram has vertices (0, 0), $(p + P, q + Q)$, $(p + 2P, q + 2Q)$, and $(P, Q)$. Notice that the dashed parallelogram has the same base as the solid one, and the same height. So it has the same area! (This is an instance of Cavalieri's principle.) If $(p + P)/(q + Q)$ were inserted as a new upper bound, the outlined dark green parallelogram would have vertices (0, 0), $(p, q)$, $(2p + P, 2q + Q)$, and $(p + P, q + Q)$, but the same area preserving observation would apply.

As our procedure continues, the parallelograms rapidly get very long, and very skinny. But this simple geometry shows that they always have area 1. So $Pq - pQ$ is always 1. We have now confirmed the surprise as well.

We have reached the end of the article, but only the beginning of the exploration. If you want to keep exploring, I suggest that you look at the rational approximations to some irrational square roots, like $\sqrt{2}$, $\sqrt{13}$, or $\sqrt{19}$. You should find plenty to explore!

Photo by and courtesy of Kenneth Libbrecht

# Snowflakes

A *huge* number of snowflakes have fallen on Boston this winter! Have you ever caught a snowflake and studied it? Can you imagine studying snowflakes as part of your *job*? Professor Kenneth Libbrecht, the physicist at the California Institute of Technology who made the photo above, studies snowflake formation and has a whole website dedicated to the topic with many more pictures. According to his website, nobody yet understands the "bizarre temperature dependence of ice crystal growth rates." Yet another reason to learn math…and physics!

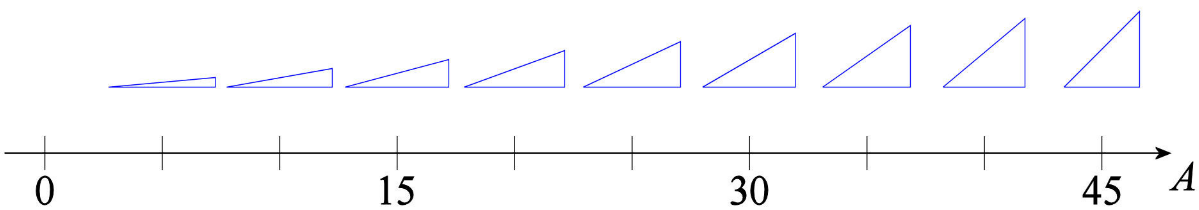
# Similarity, Part III: Trigonometry

by Ken Fan
edited by Jennifer Silva

We began this series by talking about the concept of similarity. Then, we looked at similarity and triangles. We made an abstract picture of all of the "similarity classes" of triangles. In our picture, the right triangles were represented by part of a line.

On a line, each point can be specified by providing a single number, as is done, for instance, on a number line. So, the fact that similarity classes of right triangles were represented by part of a line means that we can specify each similarity class of right triangles by a single number. If you study the abstract picture we came up with last time, you will see that one way to do this would be to specify the smallest angle of the right triangle. Furthermore, this smallest angle must be greater than 0° and less than or equal to 45°. Let's call this smallest angle $A$, which is in keeping with the notation we used last time. The other angles of the triangle will be $90° – A$ and 90°.

Here's a picture:




For values of $A$ that are multiples of 5°, we drew a triangle representing the corresponding similarity class of right triangles. *Every* right triangle is similar to a triangle represented by a point on this number line between 0 and 45 (where 45 is included but 0 is not).

Remember, each point does not represent a unique triangle. Instead, each point represents an *entire* similarity class of triangles. For example, the point corresponding to $A = 30$ represents *all* 30-60-90 right triangles, and you can imagine that there can be really itty-bitty 30-60-90 right triangles as well as enormous ones. But because they are all similar to each other, the *ratios* of corresponding lengths are equal in all of them.

Knowing these ratios can help us find many distances that otherwise could be rather difficult to measure. For example, at right is a map of Radcliffe Yard. Suppose we want to know the length of the distance marked in red. If you tried to measure that with a tape measure, you would run into all kinds of problems because you would have to go through the walls of many buildings. But if we recognize the red distance as the side length of a right triangle, all we need to know is the similarity class of this right triangle and any length associated with it. We can find the similarity class by measuring the size of angle $A$. Then, we could measure, say, the length of the side along Linnaean Street, which runs right along an unobstructed sidewalk. Next we could use similar triangles to make a

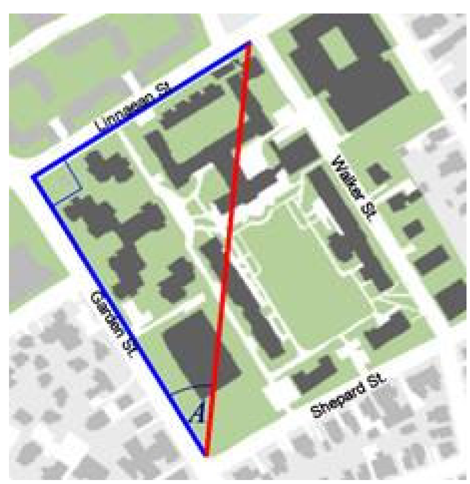



Image modified from Wikipedia image by Wikipedia user Citynoise.

scale model (like the map shown!), measure the ratio of the length of the red side to the side along Linnaean Street in the scale model, and use this to compute the actual length diagonally across Radcliffe Yard … avoiding all walls in the process!

In this problem, you could also find this distance by measuring the lengths of the two legs of the right triangle, since both legs lie along unobstructed sidewalks, and then applying the Pythagorean theorem. But in some situations, it can be very hard to measure two of the sides. A common example is in measuring the height of a very tall building. If you stand before the building at some distance, then the horizontal distance to the building and the height of the building form the legs of a right triangle whose hypotenuse is the distance from you to the top of the building. In this case, it's much easier from a practical standpoint to measure your distance from the building and the angle you have to look up to see the top of the building than it is to measure the two side lengths of the right triangle.

The ability to compute the lengths of the other sides of a right triangle given the length of one of its sides plus the measure of one of its acute angles turns out to be very useful. It is so useful that people decided to make tables showing the ratios of various side lengths in a right triangle as a function of one of its angles. These functions are known as **trigonometric** functions.

Because a right triangle has 3 sides, there are 9 ratios we can form by dividing the length of one side by the length of another. Of these, the 3 ratios in which you divide a side length by itself will always yield 1 and so are not interesting. But the other 6 are interesting and all have special names: **sine**, **cosine**, **tangent**, **cosecant, secant**, and **cotangent**, which are abbreviated **sin**, **cos**, **tan**, **csc**, **sec**, and **cot**, respectively. If we pick an acute angle in a right triangle and label it *A* as in the figure below right, it is customary to denote the leg that makes up one side of angle *A* the "adjacent" side and the remaining leg the "opposite" side. With these designations, the six trigonometric functions are defined as follows:

$$\sin A = \frac{opposite}{hypotenuse}$$

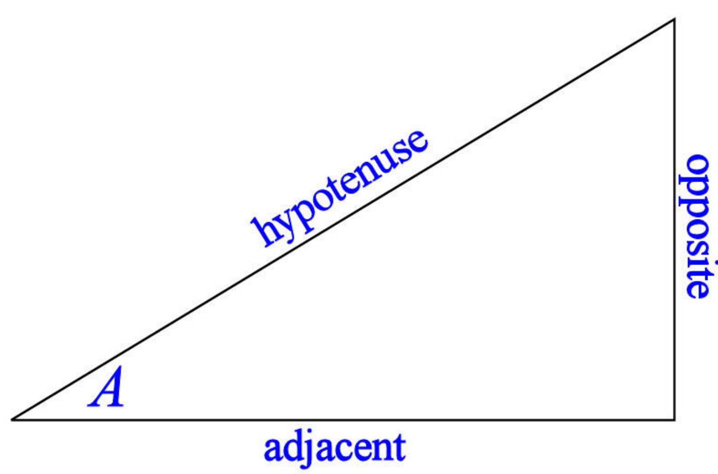
$$\cos A = \frac{adjacent}{hypotenuse}$$

$$\tan A = \frac{opposite}{adjacent}$$

$$\csc A = \frac{1}{\sin A} = \frac{hypotenuse}{opposite}$$

$$\sec A = \frac{1}{\cos A} = \frac{hypotenuse}{adjacent}$$

$$\cot A = \frac{1}{\tan A} = \frac{adjacent}{opposite}$$




The importance of these ratios has been known for a long, long time. On the following page, there is a table showing the values of these trigonometric functions, rounded to 3 decimal places, taken from a book published in 1889! The book is called *Elementary trigonometry, plane and spherical* and was written by Edwin Pliny Seaver who served as Boston's Superintendent of Public Schools from 1880 to 1904. But these functions go back even further. There are tables of these values that are from over a thousand years!

Let’s examine the table at right.

Do the entries fit well with your intuition about what the similarity classes look like?

Some of the entries read “infinite.” What do you think the meaning of that is?

All of the values in the sine and cosine columns are between 0 and 1. Why?

According to the table, the cosine of 0° and the cosine of 1° are both 1.000. Surely they cannot really be equal to each other! Explain why the table reads “1.000” for both.

Can you use the Pythagorean theorem to relate the values of sine and cosine for a given angle?

Can you determine the exact value of any of the entries? What are the exact values of the sine and cosine of 30°? 45°?

Why does the rightmost column show the angles counting down from 90° to 45°?

*TRIGONOMETRIC FUNCTIONS.* 9

TABLE I.

*Natural Values of the Trigonometric Functions.*

| Angle. | Sin. | Csc. | Tan. | Cot. | Sec. | Cos. | |
|---|---|---|---|---|---|---|---|
| 0° | 0.000 | infinite. | 0.000 | infinite. | 1.000 | 1.000 | 90° |
| 1 | 0.017 | 57.30 | 0.017 | 57.29 | 1.000 | 1.000 | 89 |
| 2 | 0.035 | 28.65 | 0.035 | 28.64 | 1.001 | 0.999 | 88 |
| 3 | 0.052 | 19.11 | 0.052 | 19.08 | 1.001 | 0.999 | 87 |
| 4 | 0.070 | 14.34 | 0.070 | 14.30 | 1.002 | 0.998 | 86 |
| 5° | 0.087 | 11.47 | 0.087 | 11.43 | 1.004 | 0.996 | 85° |
| 6 | 0.105 | 9.567 | 0.105 | 9.514 | 1.006 | 0.995 | 84 |
| 7 | 0.122 | 8.206 | 0.123 | 8.144 | 1.008 | 0.993 | 83 |
| 8 | 0.139 | 7.185 | 0.141 | 7.115 | 1.010 | 0.990 | 82 |
| 9 | 0.156 | 6.392 | 0.158 | 6.314 | 1.012 | 0.988 | 81 |
| 10° | 0.174 | 5.759 | 0.176 | 5.671 | 1.015 | 0.985 | 80° |
| 11 | 0.191 | 5.241 | 0.194 | 5.145 | 1.019 | 0.982 | 79 |
| 12 | 0.208 | 4.810 | 0.213 | 4.705 | 1.022 | 0.978 | 78 |
| 13 | 0.225 | 4.445 | 0.231 | 4.331 | 1.026 | 0.974 | 77 |
| 14 | 0.242 | 4.134 | 0.249 | 4.011 | 1.031 | 0.970 | 76 |
| 15° | 0.259 | 3.864 | 0.268 | 3.732 | 1.035 | 0.966 | 75° |
| 16 | 0.276 | 3.628 | 0.287 | 3.487 | 1.040 | 0.961 | 74 |
| 17 | 0.292 | 3.420 | 0.306 | 3.271 | 1.046 | 0.956 | 73 |
| 18 | 0.309 | 3.236 | 0.325 | 3.078 | 1.051 | 0.951 | 72 |
| 19 | 0.326 | 3.072 | 0.344 | 2.904 | 1.058 | 0.946 | 71 |
| 20° | 0.342 | 2.924 | 0.364 | 2.747 | 1.064 | 0.940 | 70° |
| 21 | 0.358 | 2.790 | 0.384 | 2.605 | 1.071 | 0.934 | 69 |
| 22 | 0.375 | 2.669 | 0.404 | 2.475 | 1.079 | 0.927 | 68 |
| 23 | 0.391 | 2.559 | 0.424 | 2.356 | 1.086 | 0.921 | 67 |
| 24 | 0.407 | 2.459 | 0.445 | 2.246 | 1.095 | 0.914 | 66 |
| 25° | 0.423 | 2.366 | 0.466 | 2.145 | 1.103 | 0.906 | 65° |
| 26 | 0.438 | 2.281 | 0.488 | 2.050 | 1.113 | 0.899 | 64 |
| 27 | 0.454 | 2.203 | 0.510 | 1.963 | 1.122 | 0.891 | 63 |
| 28 | 0.469 | 2.130 | 0.532 | 1.881 | 1.133 | 0.883 | 62 |
| 29 | 0.485 | 2.063 | 0.554 | 1.804 | 1.143 | 0.875 | 61 |
| 30° | 0.500 | 2.000 | 0.577 | 1.732 | 1.155 | 0.866 | 60° |
| 31 | 0.515 | 1.942 | 0.601 | 1.664 | 1.167 | 0.857 | 59 |
| 32 | 0.530 | 1.887 | 0.625 | 1.600 | 1.179 | 0.848 | 58 |
| 33 | 0.545 | 1.836 | 0.649 | 1.540 | 1.192 | 0.839 | 57 |
| 34 | 0.559 | 1.788 | 0.675 | 1.483 | 1.206 | 0.829 | 56 |
| 35° | 0.574 | 1.743 | 0.700 | 1.428 | 1.221 | 0.819 | 55° |
| 36 | 0.588 | 1.701 | 0.727 | 1.376 | 1.236 | 0.809 | 54 |
| 37 | 0.602 | 1.662 | 0.754 | 1.327 | 1.252 | 0.799 | 53 |
| 38 | 0.616 | 1.624 | 0.781 | 1.280 | 1.269 | 0.788 | 52 |
| 39 | 0.629 | 1.589 | 0.810 | 1.235 | 1.287 | 0.777 | 51 |
| 40° | 0.643 | 1.556 | 0.839 | 1.192 | 1.305 | 0.766 | 50° |
| 41 | 0.656 | 1.524 | 0.869 | 1.150 | 1.325 | 0.755 | 49 |
| 42 | 0.669 | 1.494 | 0.900 | 1.111 | 1.346 | 0.743 | 48 |
| 43 | 0.682 | 1.466 | 0.933 | 1.072 | 1.367 | 0.731 | 47 |
| 44 | 0.695 | 1.440 | 0.966 | 1.036 | 1.390 | 0.719 | 46 |
| 45° | 0.707 | 1.414 | 1.000 | 1.000 | 1.414 | 0.707 | 45° |
| | Cos. | Sec. | Cot. | Tan. | Csc. | Sin. | Angle. |

A page from Edwin Pliny Seaver’s book *Elementary trigonometry, plane and spherical*, published in 1898 by Taintor Brothers & Co.

There are many subtle patterns in this table. How many can you find?

# Shoot for the Moon, Part II


Written and illustrated by Julia Zimmerman

The living room was deserted: a sofa cushion lay on the floor, desolately unoccupied among open textbooks, pencils, half a cup of juice, and notes. The colorful notes - Elizabeth's were organized by color and Clara's were lovingly decorated by it - lay curled on the floor, looking forlorn and abandoned. The only sound was a low hum from the quiescent heater. But that peace and quiet was only in the living room. The den was another story entirely. It was the den that was the hub of activity (since it housed the computer) as Clara and Elizabeth tried to figure out how the distance between the earth and the moon had been measured.

The basic premise of the experiment was that by timing how long it took to send a laser beam to the moon and back to a target on Earth, scientists could compute the distance by using the time, the known constant $c$ (the speed of light) and the equation $distance = speed \times time$. Elizabeth and Clara had tried to understand why the corner reflector, whose configuration guaranteed parallel paths of incoming and outgoing light, was better than a regular plane mirror for use in the experiment; they'd done this by building a corner reflector out of the mirrored doors of the bathroom medicine cabinet, and by using Elizabeth's brother's laser pointer. To their astonishment, when they compared the two types of mirror configuration, the corner reflector had hardly seemed superior...

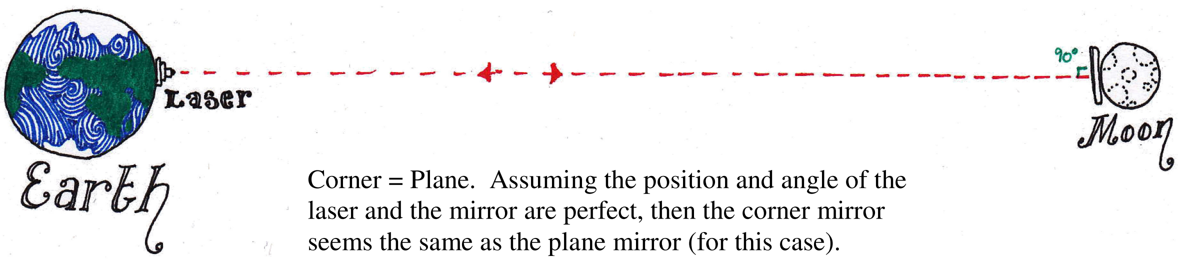
…Which is why the girls were still in the den, thoroughly immersed in animated discussion, puzzling over the outcome of their little experiment. *Why was it that a corner reflector had been chosen over a regular mirror?* Clara was at that moment gesticulating wildly, indicating that she wanted to make a point which required Elizabeth to think of the moon as being the fan pull in the back bedroom, and the earth as being the doorknob. Elizabeth pictured a tiny laser firing a minuscule beam of light from the center of the room to the door, bouncing off a minute corner reflector, and then passing her once again on its return path. She caught the light between her fingertips, mentally holding it still as she listened to Clara's excited but confusing description of the light's trip to the moon and back. *Come to think of it, though*... Elizabeth realized that neither she nor Clara knew for a fact that the target that caught the beam was the same place from which the laser was fired. She frowned, but decided to continue to assume that the laser and target were in the same place for now, so that she could keep a consistent picture in her head while she listened to Clara.

Clara continued, "...Imagine the path of the light - ideally, we want the light's path to travel from the laser to the moon and back to some area near the laser. Let's just say we want the light to return within an $r$-meter radius for some fixed value $r$. For a corner reflector facing the Earth, there are many places the laser can be for which that happens. But with a plane mirror facing the Earth, you have to wait for the laser to be in just the right place... In our experiment, we set up everything so the laser was always basically perpendicular to the plane mirror. We just assumed that our setup would represent the experiment well enough to show us why a corner reflector would be better to use than a regular mirror, but... it didn't... I mean, in real life the Earth and the moon are rotating and orbiting and stuff, right? So maybe the laser and the mirror wouldn't line up like that very often."

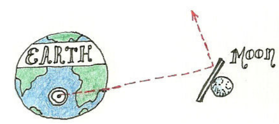

"You know," said Elizabeth, "I think you're on the right track with this idea of alignment... but it's pretty hard to talk about these ideas clearly. What if we draw it, and write out all of our thoughts?"

Clara agreed promptly, quickly volunteering all of the accessories a fashionable poster board could wish for - markers, colored pencils, glitter, sequins. Elizabeth laughed and reined Clara in. "I think we should get our thoughts straight first. Then, if you want to go crazy and add a lace border or whatever, I'm all for it." Clara could see the sense in this plan, so she nodded and brought out a large sheet of poster paper from her room. Settling down on the floor of the den, the girls started to draw out why the corner reflector was better than a plane mirror - in other words, why it was helpful to have parallel paths for the incoming and outgoing beams.

They started off by drawing the Earth and the moon lined up in the "ideal" way, the way they had lined up their reflector and the laser in their own experiment.



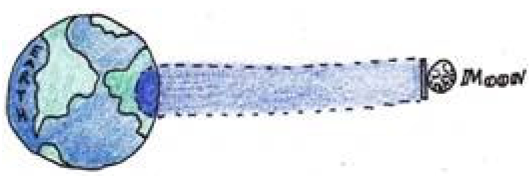

Corner = Plane. Assuming the position and angle of the laser and the mirror are perfect, then the corner mirror seems the same as the plane mirror (for this case).



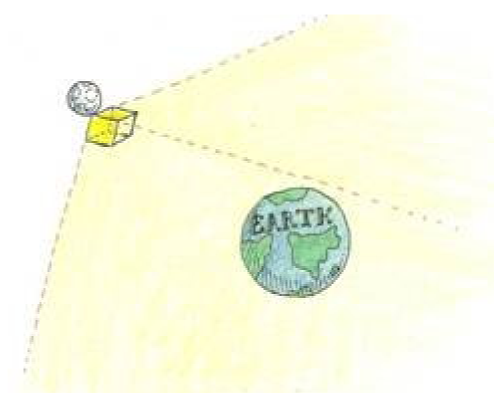

Plane Mirror

In the case that the alignment was perfect, then the corner mirror and the plane mirror acted in essentially the same way.

For cases when the alignment wasn't perfect (right), they could see that there were many possible alignments in which the outgoing beam would not even return to Earth, much less the observatory, if they used a plane mirror. They thought about the possible locations (figure at left) in which the laser could be placed so that the paths of the incoming and outgoing beams of light would be within a radius of *r* meters of each other, and came up with a zone (drawn in blue) of possible locations. In one plane, this zone was roughly a rectangle. In 3-D space, if the mirror was a square, this zone would be roughly an elongated box; if the mirror were circular, this zone would be roughly a cylinder. A laser fired from a location inside this zone (and up to about *r*/2 meters outside of this zone) could be aimed so that the laser hit the mirror at an angle close enough to perpendicular so that the incoming and outgoing paths of the light beam would be less than *r* meters apart.



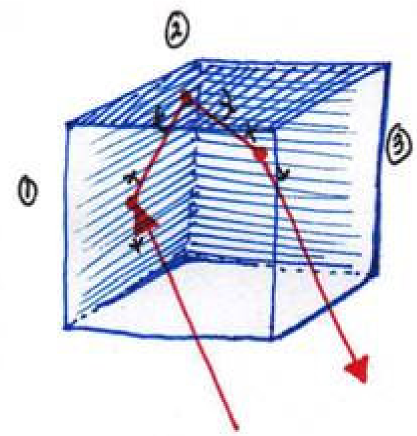

But for the 3-mirror corner reflector, the zone of possible "good" laser locations was much larger; this zone was, in one plane, a triangle, and in 3-D space, the corner of a larger cube (drawn in yellow at right). Outside this zone, the laser couldn't be aimed to hit more than one mirror - but inside this zone, it would be possible to aim the laser so that the incoming and outgoing beams would be parallel. The girls found that, if the incoming and outgoing beams were parallel, then the maximum distance between them would never be more than the distance between the two farthest corner-points of the reflector.




From these drawings, they could see that there would be no advantage to using a plane mirror instead of a corner reflector. The corner reflector allowed the paths of the incoming and outgoing beams to be parallel, guaranteeing that they would be relatively near each other and predictably located far more often than they would be using the plane mirror. In addition, they were able to see why a 3-mirror corner reflector was preferable to a 2-mirror corner reflector:

they saw that a 2-mirror reflector would act in the same way as a plane mirror with respect to any light with a velocity component parallel to the line where the two mirrors were joined. They thought that, since the goal was to get the light beam to return to Earth, in particular to a useful location (i.e. one that always falls within a relatively small fixed region, so that the equipment needed to detect the returning light wouldn't need to be moved), then the mirror configuration they should choose should be the one which guaranteed that outcome for the widest range of laser locations. This seemed to be why parallel incoming and outgoing paths were so useful. Therefore the 3-mirror reflector seemed to be the best choice (see right).




The girls felt fairly satisfied with this. Fondly, Clara patted their homemade corner reflector. To Elizabeth, she said, "Well, it looks like this baby *does* work better than the giant tape measure would have!"

Elizabeth laughed. "I think *most* things would work better than *that.*"

Clara pretended to ponder this statement. "I suppose there really are very few situations that absolutely require a giant tape measure..."

Elizabeth nodded seriously. "*Extremely* few." Then she paused, distracted by new thoughts about the experiment. "Hey, Clara, do you think our model of the laser beam as a straight line was okay? What if it should look more like a flashlight beam? You know, with the diameter of the beam getting larger as it travels - like how the circle of light from a flashlight is dimmer and larger when it hits a far wall instead of something nearby."

"I don't know," said Clara. "Now that you bring up the experiment again, why did they use a laser? Did it help them identify the light returning to the target as being the light they'd sent out? Would that be based on the color of the light, or what?"

"Hmmm" said Elizabeth, "That's a good one... And exactly what distance have we been calling "the distance from the Earth to the moon"? From the core to the core? From laser to reflector? And how did they get such a high level of accuracy?"

"Yeah! And how did they... no, wait." said Clara. "As fascinating as this is, I think it may need to be a project for another day. We should probably get back to studying for the test tomorrow."

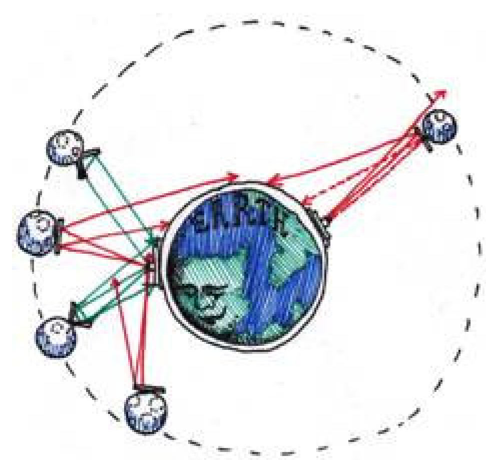
"No way," said Elizabeth.

Clara looked at her, puzzled. "What? Why not? You were certainly nagging me about it earlier!"

Elizabeth gave her a stern look. "First of all, I wasn't *nagging.* Secondly, we still have to put the doors back on the medicine cabinet before my parents get home!"

Laughing, Clara began removing tape from their corner reflector. "Okay," she said, "but don't worry about it – it'll be easy."

Elizabeth picked up the screwdriver... and crossed her fingers.

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*The author thanks Douglas Currie, Professor Emeritus of the Department of Physics at the University of Maryland for valuable and enlightening correspondence.*

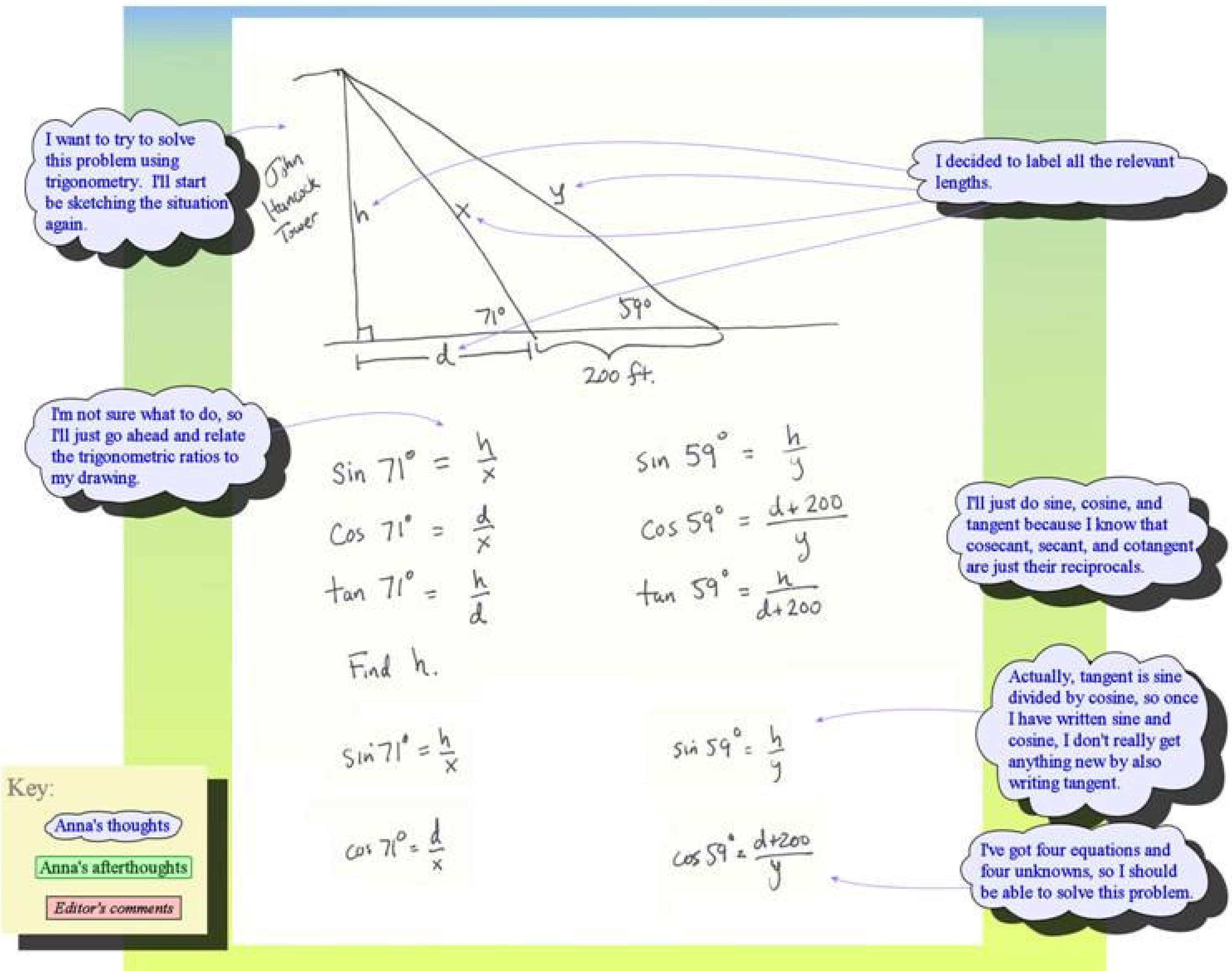
 

# Anna's Math Journal

By Anna B.

*Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.*

Anna revisits the problem from last issue: Find the height of the John Hancock building given that near the corner of Boylston and Dartmouth, her line of sight to the top of the tower made a 59° angle to horizontal and, walking 200 feet closer, the angle changed to about 71°.



Solving for $x$ and $y$ reduces four equations in four unknowns to two equations in two unknowns.

$$x = \boxed{\frac{h}{\sin 71^\circ} = \frac{d}{\cos 71^\circ}} \qquad y = \boxed{\frac{h}{\sin 59^\circ} = \frac{d+200}{\cos 59^\circ}}$$

I'll solve for $d$ in this equation and substitute into this equation.

$$d = \frac{\cos 71^\circ}{\sin 71^\circ} h = \cot 71^\circ h \quad \rightarrow \quad \frac{h}{\sin 59^\circ} = \frac{(\cot 71^\circ)h + 200}{\cos 59^\circ}$$

Now I have one equation in one variable $h$, so I can solve this equation for $h$, the height of the John Hancock tower.

$$h\left(\frac{1}{\sin 59^\circ} - \frac{\cot 71^\circ}{\cos 59^\circ}\right) = \frac{200}{\cos 59^\circ}$$

$$h = \frac{200}{\cos 59^\circ}\left(\frac{1}{\frac{1}{\sin 59^\circ} - \frac{\cot 71^\circ}{\cos 59^\circ}}\right)$$

In the denominator, if I distribute the cosine of 59 into the parentheses it combines with one over sine to make cotangent and cancels the cosine of 59 in the denominator of the second term.

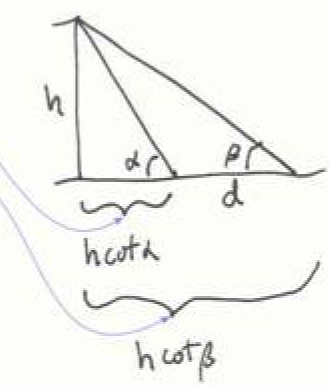
$$= 200\left(\frac{1}{\cot 59^\circ - \cot 71^\circ}\right)$$

$$\approx 779.6$$

I used a calculator to get this approximate value for the height. This is quite close to what I got last time, which was about 780 ft.

$$\underbrace{h\cot 59^\circ}_{(d+200)} - \underbrace{h\cot 71^\circ}_{d} = 200$$

If I rearrange the equation like this, I can see a direct interpretation in terms of lengths in the figure...what this tells me is that my solution was rather inefficient. With the trigonometric ratios, I can just write down an unknown length of a right triangle as a product of a known length and a suitable trigonometric ratio! I can go straight from the picture to this equation.




$$h\cot\beta - h\cot\alpha = d$$

$$\boxed{h = \frac{d}{\cot\beta - \cot\alpha}}$$

In other words, I should be able to see that this is $h$ cotangent alpha, and this is $h$ cotangent beta directly...I mean, these facts come straight from the definition of cotangent!

Using trigonometric functions spares me from having to construct a careful scale drawing.

ABB 2.23.11

The trigonometric functions enable me to avoid having to make careful scale drawings and measurements when I want to make an argument using similar figures. The problem of computing the ratios is absorbed by the trigonometric function. So the accuracy problem is transferred to the problem of computing the trigonometric functions accurately. I guess this could be done by making very, very carefully drawn right triangles...or...hmmm...is there a way to compute the values of the trigonometric functions without making any drawings??? How are tables of trigonometric functions computed?

Key:
- Anna's thoughts
- Anna's afterthoughts
- *Editor's comments*

MATH IN YOUR WORLD

Logo Design by Hana Kitasei

# The Mathematics of Voting

By Katherine Sanden

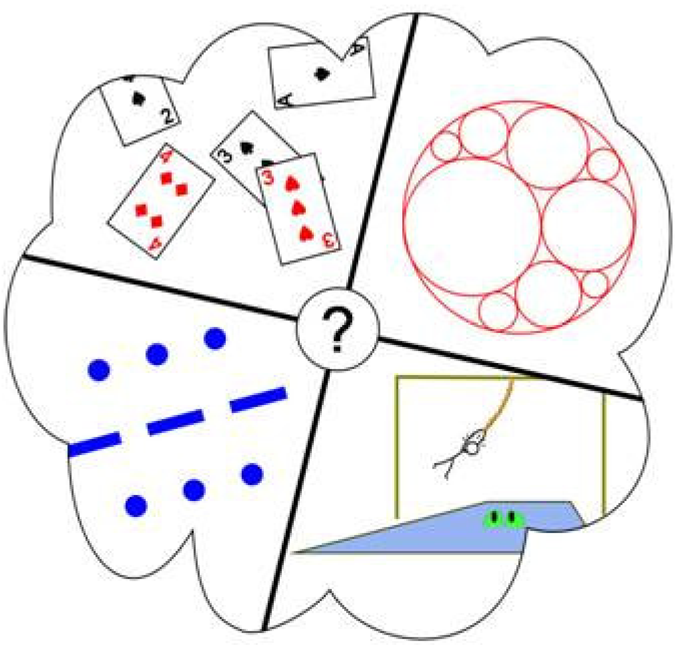
Have you done any voting recently? Perhaps you voted in a student government election. Or maybe you voted for a video in a YouTube contest, or selected an answer choice in an online poll.

Voting is a powerful social tool. It gives us a way to measure and rank things (political candidates, songs, T-shirt designs, activities, etc.). Its aim is to translate varied human preferences into a ranking that accurately reflects those preferences.

And it seems pretty simple, right? In the situations I mentioned above, each voter chooses an item, and then we count up the number of votes each item received. The item with the most votes wins. Done. We'll call this "one-choice" voting.

But is this always the best way to represent peoples' preferences? Imagine you are in a group, and you are voting on which activity to do together:

A. practice communicating in Morse code,
B. play 52-card pickup,
C. create an outdoor obstacle course, or
D. solve math problems.




Your top choice is D (math!), and C (obstacle course) is a close second. The one activity you really don't want to do is B (52-card pickup). So you vote for D. The votes are tallied, and B wins! You're disappointed. When you voted for D, you weren't able to convey the information that you *also* liked C, and that your least favorite was B. What if there was a way to include this extra information in the vote?

Instead of one-choice voting, let's have each voter rank the choices. Now, rather than simply choosing "D," you can say:

Choice 1: D
Choice 2: C
Choice 3: A
Choice 4: B

How do we tally the votes now? One option: we can assign 4 points to Choice 1, 3 points to Choice 2, 2 points to Choice 3, and 1 point to Choice 4. Then we'll add up the points going to each activity. So imagine that in a group of five people, we get the following different rankings:

| Voter | Choice 1 | Choice 2 | Choice 3 | Choice 4 |
|---|---|---|---|---|
| You | D | C | A | B |
| Friend 1 | B | C | D | A |
| Friend 2 | B | C | A | D |
| Friend 3 | B | A | C | D |
| Friend 4 | C | D | A | B |

You can check that the results come out to be 10 points for A, 14 for B, 15 for C, and 11 for D.

In this case, C wins. Notice that if we had done one-choice voting, B would have won since 3 people chose B as their top choice. Which voting system do you think would be fairer in this situation? You could argue either way. Using the ranking system, fewer people got stuck with their lowest choice. On the other hand, fewer people got their top choice. What if some of the people who voted B *really badly* wanted B, and hardly liked any of the other options? Now they're disappointed.

How could we modify our voting system to take this into account? I'll present one idea to you: let's give each person a set amount of points (say, 12 points) that she can "spend" on any activity. The people who really like B, and don't care for the other activities at all, could spend all 12 points on B. Since you like D the most, with C a close second, you might put 6 points on D, 5 points on C, 1 point on A, and 0 points on B. Do you think this idea works well?

Finally, here's a more sophisticated voting system where we modify our method of totaling the results. So far, we have been simply summing up the number of points for each activity. Instead of summing, we could square each number, then add them all together, then take the square root of the sum[1]. Let's apply this process to the original example, and see if we get the same results:

A: $\sqrt{2^2+1^2+2^2+3^2+2^2} = \sqrt{22} \approx 4.69$ points

B: $\sqrt{1^2+4^2+4^2+4^2+1^2} = \sqrt{50} \approx 7.07$ points

C: $\sqrt{3^2+3^2+3^2+2^2+4^2} = \sqrt{47} \approx 6.86$ points

D: $\sqrt{4^2+2^2+1^2+1^2+3^2} = \sqrt{31} \approx 5.57$ points

In fact, B won this time! We used the exact same rankings as before, but by squaring each ranking first, we changed the winner from C to B.

**Food for Thought**

Consider these real-world voting systems.

The early rounds of American Idol: For each singer, three judges make a binary vote: YES, you can continue, or NO, you can't.

The iTunes Top 100 most downloaded songs: You essentially cast your vote by purchasing a song, so it's similar to one-choice voting, except you have more than one vote— you can buy as many different songs as you like.

Facebook "Questions": Again, you have more than one vote to use. You can click "Recommend" on multiple answers. The most recommended answers float to the top.

Olympic Figure Skating: Research the judging system used at the Olympics. It's complex!

### Take it to your world

These are just a few ideas. There are many other modifications that would be interesting as well. Play with this. See if you can invent a voting system that you think is perfect for this situation. Test it out on a group of friends, family members, or classmates. You can survey your participants on their levels of satisfaction! Here's another challenge: can you design four different voting systems that, given one set of results, produce four different winners?

[1] Where did that idea come from?! Does it remind you of the distance formula? Can you give a visual interpretation of this method if only 2 people voted? What if only 3 people voted? Think about it! It's a juicy topic.

# Similarity and Trigonometry Problems

by Ken Fan

Prof. Meckes said, “To really understand something you have to work through it yourself.” So let’s get a good handle on similarity and trigonometry by solving some problems!

1. Each row in the table gives some of the characteristics of 3 different triangles. Two of the triangles in each row are similar to each other and one is not similar to the other two. In each row, cross out the triangle that is not similar to the other two. The number $x$ is greater than 0.

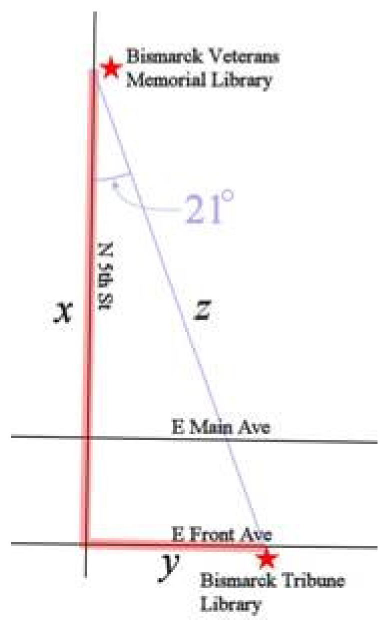
| **Triangle 1** | | | | | | **Triangle 2** | | | | | | **Triangle 3** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Side lengths** | | | **Angles** | | | **Side lengths** | | | **Angles** | | | **Side lengths** | | | **Angles** | | |
| 3 | 4 | 5 | | | | 10 | 15 | 20 | | | | 15 | 20 | 25 | | | |
| | | | 20° | 60° | | | | | 100° | | 20° | | | | | 80° | 20° |
| 7 | 7 | 7 | | | | | | | 60° | | | | 7 | 7 | 90° | | |
| 6 | | 6 | | 90° | | 9 | | 9 | | 60° | | 83 | | | | | 45° |
| | $x$ | $x$ | | | 30° | $x$ | $x$ | | | 90° | | | | $3x$ | | 120° | |

2. Draw an isosceles right triangle. Use the Pythagorean theorem to determine the sine and cosine of 45°. What are the tangent, cotangent, cosecant, and secant of 45°?

3. Draw an equilateral triangle along with one of its angle bisectors. Study this diagram and figure out what the sine and cosine of 30° and 60° are.

4. Suppose angles $\alpha$ and $\beta$ are complimentary angles (in other words, $\alpha + \beta = 90°$). Show that cos $\alpha$ and sin $\beta$ are equal to each other.

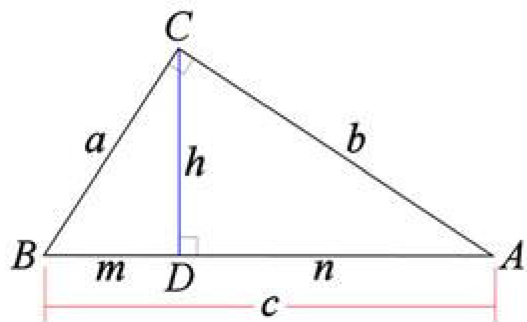
5. Elizabeth and Clara were doing research in the Bismarck Veterans Memorial Library in Bismarck, North Dakota. They couldn’t find a book they were looking for, but the librarian told them that the book was available at the Bismarck Tribune Library a short walk away. To get there, they walked down North 5th Street and turned 90° to the left at East Front Avenue. See the simplified map at right. Their total distance traveled was 840 meters, i.e. $x + y = 840$. Given that the indicated angle is 21°, determine $x$, $y$, and $z$ to the nearest meter.




6. Triangle $T_1$ and triangle $T_2$ are similar to each other. The perimeter of triangle $T_1$ is $p$ and the perimeter of triangle $T_2$ is $kp$. If the area of triangle $T_1$ is $A$, what is the area of triangle $T_2$?

7. What are the measures of the angles in a triangle whose sides have length 5, 12, and 13 units? Give your answers accurate to the nearest tenth of a degree.

The next five problems all relate to the figure below right. The figure shows a right triangle *ABC* with an altitude to hypotenuse *AB* drawn in. This altitude meets the hypotenuse at point *D* which splits *AB* into two pieces of length *m* and *n*.




8. Show that triangles *ABC*, *BCD*, and *ACD* are all similar to each other.

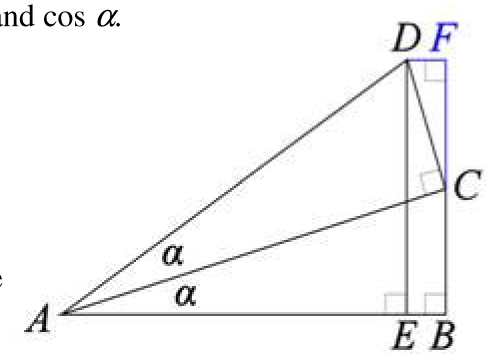
9. The length *h* represents the height of triangle *ABC* if its hypotenuse is viewed as its base. What lengths in the figure represent the heights of triangle *ABC* when each of the other sides is taken as its base?

10. Show that $h^2 = mn$.

11. Express *h* in terms of *a* and *b*. (Hint: One way to do this is to compute the area of triangle *ABC* in two different ways.)

12. Express *m* and *n* in terms of *a* and *b*.

Now you're going to use similar right triangles and the definitions of the trigonometric functions to derive formulas for $\sin(2\alpha)$ and $\cos(2\alpha)$ in terms of $\sin \alpha$ and $\cos \alpha$.




We start by constructing a new figure shown at right. In this figure, triangle *ABC* is right and the measure of angle *CAB* is $\alpha$. For convenience, we scale the figure so that the length of *AC* is one unit. We then construct another right triangle, *ACD*, so that the measure of angle *DAC* is also $\alpha$. The perpendicular from *D* to *AB* meets *AB* at point *E*. Finally, we extend *BC* so that *EBFD* is a rectangle.

13. Express the lengths *AB* and *BC* in terms of trigonometric functions of $\alpha$.

14. Using similarity, what are the lengths of *AD* and *CD*?

15. What is the angle *DCF*?

16. What are the lengths *CF* and *DF*?

17. What are the lengths *AE* and *DE*?

18. By examining triangle *ADE*, find expressions for $\sin(2\alpha)$ and $\cos(2\alpha)$ in terms of $\sin \alpha$ and $\cos \alpha$.

19. Can you find an expression for $\tan(2\alpha)$ in terms of $\tan \alpha$?

20. Can you find an exact expression for the sine and cosine of 15°?

# Errorbusters!

by Cammie Smith Barnes

Providing timely feedback is a very important part of teaching at the college level, as it is at any level. Yet grading assignments or exams is always a difficult task for me. I don't like to see my students fall into any traps.

One pitfall I've seen my students run into a lot recently when I've taught about exponents arises from a common misunderstanding of negative exponents. A student sometimes confuses the roles of minus signs in exponents and bases. That is, I frequently see something like the following:

$$2^{-3} = -2^3.$$

But $-2^3 = -(2 \cdot 2 \cdot 2) = -8$, whereas $2^{-3}$ means something else entirely. The number -3 is the opposite of 3, so $2^{-3}$ means roughly that we need to "do the opposite" of multiplying three factors of 2. But, just like the opposite of addition is subtraction, the "opposite" of multiplication is division. So $2^{-3}$ tells us that we need to divide something by three factors of 2— or equivalently, that we need to divide by 2 three times.

The number that we need to divide into is a very special number indeed: the number 1. The number 1 has the amazing property that whenever we multiply something by it, we leave our original number unchanged. (That is, for example, $53 \cdot 1 = 53$.) The number 1 also happens to be the quotient of any nonzero number with itself. (So, for instance, $\frac{53}{53} = 1$.) In fact, because of these properties, we call 1 the **multiplicative identity**.

Therefore, $2^{-3} = 1 \div 2 \div 2 \div 2$. But, since dividing by 2 is the same thing as multiplying by $\frac{1}{2}$, we actually have

$$2^{-3} = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}.$$

Thus the right-hand side is just $\left(\frac{1}{2}\right)^3$, or, equivalently, $\frac{1}{2^3}$. In other words, since

$$2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8},$$

the minus sign in the exponent is telling us to put the factor $2^3$ in the denominator of a fraction whose numerator is 1. So, for any nonzero number $x$ and any integer $n$, we have $x^{-n} = \frac{1}{x^n}$.

In fact, more generally, we have the **quotient rule for exponents**: division of terms with exponents and the same base becomes subtraction in the exponents. That is, for any nonzero number $x$ and any integers $m$ and $n$, we have

$$\frac{x^m}{x^n} = x^{m-n},$$

a rule that goes hand-in-hand with the **product rule for exponents**, which, as you may recall from the last issue, says that $x^m x^n = x^{m+n}$. It makes sense that if we replace the addition in the product rule with subtraction, then we must replace multiplication with division. Moreover, if we let $m = n$, then we see that $\frac{x^n}{x^n} = x^{n-n}$; that is, $\frac{x^n}{x^n} = x^{n-n} = x^0 = 1$ a very beautiful fact indeed.

Let's try simplifying another expression involving a negative exponent: $-4^{-5}$. Do not yield to the temptation to cancel out the two minus signs! They are playing very different roles here. Recall that the negative sign in the exponent tells us to put the factor in the denominator of a fraction. Leaving the other negative sign alone, we get

$$-4^{-5} = -\frac{1}{4^5} = -\frac{1}{1024}.$$

On the other hand, if -4 were enclosed in parentheses, we would get

$$(-4)^{-5} = \frac{1}{(-4)^5} = \frac{1}{-1024} = -\frac{1}{1024},$$

the same end result. We need to be careful, however, because when the exponent is even, the sign of the result depends on whether or not we have parentheses in the expression. Note that

$$-3^{-4} = -\frac{1}{3^4} = -\frac{1}{81},$$

whereas

$$(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}.$$

What if we raise a fraction to a negative exponent? What happens then? Suppose, for instance, that we would like to simplify $\left(\frac{3}{5}\right)^{-2}$. Well, let's recall that dividing by a fraction is equivalent to multiplying by the fraction's **reciprocal**, the new fraction obtained by switching the numerator and the denominator. In other words, $\left(\frac{3}{5}\right)^{-2}$ means that we must multiply together two factors of the reciprocal of $\frac{3}{5}$. Note that the reciprocal is $\frac{5}{3}$, so

$$\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2 = \frac{5}{3}\cdot\frac{5}{3} = \frac{5^2}{3^2} = \frac{25}{9}.$$

Moreover, if $a$ and $b$ are any nonzero numbers and $n$ is any integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}.$$

For practice, try simplifying expressions with exponents using the following exercises. Match each expression in the top row with its equivalent expression in the bottom row. One of the terms in the bottom row is used twice. The answers can be found on page 27.

$4^3$ $\quad 4^{-3} \quad -4^3 \quad -4^{-3} \quad \left(\frac{6}{7}\right)^2 \quad \left(\frac{6}{7}\right)^{-2} \quad -\left(\frac{6}{7}\right)^2 \quad -\left(\frac{6}{7}\right)^{-2} \quad \left(-\frac{6}{7}\right)^{-2}$

$-\frac{49}{36} \quad -\frac{36}{49} \quad \frac{1}{64} \quad \frac{49}{36} \quad 64 \quad \frac{36}{49} \quad -\frac{1}{64} \quad -64$

Edited by Jennifer Silva

# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

<u>Session 8 – Meet 1 – January 27, 2011</u>

Mentors: Keren Gu, Samantha Hagerman, Natasha Jensen, Lucia Mocz,
Kate Rudolph, Charmaine Sia, Liz Simon, Rediet Tesfaye

We opened Session 8 with the following icebreaker activity: All members and mentors arranged themselves into a large circle and a marker was passed around in a clockwise direction. Each person passed the marker to a person $K$ people over. The question was: For what $K$ would everyone get to hold the marker?

We also introduced an ongoing dollhouse design project. In this project, the members design a dollhouse from scratch and must give a complete specification for every part of the house. And, since this dollhouse is being designed at a math club, we're pushing math to the hilt! Hopefully, a couple issues from now, the dollhouse will be revealed.

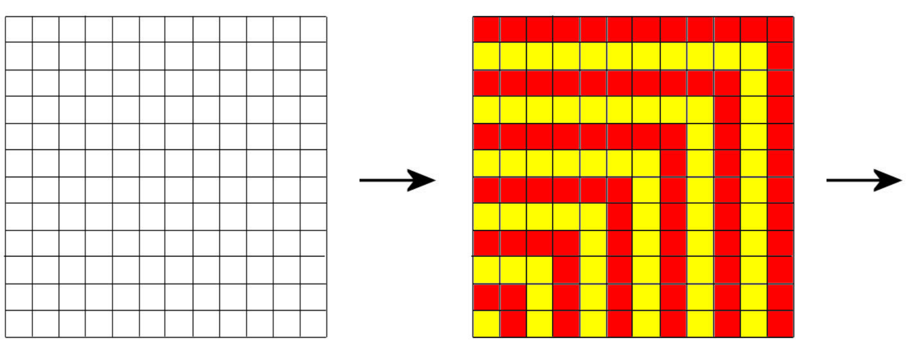
A number of girls also worked on mathematics related to David Speyer's articles on rational approximations of real numbers that appear in this and the last issues.

<u>Session 8 – Meet 2 – February 3, 2011</u>

Mentors: Samantha Hagerman, Ariana Mann, Jennifer Melot, Lucia Mocz,
Kate Rudolph, Charmaine Sia, Liz Simon, Rediet Tesfaye

The **perfect square** was one of the topics covered in Meet 2. Perfect squares are positive numbers which are the square of an integer. The first few perfect squares are 1, 4, 9, 16, 25, etc.

By systematically coloring the tiles of a perfect square, one can obtain nice arithmetic identities.

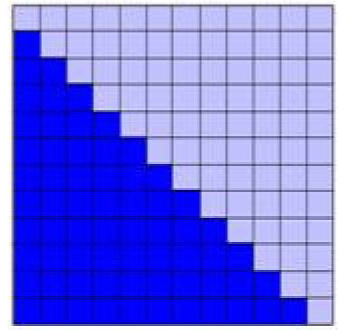


$$n^2 = 1 + 3 + 5 + \ldots + (2n-1)$$

What other ways can you think of to systematically fill in a perfect square? What arithmetic identity does it lead to? Can you think of shapes other than perfect squares to fill in?

**Lucky** noticed that she could split a perfect square into two pieces as shown at right. This led her to observe that



$$n^2 = T_{n-1} + T_n,$$

where $T_n$ is the $n$th triangular number. Do you see what she saw?

Can you prove this identity algebraically? Recall that the $n$th triangular number $T_n$ is equal to $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

The formula that **Lucky** found gives a hint about how to use geometry to find a formula for the sum of the first $n$ perfect squares: $1^2 + 2^2 + 3^2 + \ldots + n^2$. Let's call this sum $S_n$. Let's write down **Lucky**'s formula for even values of $n$:

$$2^2 = T_1 + T_2,$$
$$4^2 = T_3 + T_4,$$
$$6^2 = T_5 + T_6,$$
$$\vdots$$
$$(2m)^2 = T_{2m-1} + T_{2m}.$$

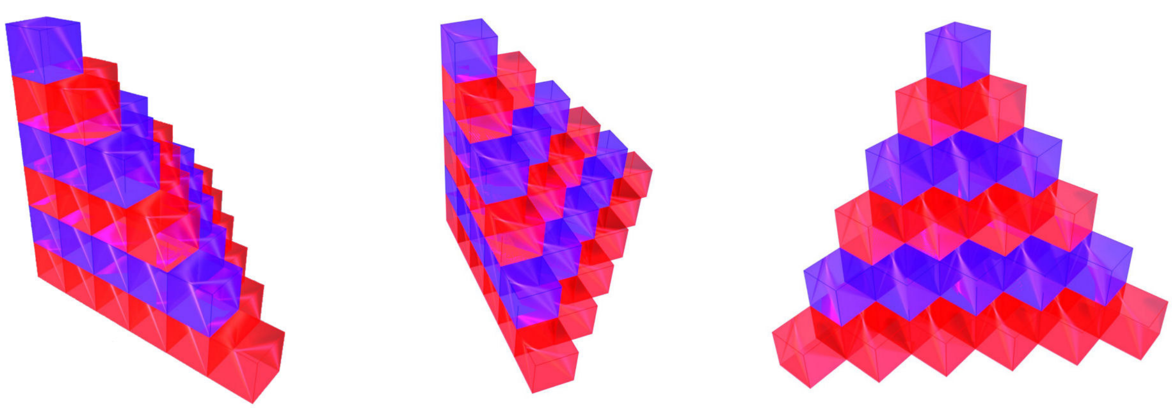
If we add up these equations, we get

$$2^2 + 4^2 + 6^2 + \ldots + (2m)^2 = T_1 + T_2 + T_3 + \ldots + T_{2m}.$$

Every term in the left hand side of the equation has a factor of 4, and if we divide it out, we are left with the sum of the first $m$ perfect squares, $S_m$. So we can write

$$4S_m = T_1 + T_2 + T_3 + \ldots + T_{2m}.$$

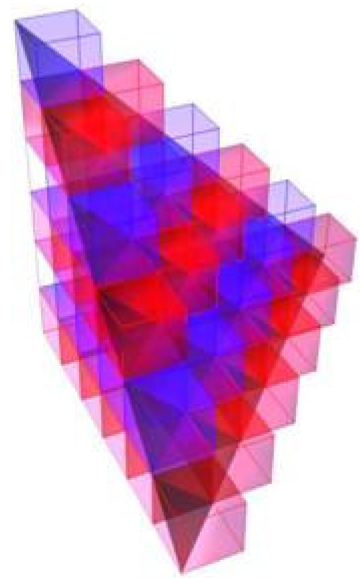
The right hand side of the equation is the sum of the first $2m$ triangular numbers. This sum has a nice geometric interpretation. It is the number of cubes in a pyramidal arrangement of cubes, as illustrated below for the case $m = 3$:



Three different views of the case $m = 3$.

If we take each cube to have unit volume, then the volume of this pyramid of blocks is equal to the number of blocks. This pyramid roughly corresponds to a pyramid with height $2m$ and base an isosceles right triangle whose legs also have length $2m$. Let's add in the triangular plane that corresponds to the diagonal face of such a pyramid. (See the image below right.)

The diagonal plane cuts through a number of cubes. It cuts through cubes on the "rough" side of the original stack of cubes as well as through the cubes one layer beneath. So if we estimate the volume of this pyramidal stack of cubes by using the volume of the pyramid below the diagonal plane, we will be missing some volume. Five sixths of the cubes on the "rough" side will be chopped off and unaccounted for. Similarly, a sixth of each cube just beneath is chopped off. If you're having trouble seeing this, think about the situation from several different vantage points. For example, try to imagine how the triangular plane passes through each cube individually.



The blue/red color scheme shows how to see the pyramid as a stack of horizontal block layers each arranged as a triangle. However, the pyramid can also be seen as a stack of triangular layers of blocks where each layer is "parallel" to the diagonal plane that we just slipped in. In this way, we can see that the number of cubes in the outermost, "rough," diagonal layer is exactly $T_{2m}$ and the number of cubes in the diagonal layer just beneath is equal to $T_{2m-1}$.

Putting all these pieces of the puzzle together, we get

$4S_m = T_1 + T_2 + T_3 + \ldots + T_{2m}$
= (volume of pyramid of blocks)

= (volume below triangular plane)
+ (total volume of parts of outermost cubes that are chopped off)
+ (total volume of parts of cubes that are cut in the layer just beneath the outermost cubes that are chopped off)

$$= \frac{1}{3}\left(\frac{1}{2}(2m)^2\right)(2m) + \frac{5}{6}T_{2m} + \frac{1}{6}T_{2m-1}.$$

Using the formula $T_n = \frac{1}{2}n(n+1)$, we find that

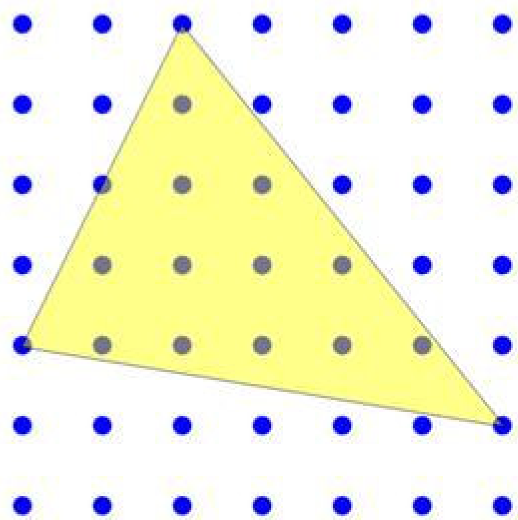
$$4S_m = T_1 + T_2 + T_3 + \ldots + T_{2m} = \frac{1}{3}\left(\frac{1}{2}(2m)^2\right)(2m) + \frac{5}{6}\left(\frac{1}{2}(2m)(2m+1)\right) + \frac{1}{6}\left(\frac{1}{2}(2m)(2m-1)\right).$$

If you simplify this and solve for $S_m$, you will find that $S_m = \frac{1}{6}m(m+1)(2m+1)$.

If you prefer algebra to geometry, there is an algebraic way of deducing the formula for the sum of the first $n$ perfect squares. It uses a technique called **telescoping sums**. In fact, the telescoping sums technique can be used to find a formula for the sum of the first $n$ perfect $k$th powers for any $k$, not just $k = 2$! If you're curious, look it up or ask about it at the club.

Session 8 – Meet 3 – February 10, 2011

Mentors: Jennifer Balakrishnan, Samantha Hagerman, Ryan Heffrin,
Ariana Mann, Jennifer Melot, Liz Simon, Bianca Viray



One thread that has arisen this session is the topic of **lattice polygons**. A lattice polygon is a polygon in the coordinate plane whose vertices all have integer coordinates.

How many ways can you think of to compute the area of the lattice triangle shown at left?

Here's one way: Use the Pythagorean theorem to compute the length of all three sides of the triangle, then use Heron's formula for the area of a triangle.

Here we go! Using the Pythagorean theorem, the side lengths of the triangle are $\sqrt{20}$, $\sqrt{41}$, and $\sqrt{37}$.

Heron's formula tells us that the area is:

$$\sqrt{\frac{\sqrt{20}+\sqrt{41}+\sqrt{37}}{2}\left(\frac{-\sqrt{20}+\sqrt{41}+\sqrt{37}}{2}\right)\left(\frac{\sqrt{20}-\sqrt{41}+\sqrt{37}}{2}\right)\left(\frac{\sqrt{20}+\sqrt{41}-\sqrt{37}}{2}\right)}.$$

If you carefully simplify this huge expression, you'll eventually find that the answer is 13 square units. Surely, there's an easier way! If you think of something, let us know.

Session 8 – Meet 4 – February 17, 2011

Mentors: Jennifer Balakrishnan, Samantha Hagerman, Ryan Heffrin,
Kate Rudolph, Liz Simon, Rediet Tesfaye, Bianca Viray

Special Guest: Felice Frankel, Photographer and Scientist

In the second half of the meet, Felice Frankel gave a presentation on photography. In her work, she seeks answers to questions such as: How can I get *you* to look? How can I get *you* to participate? How do I figure out where to stand? What is your perspective? Is there a visual metaphor? Also, foremost on her mind is how to maintain the integrity of the science. She showed a number of photographs that revealed the power images have to make the viewer think specific things, sometimes without the viewer even being consciously aware of it. Because of this power, a photographer who wishes to convey truth takes on much responsibility. For example, it often happens that the photographer feels a need to alter an image for clarity. In such cases, one always has to consider whether the alteration bends the truth inappropriately. Therefore, unlike the artist, Felice always discloses what she did to produce an image.

In an intriguing series of photos, Felice showed that images of the exact same setup can appear remarkably different just by moving the point of view. The same can be said of math problems!

# Calendar

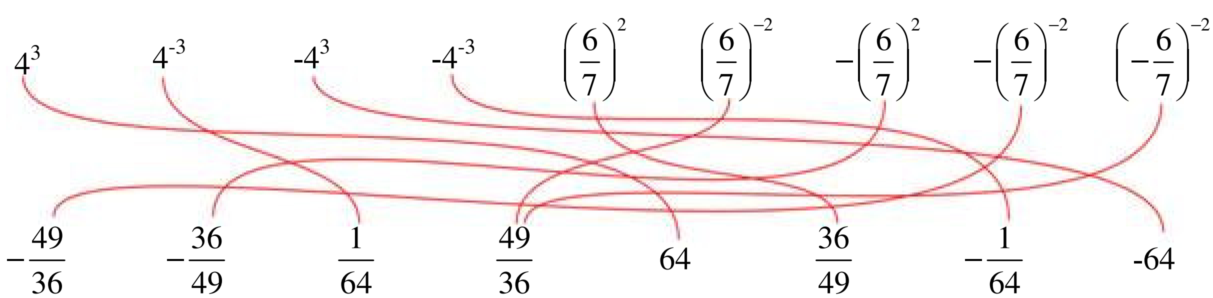
Session 8: (all dates in 2011)

| | | |
|---|---|---|
| January | 27 | Start of eighth session! |
| February | 3 | |
| | 10 | |
| | 17 | Felice Frankel, photographer and scientist |
| | 24 | No meet |
| March | 3 | |
| | 10 | |
| | 17 | Stella Yu, Boston College |
| | 24 | No meet |
| | 31 | Susan Barry, Mount Holyoke |
| April | 7 | |
| | 14 | |
| | 21 | No meet |
| | 28 | Jennifer Che, tiny urban kitchen |
| May | 5 | |

Session 9: (all dates in 2011)

| | | |
|---|---|---|
| September | 8 | Start of the ninth session! |
| | 15 | |
| | 22 | |
| | 29 | Start of Rosh Hashanah – No meet |
| October | 6 | |
| | 13 | |
| | 20 | |
| | 27 | |
| November | 3 | |
| | 10 | |
| | 17 | |
| | 24 | Thanksgiving - No meet |
| December | 1 | |
| | 8 | |

Here are answers to the *Errorbusters!* problems on page 22.



# Girls’ Angle: A Math Club for Girls

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

**What is Girls’ Angle?** Girls’ Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls’ interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls’ Angle mentors, the Girls’ Angle Support Network, the Girls’ Angle Bulletin and Community Outreach.

**Who are the Girls’ Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls’ Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

**What is the Girls’ Angle Bulletin?** The Girls’ Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

**What is Community Outreach?** Girls’ Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members’ efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

**How do I join? Membership** is granted per session and includes access to the club and extends the member’s subscription to the Girls’ Angle Bulletin to one year from the start of the current or upcoming session. You can also pay per session. If you pay per session and come for three meets, you will get a subscription to the Bulletin. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of “catching up with the group” doesn’t apply.

**Where is Girls’ Angle located?** Girls’ Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, Courant Instructor/PIRE fellow, NYU
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, Moore Instructor, MIT
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, Senior Research Scientist, Harvard Medical School.
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: A Math Club for Girls

## Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Bulletin Sponsorship

Parents/Guardians: ______________________________________________________________________

Address: ____________________________________________________________ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

**Emergency contact name and number**: ____________________________________________________

**Pick Up Info**: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ________________________________________________________________

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____________________________________________________________________________________________

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____________________________________________________ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ___________________________________________________________

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session membership
  - ☐ Bulletin Sponsorship: $60-99 Subscribing. $100-999 Bronze. $1000-2999 Silver. $3K+ Gold.*
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

*Silver and Gold sponsors may specify a recipient for their sponsored copies of the Bulletin.

# Girls' Angle: A Math Club for Girls
**Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

_______________________________________________________________________________,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____________________________________________Date: ________________

Print name of applicant/parent: ____________________________________________

Print name(s) of child(ren) in program: ____________________________________

Girls' Angle

**A Math Club for Girls**