Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

From the Director

This summer, Girls' Angle is going to conduct its first National Puzzle Contest Fundraiser. By donating $10 or more to Girls' Angle, you can enter to win prizes.

The puzzles should be fun and entertaining for the whole family, and, unlike what we do at the club, these puzzles will not involve much math. A number of the puzzles will have no math at all. We hope to make this fundraiser an annual event with more and more people playing each year.

Starting this fall, Girls' Angle will be offering a new Math Contest Prep course at the Microsoft New England Research and Development Center. This course will be designed for girls who already enjoy puzzling over contest problems and will aim to develop problem solving skills and technique.

Details for the fundraiser and the contest prep course will be posted on the Girls' Angle website by mid-July.

- Ken Fan, Founder and Director

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Girls' Angle Bulletin

The official magazine of
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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Spinning Out Mathematics*. Photograph by Ray Sidney.

An Interview with Bianca Viray

Dr. Bianca Viray received her doctoral degree in mathematics from the University of California, Berkeley. She is now Tamarkin Assistant Professor at Brown University and an NSF postdoctoral fellow. She's also a member of the Girls' Angle Advisory Board.

…research problems generally take a few months or even a few years! You have to learn to set smaller milestones for yourself to work towards each day or week…

Ken: Hi Bianca, thank you for doing this interview! Let's start with this question: How did you first become interested in mathematics?

Bianca: This is a hard question for me to answer, because I don't actually remember a time when I wasn't interested in math! But I think it came from games and puzzles. I really enjoyed doing logic puzzles and playing math-y games, like Mastermind, 24, and Set. I really liked (and still like) the open-ended and creative nature of these activities.

Ken: When did you begin thinking of mathematics as a potential career?

Bianca: I don't really remember making the decision to major in mathematics in college, but at the same time, I don't remember ever considering majoring in anything else. I didn't know what career I could pursue studying mathematics, so I started out as a double major in math and computer science. It wasn't until later in college that I realized studying math could be a career, that I could be a mathematician!

Ken: What was it like being a graduate student at UC Berkeley? Was it different from being an undergraduate?

Bianca: Being a graduate student at Berkeley was amazing. The community there is very energetic. Students are always running interesting seminars. And there are so many students in so many different areas, you can almost always find someone who knows the answer to your question, or find someone who wants to chat about some math problem that's caught your interest.

It is very different from being an undergraduate. You are much more independent. For instance, at Berkeley, there is no one course that every graduate student has to take. In addition the pace of research is very different than doing homework. Normally homework is due every week. But research problems generally take a few months or even a few years! You have to learn to set smaller milestones for yourself to work towards each day or week, and learn to keep perspective when you have a (or many!) bad research day(s).

Ken: One of your specialties is arithmetic geometry. What is that?

Bianca: Arithmetic geometry is the intersection of algebraic number theory and algebraic geometry. One aspect of number theory is the study of rational solutions (when all the variables are positive or negative fractions) to polynomial equations, say like $x^3 + 2y^3 = 5z^3$.

One way we figure out how to attack these problems is by considering the set of all solutions where x, y, and z are complex numbers (instead of just fractions), and thinking of this set of solutions as a geometric object. The idea is to use the geometric properties of this object to help you determine what methods are best suited for studying the arithmetic properties of the equations.

Some arithmetic properties you might want to know are: Is there a rational solution? If there are, are there infinitely many? If there are only finitely many, how many are there?

One important result in arithmetic geometry is called Faltings's theorem. Faltings's theorem implies that if you take any nonzero integers a, b, and c, and n is an integer larger than 3, then there are finitely many fractional solutions to $ax^n + by^n + cz^n = 0$. This is a very hard and famous result, but we could still ask for more. For instance, we could ask, if we fix n, say to be 17, and we take any a, b, and c that we want, is the number of solutions of $ax^{17} + by^{17} + cz^{17} = 0$ always less than 100? or always less than 1,000? or always less than 10 million? That question, and many others similar to it, is still open.

Ken: What are some of the most fundamental questions people are trying to answer in arithmetic geometry?

Bianca: Well, arithmetic geometry is a very large field, so I think that there are many fundamental questions. Let me just explain one of the fundamental questions in the particular subfield of arithmetic geometry that I study. As I mentioned before, one of the many areas of study is the set of fractional solutions to a polynomial equation. Before I just gave examples of polynomial equations in 3 variables, but really we'd like to consider equations in any number of variables with any degree we like. And, again as I mentioned before, one of the questions we're interested in is whether the equation has a fractional solution. Now, maybe that's not so hard for the equation $x^2 - 3 = 0$, but what if I gave you the equation

$$109x^{42} + 63y^{19}z^5 - 1111w^{41}t^3 - 79(y + w)^{103} + 1 = 0?$$

That doesn't look so easy right? We know the answer is not always "no" (consider $x - 1 = 0$) and not always "yes" ($x^2 - 3 = 0$). We also know that there are infinitely many polynomial equations, so I can't just go through the list and determine the answer for each one. So for a totally complete answer to this question, we would like a systematic way of determining, given a polynomial equation, whether the answer is "yes" or "no". In other words, we would like a set of steps that we could follow that would allow us to start with a polynomial equation, and at the end we would say "yes, this equation has fractional solutions" or "no, this equation does not have fractional solutions". A set of steps that does this is an example of an algorithm. At the moment, no one knows of an algorithm that solves this problem, i.e. that takes a polynomial equation and determines whether or not there is a solution.

OK, so maybe there hasn't been a mathematician yet who was creative enough, or hard-working enough to find the algorithm. But, actually, there's another possibility. It might be impossible for such an algorithm to exist! This would mean that no mathematician would ever be able to find an algorithm, no matter how creative or hard-working she is, because it's impossible! The question of whether or not this algorithm exists is called Hilbert's 10th problem. It is a very famous problem, and much work has been done on it. In fact one of the world's experts is Girls' Angle Advisory Board member, Bjorn Poonen.

Ken: Can you explain one of the results you proved to us?

To be continued…

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Who Won the 1989 Tour de France?

by Denis Serre

The Tour de France is the most popular cycling race in the world. Every year, it runs for three weeks in July. The race is split into about 21 day-long segments called "stages." You've probably heard of seven-time champion Lance Armstrong, but the first US cyclist to win the Tour de France was Greg LeMond. LeMond won three tours (in 1986, 1989, and 1990) and was very popular among the French audience. His victory in the 1989 Tour was amazing! After more than 3,200 kilometers, Greg won by a mere 8 seconds over Laurent Fignon (winner in 1983 and 1984). This was the closest margin in history.

Well, I just gave you the answer to the title question, so why did I ask it? Let me start by saying a few words about the actual and official running times. To rank the riders, one records the time they spend on each stage. The official records are in seconds. Compared to a 100-meter sprint where times are given in hundredths of a second, this may seem coarse. However, it makes sense because the stages last several hours and it is very unlikely that hundredths, or even tenths, of a second will separate racers. Of course, this means that the actual time a racer takes to cover a stage may be different than his official time. According to the official rules of the Tour de France, the official time is obtained by rounding the actual time *down* to the nearest second. Saying that a cyclist spent 1 h 23' 45" on a stage means that his actual time was greater than or equal to 1 h 23' 45" and less than 1 h 23' 46". After 20 stages, a rider might have been given an advantage of nearly 20 seconds, whereas his opponent might not have been advantaged at all. However, this is very unlikely, because it is reasonable to assume that all of the possible round-down quantities between 0 and 1 second are equally likely. Hence, 20 round-downs would tend to average out so that the total discrepancy between official and actual time is likely to be close to 10 seconds. (This phenomenon is known in probability as the **Law of Large Numbers**.) Nevertheless, we can't completely disregard the possibility of the cumulative round-downs flipping the ranks of two racers. If after 20 stages rider G is *officially* ahead of rider L by less than 20 seconds, there is a possibility that L *actually* rode faster than G!

So, back to the title question. While LeMond *officially* went faster than Fignon, since he won by a mere 8 seconds it is possible that Fignon *actually* was the faster cyclist. How likely is it that the official result was wrong?

Before we answer, the above description of how the racers are ranked must be appended with the following rule: When two riders finish a stage in the same peloton[1], it is declared that they spent equal time on that stage. This may seem arbitrary, but it is for good reason. In the case of a mass finish with more than 150 riders arriving together, the length of the peloton could be 30 meters or more, with the last rider in the group finishing a few seconds later than the first one. It would be dangerous if so many cyclists made a mad dash to finish first within this group, hence this rule.

This year's Tour de France kicks off on July 2!

Going back to the 1989 Tour, Greg and Laurent arrived in the same group 11 times. On these occasions, we are not concerned with the round-downs as far as our question goes because according to the above rule, they both rode these stages in equal time. Still, there remain 10 stages where they arrived in different pelotons, sometimes Greg ahead, sometimes Laurent. Each one could have been given an advantage of nearly 10 seconds. Now let us suppose that Greg got an advantage of g seconds and Laurent got an advantage of l seconds. If $g - l > 8$, then Greg's final win would have been the result of round-downs. This is possible but seems very unlikely. The question I want to address is this: How likely is it that the official result was due to round-downs?

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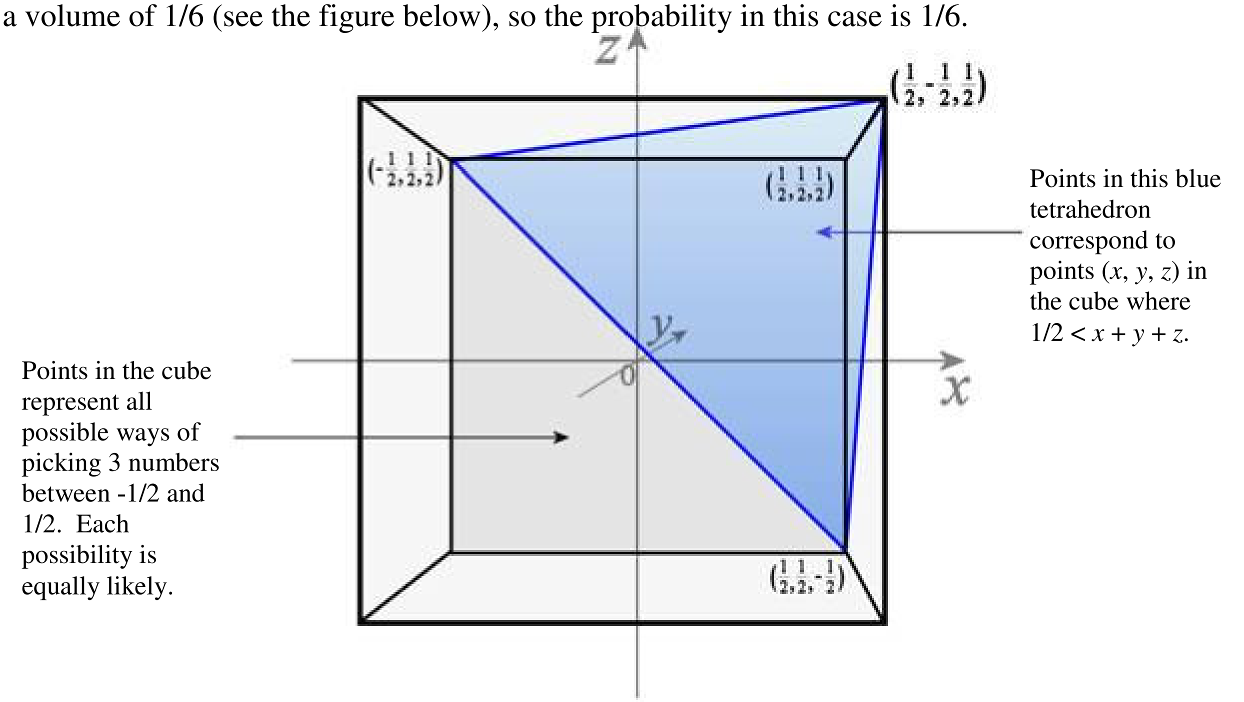
[1] A **peloton** is a group of cyclists riding together in a close formation like a flock of birds.

Transforming the Problem. If you already see how the problem amounts to computing the volume of a certain 20-dimensional polyhedron, you might enjoy trying to determine the exact answer yourself and skip to the end to check your answer. Otherwise, I will try to show how this problem relates to geometry by transforming it into an equivalent but more symmetric problem that generalizes in a way that permits us to see more easily how the geometry arises.

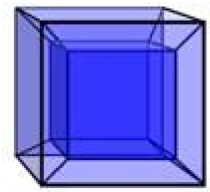
Our original problem involves 20 round-downs between 0 and 1 and determining the probability that the sum of the first 10 minus the sum of the last 10 exceeds 8. If we subtract 1/2 from each of the 20 round-downs, we can see that this is equivalent to determining the probability that the sum of the first 10 minus the sum of the last 10 still exceeds 8. Finally, if we negate the last 10 of these numbers (which are now between -1/2 and 1/2), we end up asking what the probability is that the sum of 20 random numbers between -1/2 and 1/2 adds up to a quantity greater than 8. In other words, we have a special case of the general problem: Given N numbers selected uniformly at random from between -1/2 and 1/2, what is the probability that their sum exceeds T? Our original question is the case $N = 20$ and $T = 8$.

Simplified Example. Before understanding the general situation, or even our specific one, let us work out the simpler case $N = 3$ and $T = 1/2$. Let the three random numbers be x, y, and z. We can think of them as the coordinates of a point (x, y, z) which belongs to the unit cube with vertices $(\pm 1/2, \pm 1/2, \pm 1/2)$. Because each point in this cube is equally likely to be chosen, the probability that a point falls inside some subset of this cube is equal to the volume of the subset divided by the volume of the cube. Since the volume of this cube is 1, the probability is just the volume of the subset. In our example, the event is defined by $1/2 < x + y + z$. As a subset of the cube, this event is the tetrahedron whose vertices are (1/2, 1/2, -1/2), (1/2, -1/2, 1/2), (-1/2, 1/2, 1/2), and (1/2, 1/2, 1/2). Notice that this tetrahedron is a right triangular pyramid with a volume of 1/6 (see the figure below), so the probability in this case is 1/6.



A cube is just one term in a sequence of analogous objects called **unit cubes of dimension** $\boldsymbol{N}$. The unit cube of dimension 1 is a segment of length 1. The unit cube of dimension 2 corresponds to a unit square. The unit cube of dimension 3 is illustrated above. Although it becomes harder to imagine because we live in a world with 3 spatial dimensions, one can continue this process to make the unit cube of dimension 4.

In fact, when you looked at the figure on the previous page, you saw a cube. Yet it is not really a 3-dimensional object because this magazine is printed on a sheet of paper, which is 2-dimensional. If we are able to represent a 3-dimensional object by means of a 2-dimensional figure, we can try to represent a 4-dimensional cube with a 3-dimensional construction. It will not be an actual 4-dimensional cube, but a projection of one to 3-dimensions just as a drawing of a cube is also a projection. It is in fact possible to leap another dimension and draw the projection of a 4-dimensional cube on a plane (see right).



Now, an N-dimensional cube contains a generalization of the tetrahedron, a body whose vertices have coordinates (±1/2, ... , ±1/2) with at most one minus sign. There are $N + 1$ vertices and the body is an example of an ***N*-simplex**[2]. Let's call this particular N-simplex S_N. Beginning with small dimensions, S_1 happens to be the same as the 1-dimensional cube: a line segment of unit length. Next, S_2 is an isosceles right triangle with legs of length 1. Thus the area of our 2-simplex is 1/2. The blue region in the figure on the previous page shows S_3, and so on. The fact that S_3 has volume 1/6 is a special case of the general formula for the volume of our N-simplex S_N:

$$\text{The volume of } S_N \text{ is equal to } \frac{1}{N!}.$$

Back to the 1989 Tour. The event we're considering is equivalent to the case $N = 20$ and $T = 8$:

$$x_1 + x_2 + \ldots + x_{19} + x_{20} > 8.$$

This is a subset of the 20-dimensional unit cube. It is a 20-dimensional polyhedron P whose vertices are the points (±1/2, ... , ±1/2) where at most two coordinates have a negative sign and the rest are positive. The probability that Laurent lost the Tour because of round-offs is the volume of P. Notice that the 20-dimensional unit cube still has volume 1. We give below the exact volume of P, but let us begin with an upper bound. Even though P is not a 20-simplex, it is contained in the 20-simplex whose vertices are the points $(x_1, \ldots, x_{20})$ with all coordinates equal to 1/2 except possibly one which is equal to -3/2. Let's call this 20-simplex S. Thus, the volume of P is less than that of S. This means that the volume of P is less than $\frac{2^{20}}{20!} \approx 4.31 \times 10^{-13} =$ 0.000000000000431. (Note that S is a scaled version of S_{20}. It is scaled up by a factor of 2. Since it is a 20-dimensional object, its volume is 2^{20} times the volume of S_{20}.)

This is less than a percent of a percent of a percent of a percent of a percent of a percent! We now know that it is highly unlikely that Laurent *actually* rode faster than Greg. In conclusion, as I might say in my native French, *Greg LeMond peut dormir sur ses deux oreilles*[3]. He definitely deserves his victory.

Exercise: Show that each vertex of P is either a vertex of S or the midpoint of one of its edges.

Exercise: The set P is the complement of 20 unit simplices in the simplex S. Hence its volume is exactly equal to $\frac{2^{20} - 20}{20!}$, so the upper bound given above is quite accurate!

Edited by Jennifer Silva

[2] An N-simplex is an N-dimensional polyhedron with $N + 1$ vertices.
[3] Literally translated, "Greg LeMond may sleep on his two ears." The idiomatic expression is akin to the English phrase, "To sleep like a baby."

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna searches for more exact values of sine. She stumbles upon a neat approximation for π.

After working out this trigonometric formula from last time, I decided to use it to find out what the sine of $45^\circ/2^n$ is in general.

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

I worked out the sine and cosine of 45° last time too.

$$\sin 45^\circ = \frac{\sqrt{2}}{2} \qquad \cos 45^\circ = \frac{\sqrt{2}}{2}$$

I use the formula to find the sine of 22.5°.

$$\sin\frac{45^\circ}{2} = \sqrt{\frac{1-\sqrt{2}/2}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

...And the sine of 11.25°.

$$\sin\frac{45^\circ}{4} = \sqrt{\frac{1-\cos(45^\circ/2)}{2}}$$

Oh yeah, I have to know the cosine of 22.5° if I want to use the formula to find the sine of 11.25°. But I can find the cosine of 22.5° from the sine of 22.5° by using the fact that sine squared plus cosine squared is equal to one.

$$\cos\frac{45^\circ}{2} = \sqrt{1-\sin^2\frac{45^\circ}{2}} = \sqrt{1-\frac{2-\sqrt{2}}{4}} = \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$\sin\frac{45^\circ}{4} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

Gosh, these radicals are sure piling up!

$$\cos\frac{45^\circ}{4} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

In retrospect, it's not too surprising that there are all these radicals because this formula hides repeated applications of using the Pythagorean theorem to compute a length.

And here's the sine of 5.625°!

$$\sin\frac{45^\circ}{8} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$$

$$\Rightarrow \sin\frac{45^\circ}{2^n} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2}$$

$$\cos\frac{45^\circ}{2^n} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}{2}$$

← $n+2$ twos total in expressions

I think I'm beginning to see the general pattern. In each of these expressions, there are a total of $n + 2$ twos. That's pretty important- I better write it down.

Key:

- Anna's thoughts
- Anna's afterthoughts
- *Editor's comments*

I'm pretty sure I should be able to prove my guess at the general expression using induction.

Proof: Induction on n. $n=0$ is true. $(\sin 45^\circ = \sqrt{2}/2)$ — and $\cos 45^\circ = \frac{\sqrt{2}}{2}$ too

I can use the computation I did last time as the inductive base.

...oh yeah, I need both the sine and cosine of 45° for the inductive base!

Next, I assume the formula is true for n and then prove that the formula is also true at $n + 1$.

$$\sin \frac{45^\circ}{2^{n+1}} = \sqrt{\frac{1-\cos\frac{45^\circ}{2^n}}{2}} = \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{2}}{2}}$$

$$= \frac{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2} \quad ✓$$

$$\cos\frac{45^\circ}{2^{n+1}} = \sqrt{1-\sin\frac{45^\circ}{2^{n+1}}} = \sqrt{1-\frac{\sqrt{2-\sqrt{2+\cdots+\sqrt{2}}}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}{2}$$

great...it checked out just fine!

Hmmm...as n tends to infinity, the angle tends to zero degrees, so the cosine should tend to 1.

As $n \to \infty$, $\frac{45^\circ}{2^n} \to 0$ so $\cos\frac{45^\circ}{2^n} \to 1$

So $\frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}{2} \to 1$

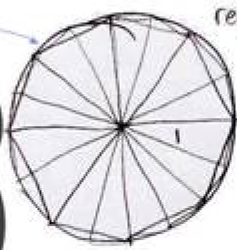
...that suggests this funny radical expression for the number two!

$$\Rightarrow 2 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$$

(What is $\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}$?)

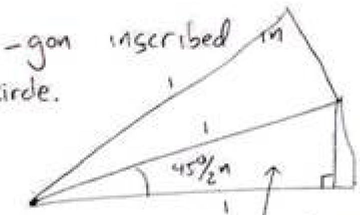
The funky expression for two makes me wonder what this evaluates to...I'll save this for later. Could it come out to be n?

Hey! I just thought of something. I can use the sines of these angles to compute the area of a regular polygon whose sides number a power of 2 and which is inscribed in a unit circle...and get a way to approximate the value of pi!



regular 2^{n+3}-gon inscribed in unit circle.

P_n



area $= A_n$.

I'll call the area of one triangular wedge of the polygon A_n.

As $n \to \infty$, area becomes close to area of unit circle, which is π.

To get the area of the polygon, I just multiply the area of each wedge by the number of wedges, which are the same in number as the number of sides.

$$\text{area } P_n = 2^{n+3} A_n$$

$$A_n = \tfrac{1}{2}(\text{base})(\text{height})$$

$$= \tfrac{1}{2}(1)\sin\frac{45^\circ}{2^n} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{4}$$

← $n+1$ twos in expression

As n grows, this formula should give better and better approximations for pi.

$$\text{area } P_n = 2^{n+1}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}$$

← $n+2$ twos in expression

Just out of curiosity, I'll make a table of the polygonal areas.

n	area P_n
0	2.8284271...
1	3.0614674...
2	3.1214451...
3	3.1365484...
4	3.1403311...

ABB 6.23.11

I used a calculator to help me compute these values. I wonder what the most efficient way to compute these values is. Also, I wonder how close to pi these numbers are supposed to be?

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Facebook and Graph Theory

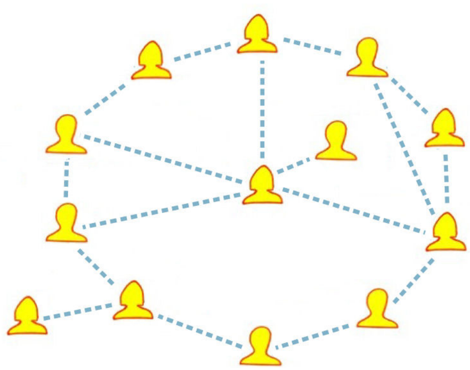
By Katherine Sanden

Logo Design by Hana Kitasei

When you log in to the popular social networking site Facebook, you will see a picture of the world dotted with people's heads. Some of the heads are connected to others with dashed lines. The image is accompanied by the words "Facebook helps you connect and share with the people in your life."[1] Below I've sketched this picture to emphasize the structure of the connections.

I love this visual representation of the online social network. The little yellow icons stand for different profiles, and the dotted lines between them depict friendships.

It hints toward the mathematical concepts at work behind the scenes. On a simplified level (we're ignoring fan pages and groups right now), Facebook consists of objects called "profiles," that can be connected to each other by links called "friendships." Somewhere at Facebook headquarters, there are engineers who figure out how to efficiently keep track of all this information.



With all the users around the world, you can imagine that the friendships alone comprise a huge amount of information to keep track of. Here's where the math comes in. Let's try to get an idea of how many friendships exist on Facebook. Imagine the picture above were the entirety of Facebook. There are 13 yellow heads, so we'll imagine Facebook only has 13 users. Now we can go ahead and count the number of dotted lines. I count 17. Easy enough. But how could we possibly estimate this number for the millions of users in real life? Without access to all the data, it seems impossible, and even if we had all the data, we surely would not want to draw a picture like the one above and start counting.

Drawing such a picture efficiently is an interesting problem in itself (see the "Fun Challenge" box on the next page), but is there a quicker way to count friendships without even drawing the picture?

Go back to the original picture with only 13 yellow icons. When you counted the dotted lines, how did you do it? The first time I did it, I counted each line individually, using a pencil to cross out lines I had already counted. Then, to check myself, I counted a different way. I went to each yellow icon and counted the number of lines coming out of it. So for the icon at the top, I wrote down "3". For the icon directly below that, I wrote "5", and for the icon on the bottom left, I wrote "1", since there is only one dotted line coming out of that yellow icon. After writing a number next to each yellow icon, I added up all the numbers and got 34 – exactly 2 times 17.

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[1] Source: www.facebook.com

Why 2 times 17? Well, notice that every dotted line connects to two yellow icons. So I am counting each dotted line twice – once when I write the number of lines coming out of the first yellow icon to which it connects, and again when I do the same thing for the other yellow icon.

So, by visiting each yellow icon, counting the number of lines coming out of it, adding each of my totals together, and dividing by two, I obtain the total number of dotted lines, i.e. friendships. Check this for yourself. Make up a different picture and check that the same technique works.

This fact enables us to estimate the number of friendships on Facebook without drawing the whole picture. We can just add up the number of friends that each profile has, and divide by two. But wait – we don't actually have a complete list of profiles and friends in real life. What we do have are the following two facts, from the Facebook Statistics Page[2]: there are more than 500 million active users, and the average user has 130 friends. (For our calculations, let's assume there are roughly 500 million active users.)

Fun challenge

Suppose you have a computer that is willing to draw a picture of all the profiles and friend connections for you. You have a huge list of profiles, and with each profile, a list of friends. How would you tell the computer to draw it? No knowledge of computer programming is necessary here. Just speak in terms of general steps. For instance, one option is to have the computer start with the first profile on your list, call it profile *A*, and draw a yellow icon. Now have it look for the first friend on *A*'s list. Call this profile *B*. Have the computer draw a dotted line with another yellow icon at the end of it, to represent profile B as well as the friendship between *B* and *A*. Or should the computer focus on drawing all the profiles first, before drawing any friendships? Or would you rather use another strategy? Compare your strategies with your friends! Different strategies may have different advantages.

It turns out this information is enough. We have just discovered that the sum of the number of each user's friends, divided by two, yields the total number of friendships. And what is an average? It is the sum of all values (in this case, the sum of the number of each user's friends), divided by the number of values (in this case, the total number of users).

$$130 = \text{Average number of friends per user} = \frac{\text{sum of the number of each user's friends}}{\text{total number of users}}$$

So let's go ahead and multiply this 130 by the number of users, 500 million:

$$\begin{aligned} 130 \times 500{,}000{,}000 &= \frac{\text{sum of the number of each user's friends}}{\text{total number of users}} \times (\text{total number of users}) \\ &= \text{sum of the number of each users' friends} \\ &= 65{,}000{,}000{,}000. \end{aligned}$$

Voila! Exactly what we wanted!

Now, as we saw before, we need to divide by two: 65,000,000,000 / 2 = 32,500,000,000.

So, there are roughly 32.5 billion friendships on Facebook. Far more friendships than users! There is actually a branch of mathematics dedicated to the study of problems like this. It's called Graph Theory. You might think of functions and *x*'s and *y*'s when you see the word "graph." But in this case, a "graph" refers to a different concept: a collection of vertices (in our case, profiles) and edges that connect those vertices to each other (in our case, friendships). It's a useful way to model any networking situation, from social networks, to transportation networks, to technological networks.

Can you use Graph Theory to model any other networks in your life? It's a powerful tool and we have just scratched the surface!

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[2] Source: www.facebook.com/press/info.php?statistics

One Number Fun?

by 1729

…and not just any number…I mean just a single, positive counting number (a. k. a. **positive integer**), like 17.

Oh, I know such numbers are used all the time to describe things, like the 82 penguins at the Aquarium, but that's a number within a context.

I'm talking about the number, all by itself. Just the number and nothing else. Is there anything interesting that can be done with a plain and simple counting number?

Yes! There are plenty of things that can be done, like play Find the Factors.

Consider 24, for instance. Can you find all eight of its factors?

The more you play this game, the better you'll get at it. You'll start noticing patterns, such as that numbers that end in 0 or 5 have 5 as a factor.

Now, I'm a number factoring veteran, so when I see a number like 555,555,555, I know almost instantly that 9 is a factor. I don't even have to perform the long division to know this, but if you don't believe me, go ahead and divide to see for yourself!

I can also see pretty quickly that the number 6,999,993 is divisible by 7.

If you play this game as long as I have, you'll build your own bag of factoring tricks. Factoring numbers is a game you can play wherever you are and whenever you want to. And, trust me, you'll never run out of numbers to factor. Hmmm, 1,419,857? That's a number I haven't factored— until just now. Can you find its 6 factors?

There's another game I like to play with a single number: *Prime* factorization!

You know what prime numbers are…yes, those numbers that have *exactly* two factors. Well, I like breaking numbers down all the way until they are expressed as a product of prime numbers. For example:

$$24 = 2 \times 2 \times 2 \times 3$$
$$63 = 3 \times 3 \times 7$$
$$128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

What's amazing is that each number has its own unique collection of prime numbers in its prime factorization. If you have two bags of prime numbers, the products of their prime numbers will be the same if and only if the two bags contain the same number of each prime number. If the bag doesn't have exactly seven 2s and nothing else, the product will not be 128. The prime factorization of a number is like a number's secret identity. Very large numbers keep their secrets well.

After playing prime factorization for a while, I started to wonder…if I've found the prime factorization of some number, say N, can I predict what the prime factorization of $N + 1$ will be without actually factoring out $N + 1$? After all, it's easy to add 1 to a number, so maybe the prime factorization of $N + 1$ is related to the prime factorization of N.

To investigate, I constructed a table of prime factorizations:

n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
prime factorization	2	3	2^2	5	$2\cdot3$	7	2^3	3^2	$2\cdot5$	11	$2^2\cdot3$	13	$2\cdot7$	$3\cdot5$	2^4	17	$2\cdot3^2$

I thought and thought. How bizarre the behavior! Is there any rhyme or reason to how the prime factorization changes from one number to the next? What patterns do you see? Any ideas? If you notice anything, feel free to write: girlsangle@gmail.com. For more on prime factorizations, see the Summer Fun problem set on page 24.

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

One of the most persistent errors that I encounter in my teaching is what I call "misuse of the equals sign." The equals sign should only be used when two things are actually equal, but many students use this special symbol as a loose way of indicating the next step in the solution of an exercise. Such usage often leads to "equations" that aren't true, such as:

$$4 + 1 = 5 \times 3 = 15,$$

which evidently is intended to mean that you want to add 1 to 4 and then multiply the result by 3. Here the equals sign on the right is used appropriately, but the equals sign on the left does not fit, since $4 + 1 \neq 5 \times 3$. (The left-hand side is 5, of course, while the right-hand side is 15.) Instead, if you want to clearly express the steps of adding 1 to 4 and then multiplying by 3, your equations should look like:

$$(4 + 1) \times 3 = 5 \times 3 = 15.$$

Or, alternatively, you could split the two equations apart to read:

$$4 + 1 = 5,$$

then

$$5 \times 3 = 15.$$

"Equals" is an example of a relation on any set of objects or numbers. Under the equals relation, any object is related only to itself. We take the equation $a = b$ to mean that a and b are really just different names for the same underlying object or number. (A number is a specific type of mathematical object.) "Equals" has three special properties. First, **reflexivity**: every object is equal to itself, that is, $a = a$ is always true regardless of what a is. Second, **symmetry**: if one object is equal to another, then the latter object is also equal to the former—in other words, any time that $a = b$ is true for some a and b, it is also true that $b = a$. Third, **transitivity**: if one object is equal to a second object, and the second object is equal to a third, then the first object is also equal to the third object—or, with mathematical symbols, whenever $a = b$ and $b = c$ for some a, b, and c, it is also true that $a = c$. We use these properties of equals intuitively all of the time. For example, $5 = 5$ is true, of course, and since $4 + 1 = 5$ we also know that $5 = 4 + 1$. Moreover, since $7 - 2 = 5$ as well, we can see that $7 - 2 = 4 + 1$ by transitivity.

It is very important, however, to use the equals sign only when two objects or quantities are actually equal. It's perfectly fine to work with an expression one step at a time, but if you connect the steps with equals signs, be sure that each equals sign is valid. Otherwise, break your steps down into a series of separate equations, each of which has a valid equals sign. For instance, if you are asked to simplify a complicated expression, such as

$$\frac{2^3 \cdot 3^{-4}}{4^2},$$

you can either start with the *entire* expression and carry each piece of it all of the way through your calculations, or you can break it down into smaller pieces and then put them all together in the end.

Let us start with the latter approach. First we have

$$2^3 = 8,$$

then

$$3^{-4} = \frac{1}{3^4}.$$

But since

$$3^4 = 81,$$

this is the same as

$$3^{-4} = \frac{1}{81}.$$

Hence

$$2^3 \cdot 3^{-4} = 8 \cdot \frac{1}{81} = \frac{8}{81}.$$

Now, since

$$4^2 = 16,$$

putting it all together we get

$$\frac{2^3 \cdot 3^{-4}}{4^2} = \frac{\frac{8}{81}}{16} = \frac{8}{81 \cdot 16} = \frac{8}{81 \cdot 2 \cdot 8} = \frac{1}{81 \cdot 2} = \frac{1}{162}.$$

Alternatively, we could have started with the whole expression and worked out our calculations from there, recalling that a negative exponent in the numerator means to bring the factor down to the denominator:

$$\frac{2^3 \cdot 3^{-4}}{4^2} = \frac{2^3}{3^4 \cdot 4^2} = \frac{8}{81 \cdot 16} = \frac{8}{81 \cdot 2 \cdot 8} = \frac{1}{81 \cdot 2} = \frac{1}{162}.$$

On the other hand, the following chain of reasoning contains equal signs that are NOT valid:

$$2^3 = 8 \cdot 3^{-4} = 8 \cdot \frac{1}{3^4} = \frac{\frac{8}{81}}{4^2} = \frac{8}{81 \cdot 16} = \frac{8}{81 \cdot 2 \cdot 8} = \frac{1}{81 \cdot 2} = \frac{1}{162}.$$

Did you spot the mistakes? The first and third equals signs in the chain are incorrect!

Let us try another example using the "break it apart" method favored by my students. Suppose we want to simplify the following expression:

$$(3 + 5)^2 \cdot (4 - 1) \div (34 - 32) + 1.$$

Resist the temptation to string all of your calculations together with equals signs, unless you are willing to recopy the entire expression at each step. Do not begin your reasoning with something like:

$$3 + 5 = 8^2 = 64 \cdot (4 - 1) = \ldots$$

because the first two equals signs are already invalid! Instead, break the expression down into several separate equations:

$$3 + 5 = 8,$$

then

$$8^2 = 64.$$

Now

$$4 - 1 = 3,$$

so

$$64 \cdot 3 = 192.$$

But

$$34 - 32 = 2,$$

so

$$192 \div 2 = 96.$$

Finally,

$$96 + 1 = 97.$$

This example is also a good review of the *order of operations*: parentheses first, then exponents, then multiplication and division, and finally addition and subtraction.

The notion of "equals" can be generalized to that of an **equivalence relation**, which is any relation that satisfies reflexivity, symmetry, and transitivity. We usually introduce a new symbol, such as ≡, ~, or ≅ to denote that two objects are **equivalent** (which has a less stringent definition than equals). For instance, we could put a relation on the integers so that all even integers are related to all other even integers, and all odd integers are related to all other odd integers. In this case, we could say $4 \equiv 22$ and $7 \equiv -31$, as well as infinitely many further equivalences. We could check that this is indeed an equivalence relation by verifying the three special properties. But, just as with the equals sign, we must be very careful to use our chosen symbol only when it will give us a true statement. Now it is true that

$$4 + 1 \equiv 5 \times 3,$$

because $4 + 1 = 5$ and $5 \times 3 = 15$, and both 5 and 15 are odd. But it is *not* true that

$$2 \cdot 4 \equiv 8 - 1 \equiv 7,$$

since the first equivalence symbol fails to be true: $2 \cdot 4$ is even whereas $8 - 1$ is odd.

For practice, try checking whether the equals signs are valid in the following "equations." Cross out the equals signs that aren't true. The answers can be found on page 26.

1. $12 + 2 = 14 \times 5 = 70$
2. $(17 - 13)^2 = 4^2 = 16$
3. $9 \div 3 = 3 + 2 = 5 - 1 = 4$
4. $17 - 13 = 4^2 = 16$
5. $9 \div 3 + 2 - 1 = 3 + 2 - 1 = 5 - 1 = 4$
6. $(12 + 2) \times 5 = 14 \times 5 = 70$

COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

OWNING IT

A nice aspect of math is that simple facts can generate complicated-looking results. This same aspect means that when tackling a hard-looking problem, one insight – one "aha!" moment – can transform a problem from seemingly impossible to obvious. Let's try doing this in both directions. For *'Impossible' to 'Obvious'* start here. See below for how to transform something from *'Obvious' to 'Impossible.'*

'IMPOSSIBLE' TO 'OBVIOUS'

Chocolate Bar

Let's consider the following situation. Your fairy godmother gives you the world's most delicious candy bar for you to share with your worst enemy (she wants you to become friends). It's a rectangle, 4¼ inches long and 2 inches wide. She also gives you a magic blade exactly 3 inches long to use as a knife and tells you to cut the bar. The blade is magical because it can be used to cut absolutely perfect straight line segments up to 3 inches in length. You must divide the bar into two portions. Your fairy godmother will let your enemy choose the first portion. The other portion will be yours.

You know your enemy will choose the bigger portion if you cut the bar into unequal pieces, so the best you can hope for is half. Of course, you could just eyeball the bar and hope you can cut it in half. But then your eagle-eyed enemy will choose the bigger piece and you will not only miss out on some of the treat, but worse – much worse – you'll have to endure the horrible gloating for which your enemy is known.

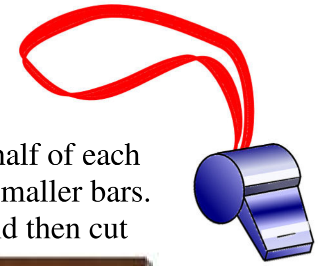
The problem is how to divide the bar exactly in half with only the blade as your tool. What do you do? Does it seem obvious, impossible, or something in between?

Not knowing how to solve a problem does not mean that you cannot get to work. One of math's many great features is that it is not dangerous. Nothing explodes if you make a false move.

If a problem seems too hard, first complain, and then make it easier! Complaints often suggest simpler versions of a problem, and tackling these can start you on the road to the answer. For instance,

"If only the rectangle were made of paper, I could fold it in half and cut along the crease."
"If only I had a ruler, I could measure where to cut the bar into two identical pieces."
"If only the blade were longer, I could cut between two diagonal corners."

Consider each of these simpler situations and see if there is anything more to say. The last complaint could be expressed as "If only the bar were smaller, I could cut it in half between two diagonal corners."



So now we know that we can cut a smaller bar in half. Does that help?

Well, can we think of the bar as two smaller bars stuck together? Yes! One half of each of the two smaller bars will combine to be half of the bigger bar made from the two smaller bars. We can cut from one corner to a reachable point on the other side of the rectangle, and then cut from that point to the other corner that shares the side with the first corner. Since the two smaller bars exist only in your imagination, it doesn't matter which point on the rectangle's side you pretend is on their border, nor whether the two halves of the smaller bars are connected. The big piece will be exactly half of the rectangle.

'OBVIOUS' TO 'IMPOSSIBLE'

The way you foiled your enemy is related to the well-known formula for the area of a triangle. You can use it to create new tricks and traps.

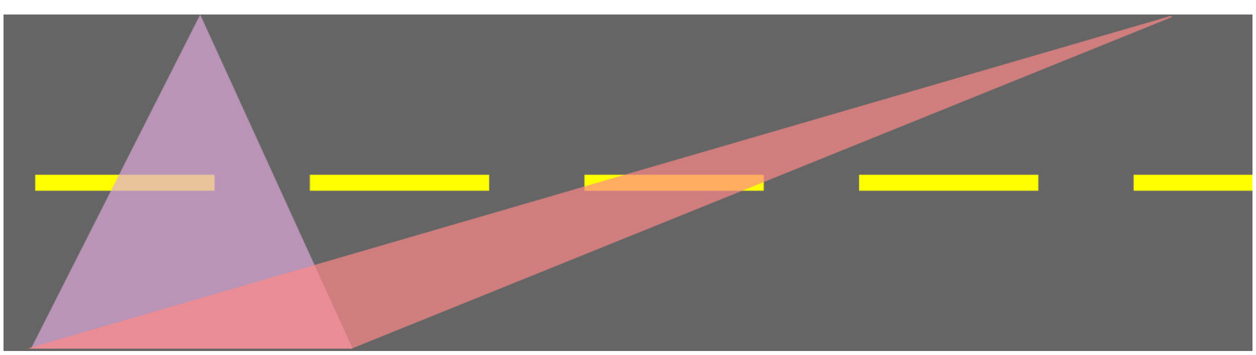
First we review: The area of a triangle is one-half the product of the lengths of its base and height. That is,

$$\text{Area of a triangle} = \frac{1}{2} \times \text{Base} \times \text{Height},$$

where the "Base" is the length of any side of the triangle and the "Height" is the length of the perpendicular line segment from the chosen base to the triangle's other vertex (that is, the vertex that isn't on the base that you chose).

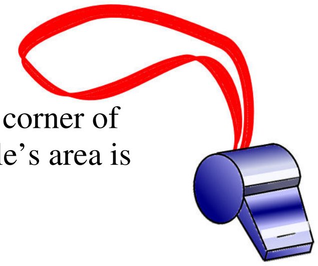
Well, so what? What does thinking about this formula get you?

First, it can tell you something surprising about triangles with very different shapes. Suppose you pick two points on one side of a long, straight road, and use those as endpoints for the base of a triangle. You and your enemy are going to see who can make the bigger triangle by picking the third vertex on the opposite side of the road. Your enemy decides she will pick a point a quarter of a mile away for her third vertex, while you nonchalantly stroll across the road and put your vertex right across from the midpoint between your two other vertices. Who wins? Your two triangles share the base. And the height is the same – it's just the width of the road. So your triangles have exactly the same area. You haven't won, but at least you've been able to walk a lot less than your enemy.



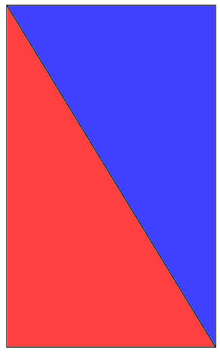
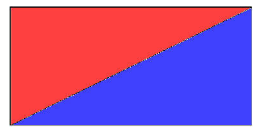
How does this relate to cutting up candy bars?

Well, we know that the rectangle with side lengths *Base* and *Height* has twice the area of the triangle with the same base and height. Consider a triangle that shares a side with a rectangle. Call that side the *Base*. Now complete the triangle by picking any point on the

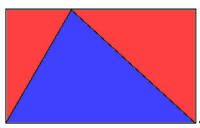
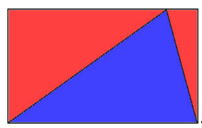
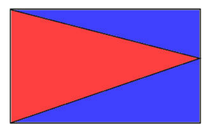


opposite side of the rectangle from the shared side. The point you picked is the third corner of the triangle. What is the height of the triangle you made? It is *Height*. So the triangle's area is half that of the rectangle's, no matter what point you picked.

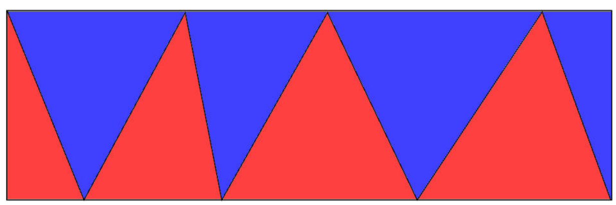
Thus, while it may be obvious just from looking that

 and 

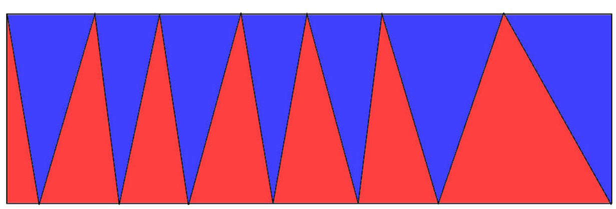
are half red, now we also know that

, , and 

are half red. And, after sticking a few of these together, we likewise know that



and



are half red. In other words, any rectangle that has zigzags across it is split exactly in half as long as the zigzags start in one corner, cross back and forth to alternate sides of the rectangle, and end in another corner of the rectangle.

Remember that you started off with a candy bar that was 4¼ inches long and 2 inches wide. You see now that even if your knife is just a little bit longer than 2 inches, you can divide the candy bar into two equal portions.

These "aha!" moments make it enjoyable to work on non-routine problems. Also, spending time thinking about obvious facts can lead to interesting complications, not to mention allow you to create tricks and traps for enemies.

Summer Fun!

The best way to learn math is to do math!

Here are the 2011 Summer Fun problem sets.

We invite Girls' Angle members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and put your solutions in the Bulletin!


The goal may be the lake, but who knows what wonders you may discover on the way there?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems before seeing solutions.

By the way, some of these problems are going to be very unlike those you will find at school. Usually, problems that you get at school are readily solvable. However, some of these problems were designed by the author to require more time and effort to solve.

If you are used to solving problems quickly, it can feel frustrating at first to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain or rock climbing. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So there's a meta-problem for those of you who feel frustrated at times doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Party Time Puzzles

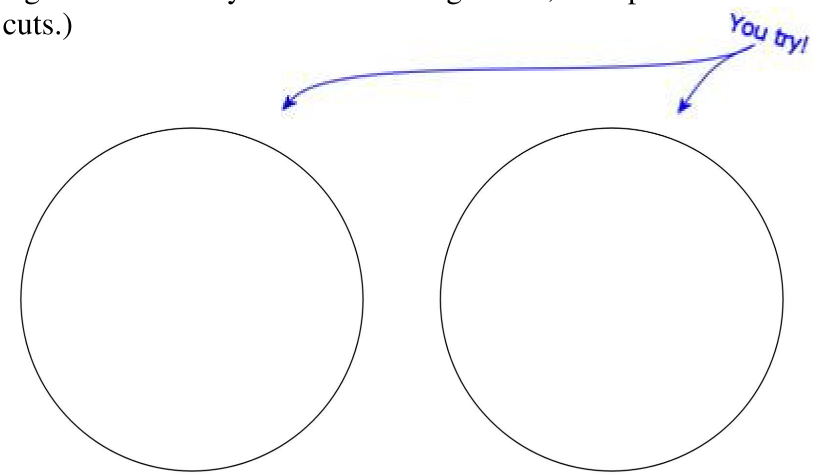
by Samantha Markowitz

It's party time! You're the host and need some help with recipes and entertainment.

1. First you must prepare the food. You want to serve pizza as the main dish, but you can only make 3 straight cuts to divide the pizza. How can you do that so that you can serve the maximum number of people? (Assume one slice per person. Also, we don't require that each slice be the same size.) How many people would that serve? Now, what if you can make 4 straight cuts; what is the maximum number of people that you can serve? Do you notice a pattern in how the cuts on the pizza are laid out to produce the maximum number of servings? Can you generalize this to any number of straight cuts? (Often, to generalize to any number of straight cuts, it helps to think systematically about 0, 1, 2, 3, 4, etc. cuts.)

Pizza courtesy of Emma's Pizza

For example, this arrangement of three cuts would serve six people. Can you create an arrangement that can serve more?



3 cuts

4 cuts

2. Pizza makes everyone thirsty, so you need to make sure there are some delicious beverages. You want to serve root beer floats at your party. Each float needs **exactly** 6 liquid ounces of root beer to balance out the scoop of vanilla ice cream. Too much root beer and the float will be too liquidy. Too little and the float will be too thick. If you have an endless supply of root beer, but only a 5 ounce cup and a 7 ounce cup, how can you measure out exactly 6 ounces of root beer? What are all the possible amounts of root beer that you can measure out?

3. Once everyone is fed, you will need to spice up the party with some entertainment. You have 100 guests and decide to play a game. The rules of the game are as follows:

Each guest must pick a number between 0 and 1,000 inclusive, and the person whose number is closest to ¼ of the arithmetic mean (the average) wins. Assuming that every person in the room is rational, what number should you choose to maximize your chance of winning?

Now, what if you have 200 guests; what number should you choose? What if you can pick a number between 0 and 1,000,000 inclusive; what number should you choose?

Summer Fun!

MIT Triathlon

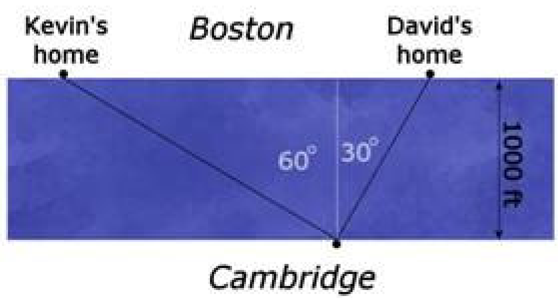
by Rachel Fraunhoffer

courtesy of the US Geological Society

The Charles River separates MIT from Boston.

At the end of the semester, MIT students start to feel a little crazy. There are just too many calculations and formulas running through their heads with the approach of final exams! Two MIT students, Kevin and David, live in Boston, though MIT is in Cambridge. These two cities are separated by the Charles River which is about 1000 feet wide. There are 5280 feet in a mile.

1. Approximately what fraction of a mile do they walk across the bridge everyday to go to school?



One hot summer day they decided to swim across the river instead of walk so that they could cool down. The guys stop just before the river and debate who will make it across first. Kevin is a much faster swimmer, but David's house appears closer. Kevin lives to the left 60 degrees, while David lives to the right 30 degrees.

2. Is Kevin's speed (4 mph) fast enough to beat David (2 mph), even though David's house is closer? How much faster does the winner get to his house (in minutes)?

3. Where along the river would they have to start their race so that they reach their homes at the same time?

Once the boys got home, David realized that he forgot a book at school. He decided to call Kevin to challenge him to another race, except this time by bike. They meet at the corner of the Massachusetts Avenue bridge and take off!

4. How long does it take for David to go cross the 1000 foot bridge if he's going 20 mph?

5. How fast is Kevin going if he makes it there 30 seconds after David?

Caution!

Please note that it is illegal to swim in the Charles River without a special permit, even on days when the water is safe to swim in, such as on those days when the Charles River Swimming Club holds its annual one mile race each June. According to their website, in 2009, the Environmental Protection Agency rated the Charles River a B+.

On the way back home, for the deciding race, the boys run back. David runs 8 mph while Kevin runs 8.5 mph.

6. If David gets a head start by five seconds, is this enough to enable him to win?

Who is the ultimate athlete?

Summer Fun!

The Road to the Stanley Cup

by Lightning Factorial

Inspired by the their record breaking three Game 7 victories and the awesome performance of goalie Tim Thomas, here's a Summer Fun problem set dedicated to the 2011 Stanley Cup Championship Boston Bruins.

During the playoffs, teams pair off into best of seven series to see which team gets to move on. Whichever team is first to win four games moves on. In a "sweep", one team wins the first four games of the series and only four games are played. But if both teams exchange wins and losses, it's possible for the teams to end up in a winner-take-all game seven.

Let's think about an N-game series. We don't want any ties, so let N be an odd number.

1. How many different sequences of wins and losses can the team that wins the series experience in an N-game series? For example, in a 3-game series, the winning team could have recorded 2 wins in a row (WW), a win followed by a loss followed by a win (WLW), or a loss followed by two wins (LWW) for a total of 3 different paths to victory. Complete the following table.

N	**1**	**3**	**5**	**7**	**9**		**N**
Number of ways to win an N-game series	1	3				. . .	

2. Suppose the two teams are evenly matched so that each team has a 50% chance of winning each game played. What is the probability that one team sweeps? What is the probability that there is a decisive Nth game?

Suppose that a team is engaged in an N-game series and has a probability of p of winning each game. In a 1-game series, the team has a probability of p of winning the series. In a 3-game series, we listed the 3 ways to win the series. WW occurs with probability p^2 and both WLW and LWW occur with probability $p^2(1 - p)$. So, the probability of winning the 3-game series is

$$p^2 + 2p^2(1 - p) = p^2(3 - 2p).$$

3. In the setup just described, complete the following table.

N	**1**	**3**	**5**	**7**	**9**		**N**
Probability of winning an N-game series	p	$p^2(3 - 2p)$				. . .	

4. Use your answer to problem 3 to deduce that

$$2^m = \sum_{k=0}^{m} \binom{m+k}{k} \frac{1}{2^k}.$$

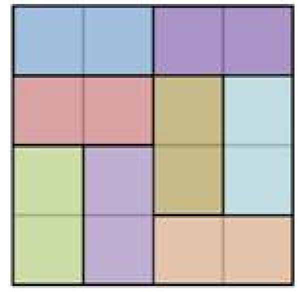
Summer Fun!

The Impossible Problems

by Kate Rudolph

Some of these problems are possible, and some of them are not! If the problem is possible, give the answer. If the problem is impossible, PROVE that there is no correct answer.

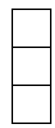
Part 1: Tiling



To tile a board, you must cover it with dominoes (or 1 by 3 tiles) with no overlaps so that the entire board is covered and no tiles are hanging off the board. Here is a tiling of a 4 by 4 board with multicolored dominoes.

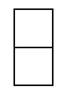
Can you tile a…

- 9 by 9 board with dominoes?
- 10 by 10 board with 1 corner removed with 1 by 3 tiles?
- 8 by 8 board with 1 corner removed with 1 by 3 tiles?
- 10 by 10 board with two opposite corners removed with dominoes?

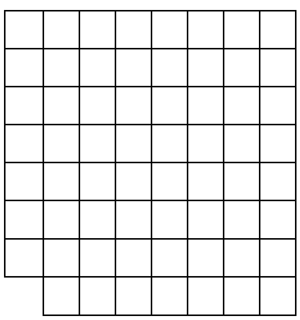


1 by 3 tile

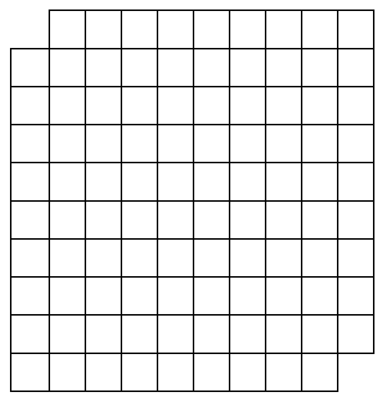
domino



8 by 8 board with 1 corner removed

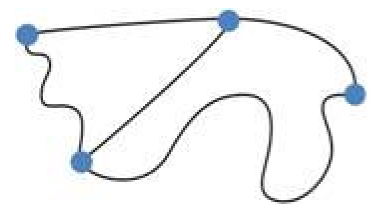


10 by 10 board with two opposite corners removed



What have you learned about tiling boards with dominoes and 1 by 3 tiles?

Part 2: Doodling



We're going to learn a new way to doodle. We start with some dots, and connect some of the dots with lines. Wherever lines cross there must be a dot. In between the lines are spaces. The doodle at right has 4 dots, 5 lines, and 2 spaces.

Can you draw a doodle with…

- 9 dots, 9 lines, and 1 space?
- 7 dots, 7 lines, and 2 spaces?
- 8 dots, 7 lines, and no spaces?
- 2 dots, 6 lines, and 4 spaces?

What have you learned about the numbers of dots, lines, and spaces a doodle can have?

Prime Factorizations

by 1729

All numbers in this Summer Fun problem set are nonnegative integers.

1. What is the prime factorization of the following numbers?

16 25 360 400 2011

2. What is the prime factorization of 10 factorial (10!)? How about 20! (that is, 20 factorial)?

3. What is the largest power of 10 that divides 100! (that is, 100 factorial)?

In the following problems, the prime factorization of n is $p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$ and the prime factorization of m is $p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_k^{m_k}$. (It may be that some n_i and some m_i are zero.)

4. How many divisors does n have?

5. If m is a perfect cube, what can you say about the exponents m_i?

6. What is the prime factorization of the product nm?

7. Give a condition on the exponents n_i and m_i that tell when n is divisible by m.

8. What is the prime factorization of the greatest common factor of n and m?

9. What is the prime factorization of the least common multiple of n and m?

10. Can you show that consecutive integers share no common prime factor?

11. Which number between 1 and 1,000 (inclusive) has the largest number of factors?

12. What is the smallest number that has all the following properties?
    It has exactly 2 different factors which are prime numbers.
    If you divide by the smaller prime factor, the result is a perfect square.
    If you divide by the larger prime factor, the result is a perfect cube.

A number is said to be **square free** if none of its factors are perfect squares except for 1.

13. How many square free numbers have no prime factor greater than 20?

14. How many square free numbers are there less than or equal to 2011?

15. Let S be the set of numbers that have prime factorizations of the form $2^x 3^y$, where both x and y are less than 10. What is the sum of the numbers in S?

Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

<u>Session 8 – Meet 11 – April 28, 2011</u>

Mentors: Rediet Abebe, Jennifer Balakrishnan, Keren Gu,
Samantha Hagerman, Kate Rudolph, Bianca Viray

Special Guest: Jennifer Che, tiny urban kitchen

Jennifer demonstrated the power of exponential growth via the Chinese art of pulling noodles. Beginning with a wad of dough, one long noodle is formed by repeatedly stretching the dough and doubling it over. In seconds, the noodle grew to 2 feet, 4 feet, 8 feet… Within a minute, the noodle, all doubled up several times, was 512 feet long! Slicing off the ends produced 256 strands of fine noodles, each about 20 inches long, ready to be cooked!

<u>Session 8 – Meet 12 – May 5, 2011</u>

Mentors: Jennifer Balakrishnan, Keren Gu, Samantha Hagerman, Ryan Heffrin,
Kate Rudolph, Charmaine Sia, Liz Simon, Bianca Viray

We hosted our traditional end-of-session Treasure Hunt. This time, the girls solved for the combinations with 7 minutes to spare.

Good job, girls!

Girls' Angle Support Network member Jane Kostick also brought in the dollhouse that was designed by the dollhouse design team. Seeing the blueprints actually realized made the dream a reality.

Furniture designed by the Girls' Angle dollhouse designers and built by Girls' Angle Support Network member Jane Kostick.

Calendar

Session 8: (all dates in 2011)

January	27	Start of eighth session!
February	3	
	10	
	17	Felice Frankel, photographer and scientist
	24	No meet
March	3	
	10	
	17	
	24	No meet
	31	Susan Barry, Mount Holyoke
April	7	Jane Kostick, woodworker
	14	
	21	No meet
	28	Jennifer Che, tiny urban kitchen
May	5	

Session 9: (all dates in 2011)

September	8	Start of the ninth session!
	15	
	22	
	29	Start of Rosh Hashanah – No meet
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	
	8	

Special Announcement: Girls' Angle will be offering a new Math Contest Prep course starting this fall. Check our website www.girlsangle.org for details.

Here are answers to the *Errorbusters!* problems on page 15.

1. $12 + 2 \neq 14 \times 5 = 70$
2. $(17 - 13)^2 = 4^2 = 16$
3. $9 \div 3 \neq 3 + 2 \neq 5 - 1 = 4$
4. $17 - 13 \neq 4^2 = 16$
5. $9 \div 3 + 2 - 1 = 3 + 2 - 1 = 5 - 1 = 4$
6. $(12 + 2) \times 5 = 14 \times 5 = 70$

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club and receive a printed copy of the Girls' Angle Bulletin for the duration of the membership. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin and a 25% discount for any club meet attended. Remote members may email us math questions (although we won't do people's homework!).

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, Courant Instructor/PIRE fellow, NYU
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, Moore Instructor, MIT
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, Senior Research Scientist, Harvard Medical School.
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership
  - ☐ $108 for a one year Remote Membership
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls
Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________

Girls' Angle

A Math Club for Girls