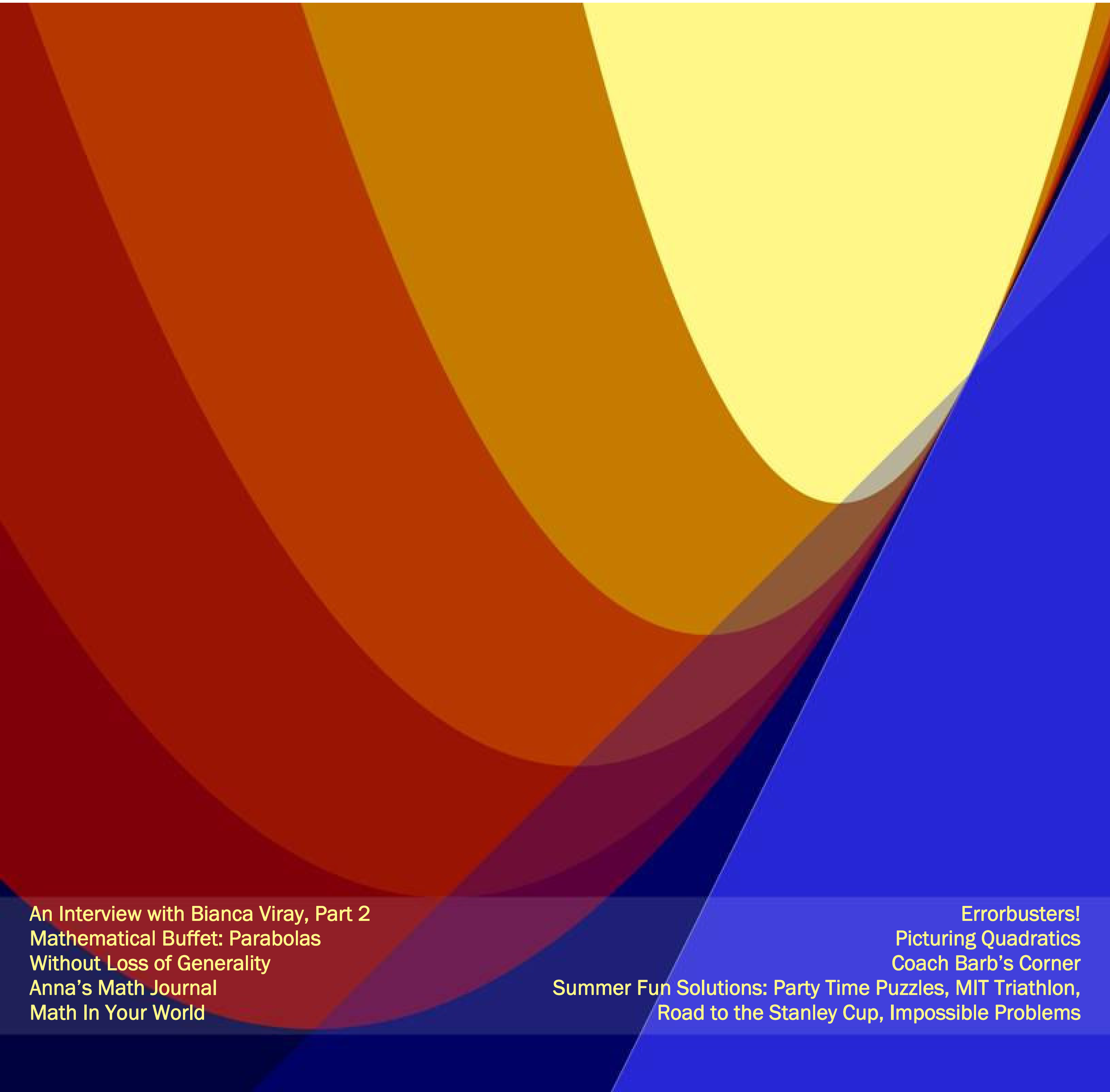
Girls' Angle Bulletin

August 2011 • Volume 4 • Number 6

To Foster and Nurture Girls' Interest in Mathematics



From the Director

The Ninth session of Girls' Angle begins September 8 and we're looking forward to seeing our current members and meeting new members.

We're also offering a new course that targets girls, primarily in high school, who enjoy solving math contest problems. When I first announced this, it caused some controversy because some people thought we were going to make some kind of math contest boot camp. Instead, our intention is to create a program that keeps the good things about math competitions while avoiding the bad. For more information, please see our website and follow the links to "courses" and related postings.

If you're a girl who thinks learning math is a good idea, come on and meet our fabulous mentors and join Girls' Angle!

- Ken Fan, Founder and Director

Girls' Angle Donors

Girls' Angle thanks the following for their generous contribution:

Individuals

Marta Bergamaschi
Doreen Kelly-Carney
Robert Carney
Merit Cudkowicz
Ingrid Daubechies
Anda Degeratu
Eleanor Duckworth
Julee Kim
Brian and Darlene Matthews
Toshia McCabe
Alison Miller
Mary O'Keefe
Beth O'Sullivan
Elissa Ozanne
Craig and Sally Savelle
Eugene Sorets
Patsy Wang-Iverson
Mary and Frank Zeven
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathematical Sciences Research Institute

Corporate Donors

Big George Ventures
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
Microsoft
Nature America, Inc.
Oracle
Science House
State Street

For Bulletin Sponsors, please visit our website.

Girls' Angle Bulletin
The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Kissing Parabolas* or *From the Vertex, Go Northeast to Find the Number Two*, by Ken Fan.

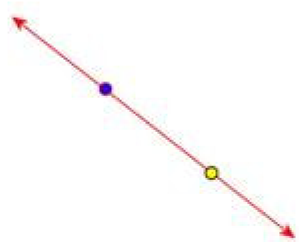
An Interview with Bianca Viray, Part 2

This is the second half of our interview with Dr. Bianca Viray, Tamarkin Assistant Professor at Brown University and an NSF postdoctoral fellow. She's also a member of the Girls' Angle Advisory Board.

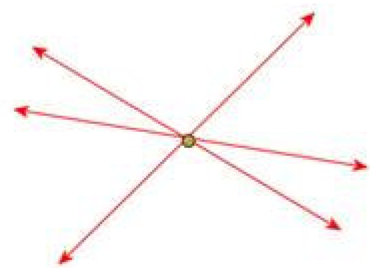
Ken: Can you explain one of the results you proved to us?

Bianca: One of the results I proved was about the reducibility of the Hilbert scheme of points. You can think of this problem as studying certain arrangements of points in *d*-dimensional space— except, you don't just study points, you study points with extra information.

The exact extra information is a little bit technical to describe, so let me describe something close. You may not be able to understand this description completely, and that's ok because I'm not trying to be precise here; I'm just trying to give something of the flavor of what I did. Let's start with considering 2 points in the plane. We're going to study pairs of points in the plane and a line going through the 2 points. Now, if you have two distinct points in the plane, there is a unique line that goes through both. So really, if you have distinct points, all you need to remember are the points, and you can determine the line. However, if your pair of points is the same point twice, then there are many possible lines.



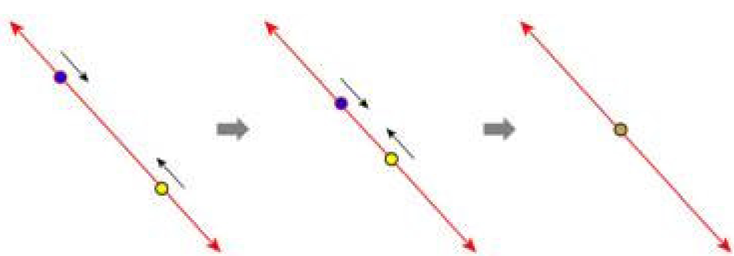
There is a unique line defined by the 2 distinct points.



There are many lines that go through one point.

Even though a set consisting of the same point twice and a line through them may not look like a set of 2 distinct points, it's not too far off. We can see this set as a limit of distinct points, as shown in the picture below.

Now, you could play a similar game with any number of points. You can consider a list of n points in the plane and a list of $n(n-1)/2$ lines such that any two points lie on a line. Just as before, if the n points are distinct, then the lines are uniquely determined. And also just as before, there are funny things that happen when the points aren't distinct. And again just as before, these funny things aren't *too* bad. The funny things are still limits of the distinct points.



When two points are in the same place, each line through them can be seen as a limiting case of two distincts points and the line they define.

Now let's make the game a little more complicated by leaving the plane and going to a *d*-dimensional space. Take n points in d-dimensional space and a list of $n(n-1)/2$ lines such that any two points lie on a line. You probably know some of what's coming. When we have n distinct points, the lines are uniquely determined, and when the points aren't distinct funny things happen. But unlike what happened before, now the funny things can be pretty bad. The funny things are no longer necessarily limits of distinct points.

The first time this really bad stuff happens is when you look at 8 points in 4-dimensional space. Three co-authors and I completely classified the bad stuff that happens in this case. We also figured out which funny things are not too bad, i.e. which ones are limits of distinct points, and which funny ones are pretty bad, i.e. *not* limits of distinct points.

Ken: That's hard for me to picture, but at least I think I can sense that something unusual could happen. I would definitely have to study your paper carefully to really see what exactly is this bad stuff is that you're hinting at. I understand you worked for a summer at Microsoft. What did you do there?

Bianca: That's right. That summer I worked as an intern at the Cryptography group at Microsoft Research. While some of you may know Microsoft from using Windows, or Office, or Xbox, you may not know that Microsoft also has a research division, called Microsoft Research, which is pretty similar to departments at a university. While most of the research is in computer science, there are two groups, the Cryptography group and the Theory group, that do quite a bit of mathematical research.

Cryptography studies methods for passing information, like a password for your email account, securely. Many of these methods rely on math problems that are hard to solve, but easy to verify that the solution is correct. For example, some of these methods rely on the difficulty of factoring. Factoring a number like 143, isn't that hard, but factoring a number that is the product of two very large primes, such as

$$N = 1881988129206079638386972394616504398071635633794173827007633564229888597152346654853190606065047430453173880113033967161996923212057340318795506569962213051687593076502570 59,$$

is very difficult! However, if I give you two prime numbers p and q, it is easy to check if I gave the correct factors; you just multiply the primes together and see if N equals pq. The methods of protecting information that rely on the difficulty of factoring work well now, but computers keep getting faster and better at factoring, so in the future, we won't be able to use these methods anymore. At Microsoft, I worked with Kristin Lauter on the mathematics behind a different hard problem that could be used instead of these factoring-based methods sometime in the future.

Ken: Do you have any advice for our members about how best to study mathematics?

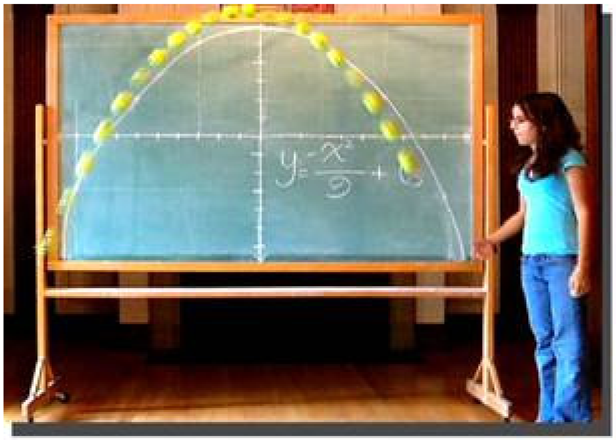
Bianca: I think there is no universal "best" way to study mathematics; it depends so much on the individual. For instance, I learn best by talking to people. For some reason it's often hard for me to make my thoughts precise when I'm just thinking about them. I need to explain the difficulties I'm having to someone else. Often, when I did this, I would figure out my difficulty in the middle of explaining it to someone else! It happened so often that I used to joke with my officemate that I needed a cardboard cutout of him, so I would feel like I was explaining my ideas and problems to a person, without having to constantly distract him! One of my good friends learns best by sitting down and reading books or papers almost cover to cover. But when I read books or papers, I practically read them backwards, starting at the theorem I want to understand, then paging back to the previous arguments that the theorem relies on. There are many different ways to study mathematics— you should use the one that works for you!

Ken: Thank you so much for this interview!

Mathematical Buffet

Parabolas by Ken Fan

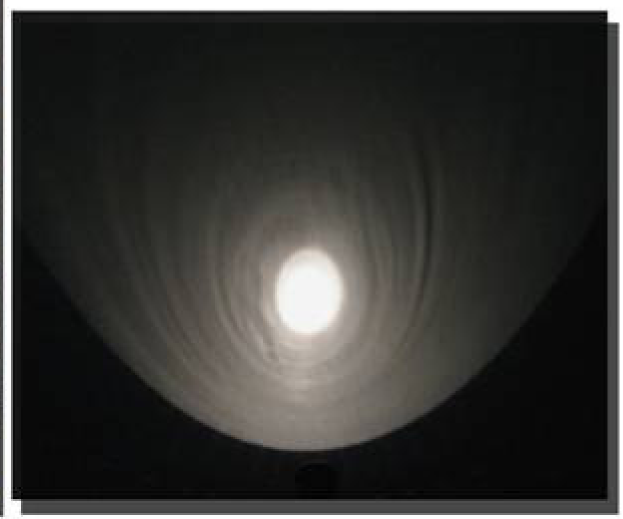
Here's a showcase of parabolas. Where else have you seen parabolas?



For short distances, tossed balls trace out parabolas, as seen in the Girls' Angle video demo *The Ball and the Parabola* by Hana Kitasei and Gabriela Acevedo.

Water fountains shoot out parabolas, like this one in Trafalgar Square, London.

Parabolas appear in the facade of the Abbey Church in Creve Coeur, Missouri which was designed by Gyo Obata.



The light boundary created by a flashlight when shone at a specific angle against a wall creates a parabola.

Photo credits: Still from *The Ball and the Parabola*, produced by Girls' Angle. Trafalgar Square LED Fountains by David Iliff courtesy of http://en.wikipedia.org/wiki/File:Trafalgar_Square_LED_Fountains_-_June_2009.jpg. Abbey Church by Br. Mark Kammerer, O.S.B., licensed under Creative Commons Attribution-ShareAlike 3.0 License. Flashlight Parabola by Ken Fan.

Without Loss of Generality

by Timothy Chow

Have you ever wondered what mathematicians mean when they say, “without loss of generality”? Wonder no more! After reading this article, you will not only understand what it means, but will be able to apply the idea yourself when you solve mathematical problems.

All good problem solvers know that if they don’t see immediately how to solve a problem, then a good way to get started is to *solve a special case first*. For example, suppose you are trying to figure out the maximum number of (not necessarily identical) pieces that you can cut a pizza into with n straight cuts. A good way to start is to consider the special cases $n = 1$, $n = 2$, $n = 3$, etc. With one straight cut, obviously you get 2 pieces; with two straight cuts, you can get 4 pieces; with three straight cuts, you can get 7 pieces if you do it right[1]. These *special cases* for *specific* values of n are easy to solve, and by studying them, you may get some insight into how to find a formula for the *general case* that works for *all* n.

Of course, a solution to a special case of a problem doesn’t automatically lead to a solution of the general case. Sometimes you can see how to solve a special case but are still left scratching your head about how to solve the general case. Every mathematician has had this experience. There are many famous problems in mathematics that nobody knows how to solve in the general case, even though many special cases have been solved.

Finding an easy special case that you can solve is one trick you can use to attack a hard problem, but if this doesn’t work, there is another trick that you can sometimes use. Sometimes you can find a special case of your hard problem that you still can’t solve, but you can see that *if you could solve the special case, then you could solve the general case*.

Here is an example. By definition, a **permutation** of the numbers 1 to n is obtained by writing down the numbers from 1 to n in some arbitrary order. For example, there are exactly six permutations of the numbers 1 to 3, namely 123, 132, 213, 231, 312, and 321. Let me now pose a problem. Suppose I give you two permutations of the numbers 1 to n, and I ask you to transform one of them to the other by repeatedly swapping adjacent numbers. Can you always do so in at most $n(n – 1)/2$ steps?

In case the question isn’t clear, let me give an example. Let me give you two permutations of the numbers from 1 to 5: 35142 and 41235. I can transform the first permutation to the second permutation in 7 steps, at each step swapping two adjacent numbers:

$$35142 \to 31542 \to 31452 \to 13452 \to 14352 \to 41352 \to 41325 \to 41235.$$

The question is whether, given *any* two permutations of the numbers from 1 to 5, you can transform the first to the second in at most $5(5 – 1)/2 = 10$ steps.

Following the previous example of our pizza-cutting problem, we might start working on the special cases $n = 1$, $n = 2$, etc. This is a reasonable approach, but there is another special case we can consider. Namely, let us think about *the special case when the first permutation is* $1234 \cdots n$. That is, the first permutation is just the numbers 1 to n written in their usual order.

[1] See page 26 for an illustration.

Even though this is a special case, it still may not be obvious to you how to solve this special case. However, this particular special case has the important feature that *if* we could solve it, *then we could immediately solve the general case!*

Why is that? Well, let's look at it this way. Suppose that instead of the numbers from 1 to 5, we had used five people instead: Isabella, Ryan, Madison, David, and Olivia. Clearly, transforming the sequence

Madison, Olivia, Isabella, David, Ryan

to the sequence

David, Isabella, Ryan, Madison, Olivia

by swapping adjacent people is exactly the same problem as transforming 35142 to 41235 by swapping adjacent numbers. In other words, as long as we use the *same label throughout* for each of our 5 objects, it doesn't matter what the labels are. In particular, if we give 3 the name "One," 5 the name "Two," 1 the name "Three," 4 the name "Four," and 2 the name "Five," then transforming 35142 to 41234 is the same as transforming

One, Two, Three, Four, Five

to

Four, Three, Five, One, Two

So if we could solve our problem in the special case where the first permutation is One, Two, Three, Four, Five, then we could also solve the case where the first permutation is 35142. In other words, solving the special case where the first permutation is $1234 \cdots n$ allows us to solve *any* case, just by renaming the numbers appropriately.

This kind of situation arises frequently enough that mathematicians have a special phrase for it: **without loss of generality**. In our permutation problem, a mathematician would say, "Without loss of generality, we may assume that the first permutation is $1234 \cdots n$." What they mean is that *a solution of this special case gives us a solution of the general case*. Therefore we can focus on solving the special case, knowing that that is all we need to do to solve the original problem.

After explaining the renaming trick, a mathematician might also say, "This renaming trick *reduces* the problem to the special case where the first permutation is $1234 \cdots n$." We use the word "reduce" because the amount of work we have to do is reduced from what it seemed we had to do at first. In the case $n = 3$, there are 6 permutations of the numbers from 1 to 3, so it might seem at first that there are 6 choices for the first permutation and 6 choices for the second permutation, for a total of $6 \times 6 = 36$ cases to check. But since we may assume that the first permutation is 123, we really only need to check 6 cases. The amount of checking we need to do has been *reduced* by a factor of 6.

To summarize: When confronted with a hard problem that you do not immediately see how to solve, look for ways to *reduce* the problem by finding a simpler special case that, *if* you could solve it, would yield a solution to the original problem. If you can find such a special case, then you can focus exclusively on that special case *without loss of generality*.

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna thinks about the nested square root question from her last installment.

I decided to think about the nested square roots question that I stumbled upon last time.

$$x = \sqrt{n + \sqrt{n + \sqrt{n + \cdots}}}$$

$$x^2 = n + \sqrt{n + \sqrt{n + \cdots}}$$

$$= n + x$$

It comes down to a quadratic equation which I can solve using the quadratic formula.

$$x^2 - x - n = 0$$

$$x = \frac{1 \pm \sqrt{1+4n}}{2}$$

For a derivation of the quadratic formula, see this issue's Errorbusters!

$$x > 0 \Rightarrow x = \frac{1 + \sqrt{1+4n}}{2}$$

I wonder when x is an integer.

When is x an integer?

If $x \in \mathbb{Z}$, then $\sqrt{1+4n}$ must be odd.

So $1+4n$ is an odd perfect square.

Hmmm...maybe this is a little amusing, but I'm not sure what to do with it.

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

$$3 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$$

$$4 = \sqrt{12 + \sqrt{12 + \sqrt{12 + \cdots}}}$$

$$5 = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}$$

Maybe this will spice things up.

What about $\sqrt{n + \sqrt{m + \sqrt{n + \sqrt{m + \sqrt{n + \sqrt{m + \cdots}}}}}} = y$?

$$y^2 - n = \sqrt{m + \sqrt{n + \sqrt{m + \sqrt{n + \sqrt{m + \cdots}}}}}$$

$$(y^2 - n)^2 = m + y$$

$$y^4 - 2ny^2 + n^2 = m + y$$

$$y^4 - 2ny^2 - y + n^2 - m = 0$$

It leads to a quartic equation. I know there is a formula for the roots of a quartic, but I wonder if I can make anything of this without the formula.

Key:
- Anna's thoughts
- Anna's afterthoughts
- *Editor's comments*

I know what should happen if $n = m$. I should get x as a root of the quartic.

If $n = m$, should recover x.

I decided to verify that x is a root of the quartic when $n = m$. Another way of putting it is that the polynomial for x should divide into the quartic for y when $n = m$.

$$\begin{array}{r} y^2 + y + (1-n) \\ y^2 - y - n \,\overline{\big)\, y^4 - 2ny^2 - y + n^2 - n} \\ \underline{y^4 - y^3 - ny^2} \\ y^3 - ny^2 - y \\ \underline{y^3 - y^2 - ny} \\ (1-n)y^2 + (n-1)y + n^2 - n \\ \underline{(1-n)y^2 + (n-1)y + n(n-1)} \\ 0 \end{array}$$

ooops!

So, we get this factorization of a quartic.

$$y^4 - 2ny^2 - y + n^2 - n = (y^2 - y - n)(y^2 + y + 1 - n)$$

I'll try to find another expression for 2.

$$2 = \sqrt{n + \sqrt{m + \sqrt{n + \cdots}}}$$

If I force n to be 1, then this must be 3 since the square root of 4 is 2.

$$n = 1 \Rightarrow \sqrt{m + \sqrt{n + \sqrt{m + \cdots}}} = 3$$

That means that this must be 9...

$$m + \underbrace{\sqrt{n + \sqrt{m + \cdots}}}_{2} = 9$$

...but this should be 2...

...and so m must be 7.

$$\Rightarrow m = 7$$

$$2 = \sqrt{1 + \sqrt{7 + \sqrt{1 + \sqrt{7 + \cdots}}}}$$

I'll double check to make sure that 2 is a root of the quartic when $n = 1$ and $m = 7$.

$$2^4 - 2 \cdot 2^2 - 2 + 1 - 7 = 0 \quad \checkmark$$

I should be able to figure out what m is for any n to make the funky radical come out to be equal to 2.

$$\sqrt{m + \sqrt{n + \sqrt{m + \cdots}}} = 4 - n$$

$$m + \underbrace{\sqrt{n + \sqrt{m + \sqrt{n + \cdots}}}}_{2} = 16 - 8n + n^2$$

$$\Rightarrow m = 14 - 8n + n^2$$

If $n = 0$, then $m = 14$. This is like taking nested fourth roots...

$$n = 0,\ m = 14:\quad 2 = \sqrt{\sqrt{14 + \sqrt{\sqrt{14 + \sqrt{\sqrt{14 + \cdots}}}}}}$$

Actually, this is silly! The original quartic tells me what m should be for any given y and n!

$$m = y^4 - 2ny^2 - y + n^2$$

Hmmm...now I wonder about this...does this even equal something finite?

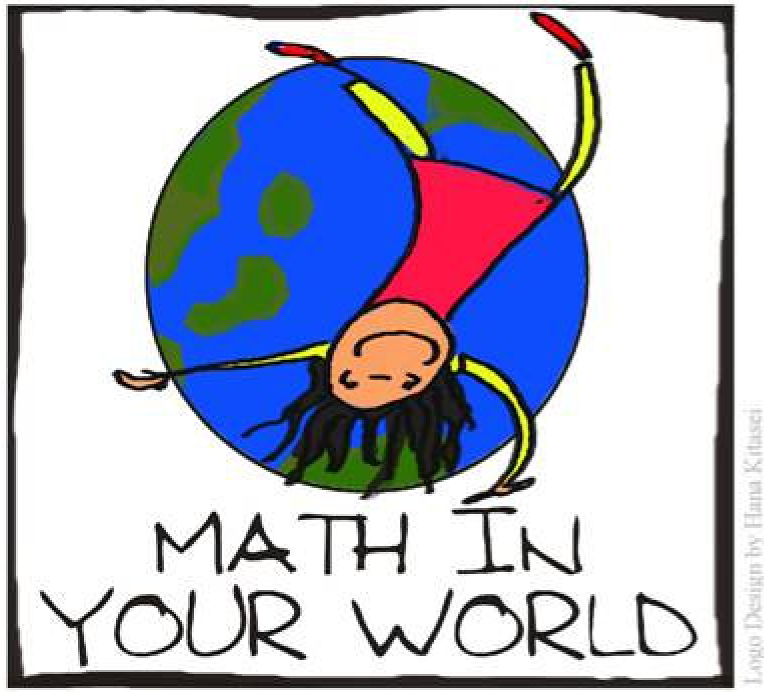
What about $\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \cdots}}}}$?

ABB 8.27.11

Key:
- Anna's thoughts
- Anna's afterthoughts
- *Editor's comments*

Six Degrees of Separation

By Katherine Sanden



Have you ever heard of the "six degrees of separation" theory? It refers to the idea that each of us is connected to any other human in the world by six steps or fewer. The people you know personally, such as your classmates, teachers, friends, and family, all have one degree of separation from you. Someone who doesn't know you personally, but knows someone that you do know personally, would have two degrees of separation from you. *I* probably have two degrees of separation from *you*.

It's a fun concept to play with – the idea that someone on a different continent in a different culture could be connected to you by six (or fewer) steps. Of course there are plenty of exceptions. And there could even be people in the world who aren't connected to you at all – not even by ten steps, or a hundred steps.

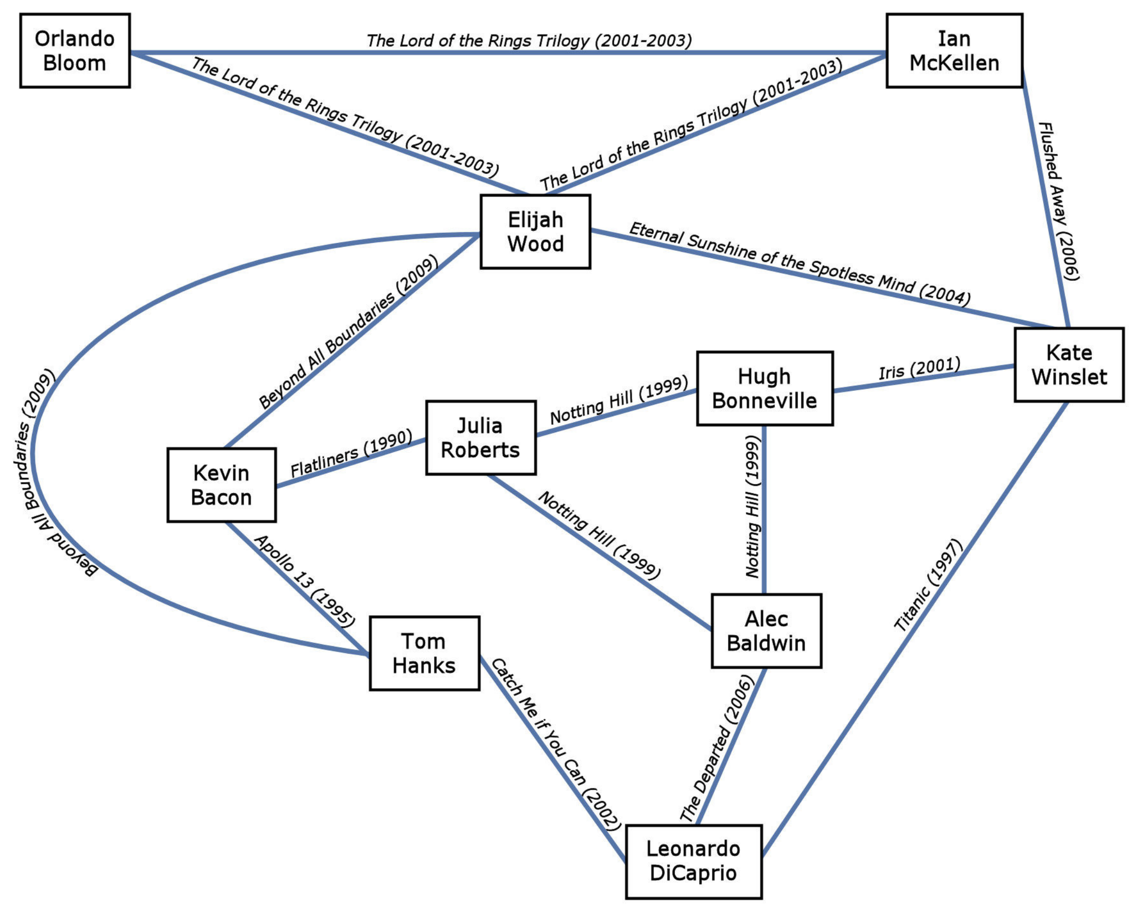
The "six degrees of separation" theory is often applied to famous actors, and has been popularized as the "Kevin Bacon game," where the goal is to find the shortest "path" between movie actor Kevin Bacon and any other movie actor. Actors are linked by the movies they act in; for instance, Kevin Bacon and Tom Hanks would have a shortest path of length one (i.e. one degree of separation) since they acted in *Apollo 13* (1995) together. Tom Hanks and Kate Winslet would have a shortest path of length two, since Tom Hanks and Leonardo DiCaprio acted together in *Catch Me if You Can* (2002) and Leonardo DiCaprio and Kate Winslet acted together in *Titanic* (1997) (among other movies), but Tom Hanks and Kate Winslet never acted in the same movie.

I recently came across a cool website to play with this theory: "The Oracle of Bacon" (oracleofbacon.org). Check it out. The site prompts you to enter two movie actors, and then finds the shortest path between them. You can input any actors included in the Internet Movie Database (www.imdb.com), where "The Oracle of Bacon" gets its data. See if you can input two actors who have a degree of separation of more than six. It's hard. (But it's possible!)

As you may have guessed, there's a healthy dose of math behind the scenes on this website. In order to hunt down the shortest path (not just any path, but the *shortest* path) between two actors, "The Oracle of Bacon" sifts through hundreds of megabytes of information stored in the Internet Movie Database using a **Breadth-First Search (BFS)**.[1]

The concept of Breadth-First Search falls under the same branch of mathematics that we touched on in last issue's article about social networks: **Graph Theory**. The Internet Movie Database and social networks such as Facebook share an underlying structure. They have collections of vertices (people, profiles, or actors) connected to one another by edges (relationships, friendships, or shared movies). Such a structure is called a **graph**[2]. Breadth-First Search is an algorithm – a set of steps – that can be used to find the shortest path between any two vertices (in our case, actors) in a graph.

To illustrate this, let's consider a simplified example. Pictured below is a small piece of the Internet Movie Database. You'll note that thousands of actors and movies are missing! But for now, let's imagine this were the whole database. Let's apply Breadth-First Search to find the shortest path between Kevin Bacon and Leonardo DiCaprio.



---

[1] Source: oracleofbacon.org/how.php

[2] Recap from last issue's Math in Your World: "You might think of functions and x's and y's when you see the word 'graph.' But in this case, a 'graph' refers to a different concept: a collection of *vertices* and *edges* that connect those vertices to each other. It's a useful way to model any networking situation, from social networks, to transportation networks, to technological networks." For more information, visit the Wikipedia page on graphs: en.wikipedia.org/wiki/Graph_(mathematics).

The central idea of Breadth-First Search is that we consider *breadth* before considering *depth.* So we look at *all* the connections of each actor before reaching deeper and pursuing connections of those connections. As we go deeper, we ignore any actors that we've already encountered.

We begin at Kevin Bacon, and look at each of his connections. For this example, let's consider them in alphabetical order by first name. We look at Elijah Wood, then Julia Roberts, then Tom Hanks. We have looked at all paths of length one and we haven't found Leonardo DiCaprio yet. So we begin looking at paths of length two, starting with Elijah Wood's connections. We look at Ian McKellen, then Kate Winslet, then Orlando Bloom. Now that we have looked at all of Elijah Wood's connections without finding Leonardo DiCaprio, we consider Julia Robert's connections. We look at Alec Baldwin, then Hugh Bonneville. No Leo. So we consider Tom Hanks' connections. His first and only connection other than Kevin Bacon is Leonardo DiCaprio. We're done. We have found that Leonardo DiCaprio and Kevin Bacon have two degrees of separation, and we have also found a shortest path that connects them.

Take it to Your World

- Not only did we find a shortest path between Kevin Bacon and Leonardo DiCaprio in this example – we also found out exactly which actors have one degree of separation from Kevin Bacon and which actors have two degrees of separation from Kevin Bacon. If you start at a given actor and let Breadth-First Search run without a specific "destination," you can find the lengths of the shortest paths between that actor and any other actor. Test it for yourself using different actors in the picture above. The "Oracle of Bacon" website has a page devoted to this concept, called, "How good a 'center' is a given actor?" Check it out.[3]

- There is another search algorithm called **Depth-First Search** that follows the path of each connection as far as possible before considering other connections. For instance, in finding the path between Kevin Bacon and Leonardo DiCaprio, Depth-First Search would visit Elijah Wood, then Ian McKellen, then Kate Winslet, then Hugh Bonneville, then Alec Baldwin, then Julia Roberts. From Julia Roberts there is nowhere "new" to go, and Leo still hasn't been found. So Depth-First Search backtracks to Alec Baldwin, and visits the next of his connections: Leonardo DiCaprio. Done. Hmm. Well, it sure didn't find the shortest path, but it did find *a* path, and in this case it did so more quickly than Breadth-First Search – it only looked at eight actors, whereas Breadth-First Search had to look at all ten. Can you think of situations where you would want to use Depth-First Search instead of Breadth-First Search? What are the advantages and disadvantages of each?

What other types of graphs could you use these search algorithms on? How about websites and links? Or cities and flights between them? Finally, can you *prove* that Breadth-First Search will always succeed in finding a shortest path?

[3] oracleofbacon.org/cgi-bin/center-cgi

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

A common error that I see in my students' work is what I call "forgetting the negative root." Often I ask my students to solve **quadratic equations**—that is, equations that have a squared term in them, such as $x^2 - 2x + 1 = 49$ or $x^2 = 81$. Let's start with the latter equation since it is simpler: let's solve for x in $x^2 = 81$. My students usually remember to take the square root of both sides of the equation, like this:

$$\sqrt{x^2} = \sqrt{81}\,.$$

Next they write

$$x = 9,$$

which certainly is one of the solutions to our original equation since $9^2 = 81$. But what about the negative root? Recall that it is also true that $(-9)^2 = 81$. How did that solution get lost? We have to remember that, by definition, $\sqrt{a}$ stands for the *positive* square root of a. So when we simplify $\sqrt{x^2}$ to x, we have implicitly assumed that $x \geq 0$. Since in reality x can be negative, $\sqrt{x^2}$ really simplifies to $|x|$. Thus going from $\sqrt{x^2} = \sqrt{81}$ to $x = 9$, while not wrong, is only half of the truth. For the complete truth, the next step should be $|x| = 9$, and this yields

$$x = \pm 9,$$

which means that x is either 9 or -9. That's exactly what we want.

Let's solve the other equation, $x^2 - 2x + 1 = 49$. We can factor the left-hand side to get

$$(x - 1)^2 = 49$$

since $x^2 - 2x + 1$ is a **perfect square trinomial** (an expression in powers of a variable that has three terms and is the perfect square of another expression). We can then take the square root of both sides, but we must not forget about the absolute value when we simplify it:

$$\begin{aligned} \sqrt{(x-1)^2} &= \sqrt{49} \\ |x - 1| &= 7 \\ x - 1 &= \pm 7 \end{aligned}$$

Then solving for x we get $x = 1 \pm 7$, in other words $x = 8$ or $x = -6$. If we forgot the absolute value signs we would lose the negative solution $x = -6$, so it's very important that we remember them.

If you've seen the "plus or minus" sign before, it was likely in the context of the **quadratic formula**. Let's review and derive this famous and useful formula, being careful not to neglect any negative roots! Given any quadratic equation, we can manipulate it until we get it into the form $ax^2 + bx + c = 0$, where a, b, and c are constants. For instance, we can change the equation above, $x^2 - 2x + 1 = 49$, into $x^2 - 2x - 48 = 0$ by subtracting 49 from both sides. Here

$a = 1$, $b = -2$, and $c = -48$. One way to solve this equation for x is to factor the left-hand side to get $(x + 6)(x - 8) = 0$, but being able to factor like this means that you are able to figure out what the roots are in the first place! When you can't easily see the roots, you can always use the tried and true quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note the use of the "plus or minus" sign in front of the radical. That is there to acknowledge the existence of the negative root.

Though this important formula may look complicated, students are usually asked to memorize it. I find things much easier to memorize when I understand them. Let's first see how to use the quadratic formula and then look at where it comes from, so that we can get a better understanding of it. We'll use the quadratic formula to solve the equation $x^2 - 2x - 48 = 0$ as given above. Since we already know the solutions to this equation, this will be a good way to test whether we're using the quadratic formula properly. So, as previously mentioned, we set $a = 1$, $b = -2$, and $c = -48$, since these are the **coefficients** of the x^2, x^1, and x^0, respectively. (A coefficient is a number that is multiplied by a given term.) Now we substitute a, b, and c into the formula to get

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-48)}}{2(1)}.$$

This simplifies to $x = \frac{2 \pm \sqrt{4+192}}{2} = \frac{2 \pm \sqrt{196}}{2} = \frac{2 \pm 14}{2}$, since 14 is the principal square root of 196. To finish simplifying, we must take into account the "plus or minus" sign in front of the 14. We do this by first adding 14 to 2 to get $x = \frac{2+14}{2} = \frac{16}{2} = 8$, and then subtracting 14 from 2 to get $x = \frac{2-14}{2} = \frac{-12}{2} = -6$. So we do indeed obtain both of the solutions that we saw earlier.

Now we shall see where the quadratic formula comes from. An important process involved in deriving this formula is called **completing the square**. Let's look at an example of completing the square before we begin our derivation. Suppose that we want to solve the equation $x^2 + 12x = 28$. In addition to factoring or using the quadratic formula, there is another method we can try: manipulating the equation until we get a perfect square trinomial on the left-hand side. We can do this by adding a carefully chosen constant term to both sides of the equation. What number do we need to add? To find out, let's imagine that $x^2 + 12x + C$ is a perfect square trinomial equal to $(x + K)^2$, where C and K are constants. We would then have $x^2 + 12x + C = (x + K)^2 = x^2 + 2Kx + K^2$. For these two quadratics to be equal their coefficients must be equal, so we know that $12 = 2K$ and $C = K^2$. This tells us that to find the constant that is added (that is, the constant C) we take the coefficient of x, which in this case is 12, and divide it by 2 to get 6. Then we square 6 to get 36, the number we need to add. So let's add 36 to both sides of the equation:

$$x^2 + 12x + 36 = 28 + 36 = 64.$$

But now the left-hand side is a perfect square trinomial, so we can factor it to get

$$(x + 6)^2 = 64,$$

an equation that we know how to solve by taking the square root of both sides. And remember: don't neglect the negative root! When we take square roots of both sides, we come up with the equation $x + 6 = \pm 8$. Solving for x, we get $x = -6 \pm 8$ or, in other words, $x = 2$ or $x = -14$.

Let us now solve the general quadratic equation $ax^2 + bx + c = 0$ by completing the square. First we need the coefficient of x^2 to be 1, so we divide both sides by a. (We'd better have $a \neq 0$ in order to divide, but if $a = 0$ then we wouldn't have had an actual quadratic equation in the first place!) So our first step gives us

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Let's move the constant term, c/a, to the right-hand side by subtracting it from both sides to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Now we can complete the square on the left-hand side. We divide b/a by 2 and then square the result to get $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \frac{b^2}{4a^2}$. This is the number that we need to add to both sides if we want to make the left-hand side a perfect square trinomial. Doing this yields

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}.$$

Factoring the left-hand side and using a common denominator on the right-hand side yields

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2},$$

which we can solve by taking the square roots of both sides and inserting a "plus or minus" sign (so we don't neglect the negative root!):

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Notice that in the above equation, when going from the second expression to the third, we don't have to introduce a second "plus or minus" sign when we take the square root of the denominator. That's because dividing +1 or -1 by +1 or -1 also yields either +1 or -1, so the single "plus or minus" sign in the numerator still accounts for all cases. Isolating x results in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For practice, try solving the following quadratic equations. Whichever method you use, don't forget any solutions! The answers can be found on page 33.

1. $x^2 - 4x + 4 = 64$
2. $x^2 + 6x - 27 = 0$
3. $x^2 - 8x = -15$
4. $2x^2 + 10x = 28$
5. $3x^2 + 27 = 18x$
6. $5x^2 - 15x + 25 = 15$

Picturing Quadratics

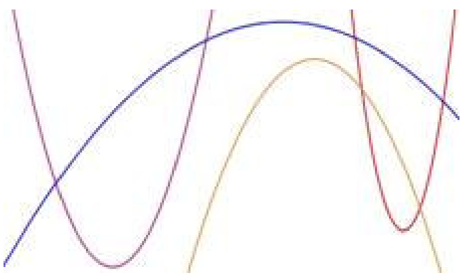
by Ken Fan / edited by Jennifer Silva

In this issue's *Errorbusters!*, Cammie derives the famous **quadratic formula** (see page 15). The quadratic formula is used to find solutions to quadratic equations, like the one that arises in problem 3 of Rachel Fraunhoffer's *Summer Fun* problem set (see page 27). To derive the quadratic formula, Cammie uses an algebraic technique known as "completing the square." Here, I'm going to explain how all of these things can be visualized.

To begin, let's recall that a quadratic function is a function that can be put into the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are constants. If $a = 0$, then the function becomes a **linear function** and its graph would be a straight line. Since we want to study quadratics, let's assume that $a \neq 0$.



Let's imagine that we've never, ever seen the graph of such a function. How would we figure out what it looks like? We could start by carefully graphing some examples (see left). But I'm going to take a different approach. I'll try to transform the graph of the above function into something simple. By undoing all of the changes made to reduce it to this simple case, we will understand what the general graph looks like.

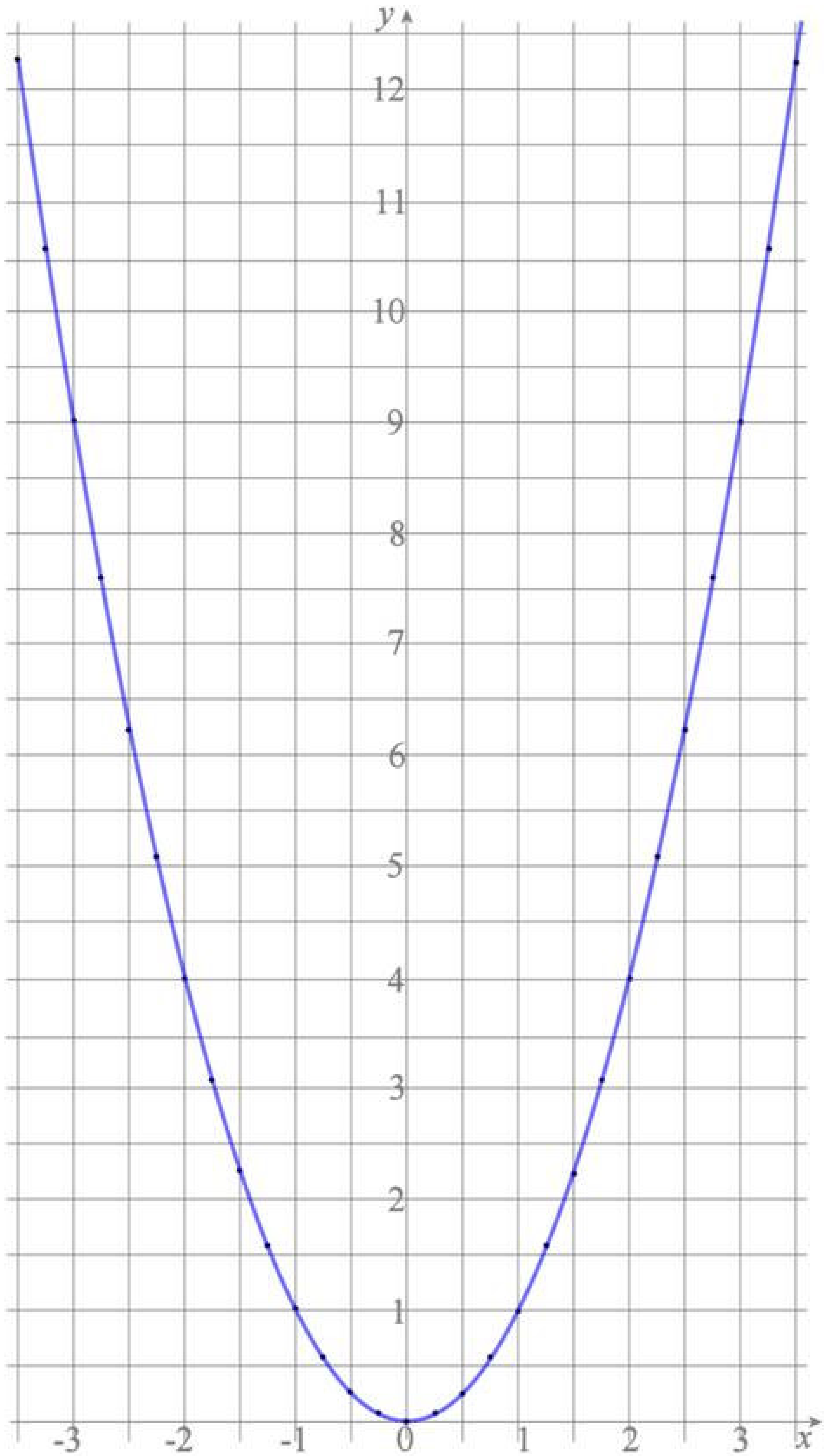
To begin with, shifting the graph up and down is easy enough to imagine. Mathematically, this corresponds to adding or subtracting a constant to the function. So, let's subtract c from the function to obtain the function $g(x) = ax^2 + bx$. Our quadratic is now just two terms instead of three. If we know what the graph of $y = g(x)$ looks like, then to get the graph of $y = f(x)$ we just have to shift the graph up or down.

Next, let's vertically stretch the graph by dividing by the nonzero factor a to get the function $h(x) = x^2 + (b/a)x$.

This previous function can be rewritten as $h(x) = x(x + b/a)$; we just factor out an x. Let's think about this function. It says that to evaluate the function for a given value of x, we take the two numbers x and $x + b/a$ and multiply them together. These two numbers, x and $x + b/a$, are separated by a fixed distance $|\, b/a \,|$. Let's look at this function from the point of view of the average of these two numbers. In other words, let's horizontally shift the graph of $y = h(x)$ by the amount $-\frac{b}{2a}$. After the shift, we get a function $i(x) = (x - \frac{b}{2a})(x + \frac{b}{2a}) = x^2 - (\frac{b}{2a})^2$.

The graph of this last function, $i(x)$, is just the graph of $y = x^2$ shifted vertically by the amount $-(\frac{b}{2a})^2$. So, by performing four simple transformations, we have reduced the problem to visualizing the graph of $y = x^2$. Because this is just one very specific function, we can get an excellent idea of how the graph looks by making a table of values and plotting them. We can also save ourselves half of the work if we observe that $(-x)^2 = x^2$. That is, if we draw the graph for positive values of x, we can get the whole graph by reflecting it around the y-axis. Also, for positive x, notice that if $x_1 < x_2$, then $x_1^2 < x_2^2$ (that is, squares with longer side lengths have more area). This tells us that between consecutive plotted points, the graph will pass within a rectangle whose corners are defined by those two points, so simply connecting consecutive plotted points with line segments will give a reasonably close idea of what the graph looks like.

x	x^2
0.00	0
0.25	0.0625
0.50	0.25
0.75	0.5625
1.00	1
1.25	1.5625
1.50	2.25
1.75	3.0625
2.00	4
2.25	5.0625
2.50	6.25
2.75	7.5625
3.00	9
3.25	10.5625
3.50	12.25



A graph of the equation $y = x^2$.

To get the look of the general graph of a quadratic, we just have to unravel what we did to transform the general quadratic to the specific quadratic $y = x^2$. To review, we shifted it vertically, vertically dilated it, shifted it horizontally, and then shifted it vertically again. If you think about this, all of these moves can be reduced to a single vertical dilation followed by a translation.

Thus, the graph of any quadratic looks like the graph shown on the previous page, only vertically dilated and shifted to some other location of the plane. One consequence of this fact is that the graph of every quadratic function has either a global maximum or a global minimum (depending on whether a is negative or positive, respectively). And if we carefully trace through the transformations, we see that the location of this maximum or minimum occurs where $x = -\frac{b}{2a}$. The point on the graph corresponding to this maximum or minimum is called the **vertex** of the parabola, and the parabola is symmetric about the line $x = -\frac{b}{2a}$.

Another consequence of our geometric understanding of quadratic functions is that we can reinterpret the meaning of completing the square visually. On page 17, starting with the general quadratic equation $ax^2 + bx + c = 0$, Cammie first divided both sides by a and then subtracted c/a from both sides to arrive at the equation

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Next, she "completed the square" by adding $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \frac{b^2}{4a^2}$ to both sides of the equation to turn the left side into a perfect square trinomial.

Algebraically, the left-hand side changes from $x^2 + \frac{b}{a}x$ to $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = (x - \frac{b}{2a})^2$. Geometrically, what this corresponds to is vertically shifting the graph so that its vertex sits on the x-axis, and the graph of $y = (x - \frac{b}{2a})^2$ is just the graph of $y = x^2$ shifted horizontally.

So even if algebra isn't your thing, you still could have discovered the notion of completing the square by thinking geometrically. The idea that would lead you to this is to think about shifting the graph of the quadratic so that its vertex lands on the origin of the coordinate system! Doesn't that seem like a natural thing to try to do?

Incidentally, knowing the location of the global maximum or minimum of a quadratic function can be very useful. Here's an application: suppose you have 40 meters of fencing and want to use it to create a rectangular pen in your backyard. You decide to use the wall of your home as one of the walls of the enclosure, so that the fencing will comprise only three of the sides of the rectangle. What should the dimensions be in order to maximize the area of the pen?

To solve this, suppose that the rectangle is an L by W rectangle and that the back wall of your home makes up one of the sides of the rectangle with length L. The area of the pen is given by LW. But we also know that $L + 2W = 40$. This means that $L = 40 - 2W$, so the area can be expressed as a function of W alone: $(40 - 2W)W = -2W^2 + 40W$. This is a quadratic function in W, and we know that the global maximum occurs where $W = -\frac{40}{2(-2)} = 10$. Therefore, the pen should be 10 by 20 meters.

For more pictures of parabolas, see this issue's *Mathematical Buffet* (on page 6).

COACH BARB'S CORNER

by Barbara Remmers

Why 6?

Coach Barb wants to tell you a story, a satisfying story. It's about you, so it will be good.

One afternoon, two sisters who you are baby sitting tire of using their many matching hula hoops for their intended purpose.   Instead they want to "draw" a flower on their lawn with hula hoops. The younger one puts a hoop down and says, "That's the center. Now we need petals." She goes off to gather hoops to use as petals.

The older one, who also has very particular ideas, says, "The petals should form a ring around the center. Petal hoops should touch the petal hoops on either side, but they should not overlap. Nor should the petals overlap with the center, although touching it is fine."

The little one returns, burdened with hoops, and says, "How about 100?"

The big one shrieks, "No, Silly! Then the ring of petals would be too far from the center to look like a flower!"

The little one starts to get a violent look in her eye, and you know where this is headed if you don't do something quickly.

With faked enthusiasm, you say, "Hey, Want to make as many flowers as possible? Let's all work together to figure out the smallest number of hula hoops that works for a ring of petals."

They stare at you dully. Can they smell fake enthusiasm? In any case, you need a new plan.
But while you're staring back at them wracking your brain for ideas, the fabulous wall behind them distracts you.

Cylindrical blocks are not common building materials. They should be.   You notice how nicely the cylinders fit together. Although the wall has holes in it, the blocks don't rattle around. You see lovely patterns: triangles, diamonds, flowers— and the answer to the question.

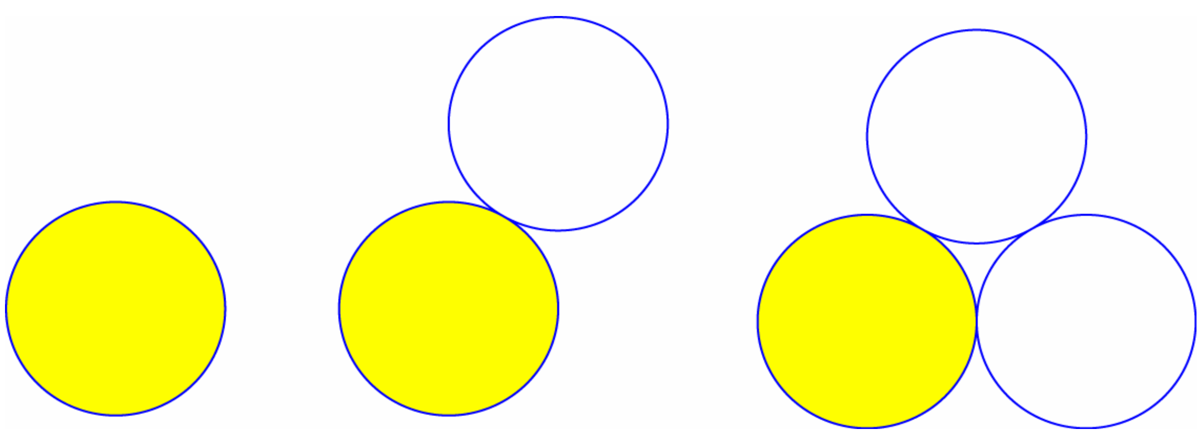
"Six!" you holler, and set the girls to work, first dividing the hula hoops into groups of seven— a center and six petals— and then merrily covering the lawn with flowers. Satisfied as you are to have forestalled World War Three, you won't be completely satisfied until you know the answer to the question: Why Six? How come six of the round blocks fit exactly around a seventh? Why not five? Why not seven? Why is the fit perfect?

You have an overwhelming urge to fiddle around to try to understand the situation. But fiddle around with *what*? Cement cylinders are an obvious choice. But they are heavy. The hula hoops are taken. Anything with a circular face— even a circle itself— will do. Coins are short cylinders, and convenient.

So you fiddle around with seven pennies, seven nickels, seven dimes, and seven quarters. You make the same six-petal-flower pattern with each of them.

Then you have an unsettling thought. If you can't explain "Why Six?" you can't even say for sure that it always *is* six. Checking several examples may heighten your suspicion that it's true for all circles, but it doesn't prove anything. After all, perhaps the sizes of the coins were chosen precisely *because* they were special circles that have the quality of six others fitting neatly around a seventh. Maybe vending machines rely on this special feature to work properly.

Now what? It's hard to see where to go from here, but, as with dealing with potentially violent children, it's best to do *something*. You decide to try positioning circles one by one, and listening to your thoughts. Maybe that will stir up something helpful. It cannot hurt. Remember: math is not dangerous.

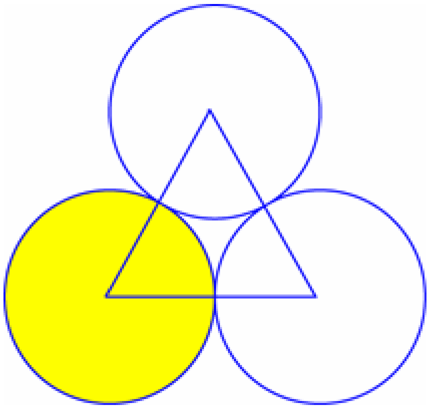


Here's the center circle. Okay. Keep going.

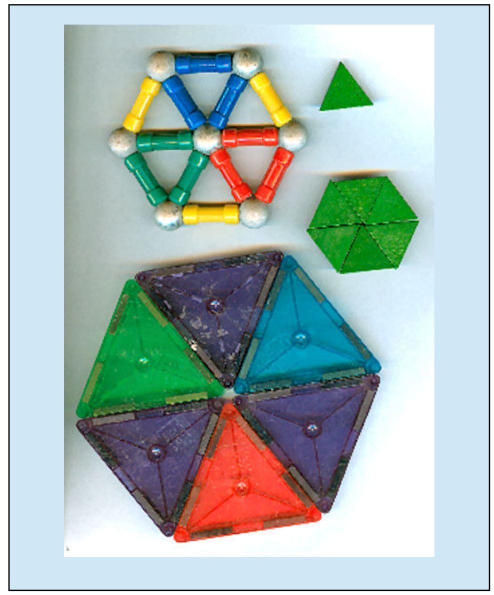
Here's another one, touching it. Still not very exciting.

Here's a third; A bit interesting.

The last one reminds you of a triangle, even though it is not a triangle. So you draw a triangle using the centers of all three circles. Why? Why not! This is math: No one gets hurt.

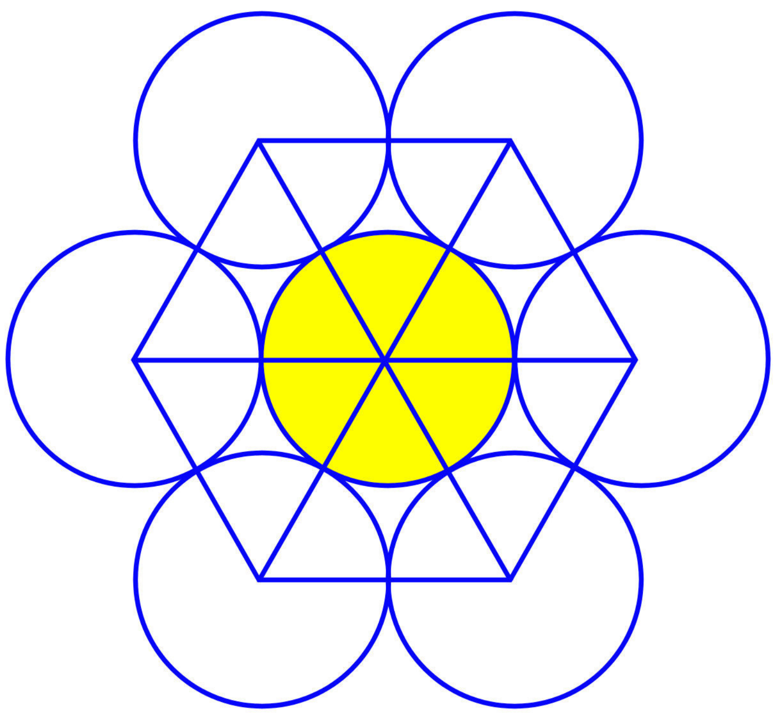


You take a good look at the triangle, and notice that each side of the triangle is made from the radius of one circle connected to the radius of another circle. The circles are identical, so the triangle is equilateral. Now you are on firm ground: equilateral triangles just so happen to be one of your all-time favorite things to fiddle with.



By playing around with them over the years, you discovered that six equilateral triangles fit snugly together, to form a regular hexagon. "Regular" means that all the sides have the same length and all the corner angles have the same measure. Equilateral triangles are regular triangles, but no one seems to call them that.

You realize that you can make a regular hexagon from six equilateral triangles of the sort you drew on the three circles above. The center of the hexagon is the center of the center circle. At each corner of the hexagon is the center of one of the six circles that exactly fit around the center circle.



Now you own it. The fact that six identical circles fit exactly around a seventh is no longer an isolated fact. That is satisfaction.

Summer Fun!

In the last issue, we invited members to submit solutions to a number of Summer Fun problem sets.

In this issue, solutions to many of the problems are provided. These solutions will sometimes be rather terse and, in some cases, are more of a hint than a solution. We prefer not to give completely detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics really well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil and move your mind! Your mind may just end up somewhere no one has been before.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially curt will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Your
Ad
Here

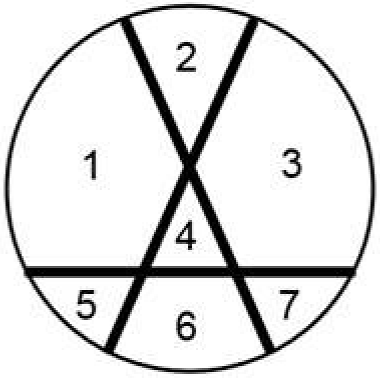
Girls' Angle is now selling advertising space in the electronic version of the Bulletin.

The print version is ad free.

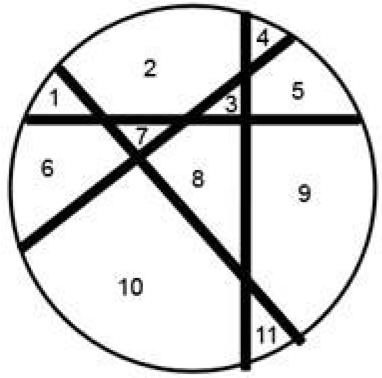
Party Time Puzzles

Solutions (Samantha Markowitz)

1. **3 cuts**



4 cuts



With 3 cuts, the maximum amount of servings is 7. With 4 cuts, the maximum amount of servings is 11.

There is a pattern: the way to maximize the number of servings is for every cut to intersect with every other cut, but no more than two cuts should ever intersect at the same point.

In other words, if we let a_n be the maximum number of pieces you can cut a pizza into using n cuts, then $a_0 = 1$ and $a_n = a_{n-1} + n$. For a non-recursive formula, $a_n = 1 + n(n + 1)/2$.

2. You can get every integer amount from 1 to 12 oz of root beer. Here's how to get 6 oz:

Step 1	Fill up the 5 oz cup with root beer and pour the 5 oz of root beer into the 7 oz cup. At the end of this step, the 5 oz cup will be empty and the 7 oz cup will contain 5 oz root beer.
Step 2	Refill the 5 oz cup with root beer and pour it into the 7 oz cup until the 7 oz cup is full. At the end of this step, the 5 oz cup will contain 3 oz of root beer and the 7 oz cup will contain 7 oz root beer.
Step 3	Pour out the 7 oz cup. Then pour the 3 oz of root beer left in the 5 oz cup into the 7 oz cup. At the end of this step, the 5 oz cup will be empty and the 7 oz cup will contain 3 oz root beer.
Step 4	Refill the 5 oz cup with root beer and pour it into the 7 oz cup until the 7 oz cup is full. At the end of this step, the 5 oz cup will contain 1 oz of root beer and the 7 oz cup will contain 7 oz root beer.
Step 5	Pour out the 7 oz cup. Then pour the 1 oz of root beer left in the 5 oz cup into the 7 oz cup. At the end of this step, the 5 oz cup will be empty and the 7 oz cup will contain 1 oz root beer.
Step 6	Refill the 5 oz cup with root beer and pour it into the 7 oz cup. The 7 oz cup will now contain 6 oz, which is the exact amount of root beer to make a perfect root beer float!

3. Each person should choose the number zero, and everyone will win. Think about it like this: the highest mean possible would be 1000 (if every person chose the number 1000), so the number that will win the game will be 250 (¼ of 1000).

Knowing this, no rational person should choose a number higher than 250, because it is impossible for ¼ of the mean to be greater than 250. Now, each guest will be choosing a number between 0 and 250 inclusive; thus, the highest mean now possible is 250 (every person chooses the number 250), and the number that will win the game is 62.5 (¼ of 250).

Knowing this, no rational person should choose a number higher than 62.5, because it is impossible for ¼ of the mean to be greater than 62.5. Reasoning in this way, we conclude that all of the players should choose the number zero to win the game.

To answer the second and third part of the question, regardless of the number of players or the boundary of the highest possible number choice, the game will play out the same way and every player should still choose the number zero.

Summer Fun!

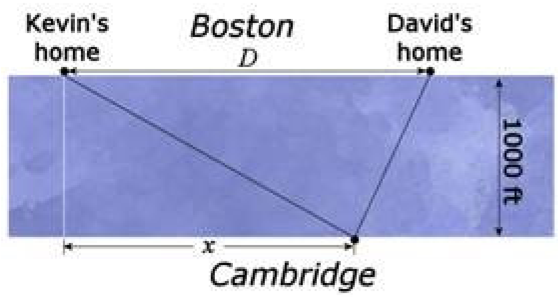
MIT Triathlon

Solutions (Rachel Fraunhoffer and Ken Fan)

1. 1000/5280 = 25/132, which is approximately 1/5 of a mile.

2. The time it takes to travel a distance at a certain rate is equal to the distance divided by the rate. David travels 1000 / cos 30° feet at 2 mph, which results in a travel time of approximately 6.56 minutes. Kevin travels 1000 / cos 60° feet at 4 mph, which results in a travel time of approximately 5.68 minutes. (Recall that cos 30° = $\frac{\sqrt{3}}{2}$ and cos 60° = $\frac{1}{2}$.) Even though slower, Kevin still beats David by about 53 seconds!

3. Since Kevin swims twice as fast as David, we need to find the points where the distance to Kevin's home is twice as far as the distance to David's home. Let D represent the distance between Kevin's home and David's home and let W = 1000 feet be the width of the river. Suppose the starting point is a distance x to the right of the point across the river from Kevin's home. Using the Pythagorean theorem, we want



$$\sqrt{x^2 + W^2} = 2\sqrt{(D-x)^2 + W^2} \ .$$

If we square both sides and simplify, we get $3x^2 - 8Dx + 4D^2 + 3W^2 = 0$, a quadratic equation. Using trigonometry, D = 1000(tan 30° + tan 60°) = 4000/$\sqrt{3}$. We solve for x by using the quadratic formula (see page 15 for a derivation of the quadratic formula) and find two solutions: $x \approx$ 1910 feet and 4250 feet.

4. It takes about 34 seconds.

5. David would have to go about 1000 feet in 64 seconds, which is about $10\frac{2}{3}$ miles per hour.

6. Kevin takes about $\frac{1000 \text{ ft}}{5280 \text{ ft/mi}} \frac{1}{8.5 \text{ mph}} \frac{3600 \text{ s}}{1 \text{ hr}} \approx 80.21 \text{ s}$. Since David runs 8 mph, he will take about $\frac{8}{8.5} 80.21 \text{ s} \approx 75.49 \text{ s}$ to cross the river. So, even given the 5-second handicap, David still gets there about a fifth of a second after Kevin!

Kevin won all 3 events so he is the ultimate athlete!

Summer Fun!

The Road to the Stanley Cup

Solutions (Lightning Factorial)

During the playoffs, teams pair off into best of seven series to see which team gets to move on. Whichever team is first to win four games moves on. In a "sweep", one team wins the first four games of the series and only four games are played. But if both teams exchange wins and losses, it's possible for the teams to end up in a winner-take-all game seven.

Let's think about an N-game series. We don't want any ties, so let N be an odd number.

1. How many different sequences of wins and losses can the team that wins the series experience in an N-game series?

N	**1**	**3**	**5**	**7**	**9**		**N**
Number of ways to win an N-game series	1	3	10	35	126	. . .	$\binom{N}{\frac{N-1}{2}}$

2. If teams are evenly matched, then the probability that one team sweeps is equal to $\left(\frac{1}{2}\right)^{\frac{N-1}{2}}$ and the probability that there is a decisive Nth game is $\binom{N-1}{\frac{N-1}{2}}\left(\frac{1}{2}\right)^{N-1}$.

Suppose that a team is engaged in an N-game series and has a probability of p of winning each game. Since N is odd, there is an integer m such that $N = 2m + 1$.

3. The probability that the team will win an N-game series is:

$$p^{m+1}\left(1+\binom{m+1}{1}q+\binom{m+2}{2}q^2+\ldots+\binom{m+m}{m}q^m\right)=p^{m+1}\sum_{k=0}^{m}\binom{m+k}{k}(1-p)^k .$$

4. If $p = 0.5$, then the team has a 50-50 chance of winning the series. Therefore, if we substitute $p = 0.5$ into the formula for problem 3, the result should be 0.5. That is,

$$\frac{1}{2}=\left(\frac{1}{2}\right)^{m+1}\sum_{k=0}^{m}\binom{m+k}{k}\left(\frac{1}{2}\right)^k ,$$

which can be rearranged to

$$2^m=\sum_{k=0}^{m}\binom{m+k}{k}\frac{1}{2^k} .$$

Summer Fun!

The Impossible Problems

Solutions (Kate Rudolph)

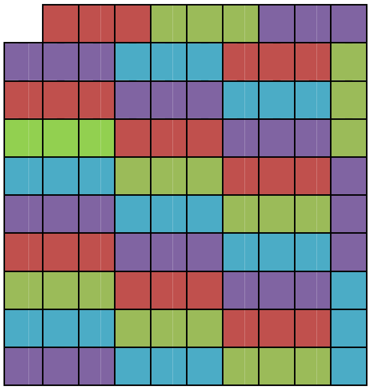
Part 1: Tiling

Can you tile a…

- 9 by 9 board with dominoes?

Impossible! The area of a 9 by 9 board is 81. The area of a domino (1 by 2) is 2. How many dominoes would we need to tile the board? Well, 81/2 = 40.5, so we would need 40 and a half dominoes to tile the entire board. But if we need to use a half domino it's not a tiling. The problem was that 81 is not divisible by 2. Thus it is impossible to tile a 9 by 9 board with dominoes!

- 10 by 10 board with 1 corner removed with 1 by 3 tiles?



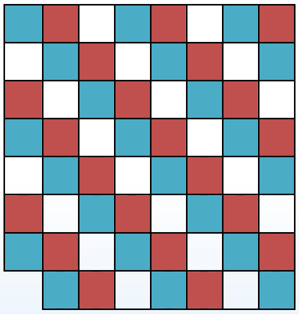
Possible! The first step is to check the areas, just like the first problem. The area of a 10 by 10 board is 100, but one corner is missing so it is 99. A 1 by 3 tile has an area of 3. Since 99 is evenly divisible by three, the area calculations worked out!

Now all we need to do is find an arrangement of tiles that will work. There are many possible answers. One example is on the right—is yours different?

- 8 by 8 board with 1 corner removed with 1 by 3 tiles?

Impossible! We start by finding the areas again: an 8 by 8 board has area 64, and with one corner removed the area is 63. We know 63 is evenly divisibly by 3, so the areas will work! But this is misleading. Once we start trying to tile our board, we discover that it's really hard to find a tiling that works.

After we bang our heads against the wall in frustration for a little while, it's time to find a different approach. Maybe having the areas evenly divisible isn't enough, and this tiling is impossible after all. Here's a neat proof:

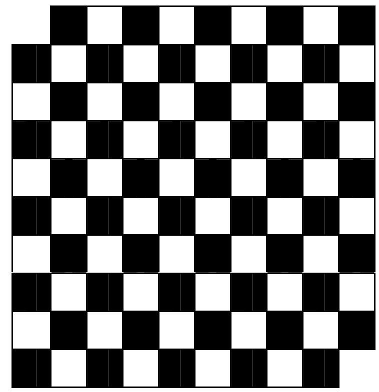


Look at the figure to the left. Anywhere you put a 1 by 3 tile on the board, it must cover exactly one blue square, one red square, and one white square (check this yourself!). But there are 22 blue squares, 21 red squares, and 20 white squares. If we could tile the board with 1 by 3 tiles, there would have to be an equal number of red, white, and blue squares. This is a contradiction, so the tiling is impossible!

Summer Fun!

- 10 by 10 board with two opposite corners removed with dominoes?

Impossible! We start by computing the area. Our board has area 100 – 2 = 98, and a domino has area 2. Since 98 is divisible by 2, the areas work. But as we learned in the last problem, that doesn't mean the tiling is possible. We might try some tilings, but before we get too far, let's try the approach from the previous solution. This time instead of using three colors we'll only use two, since the area of a domino is 2.



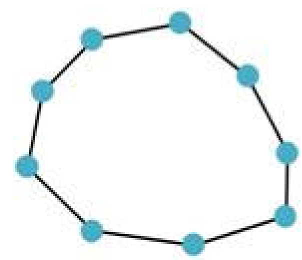
In the checkerboard pattern, no matter where you put a domino it has to cover one black square and one white square. So, if there is a tiling that works, then there must be the same number of black squares as white squares: one for every domino. But if we count the squares on the board, there are 50 black squares and 48 white squares, a contradiction! So it's impossible to tile this board with dominoes!

Part 2: Doodling

Can you draw a doodle with…

- 9 dots, 9 lines, and 1 space?

Possible! Draw 9 dots and connect them in one big circle, like this:

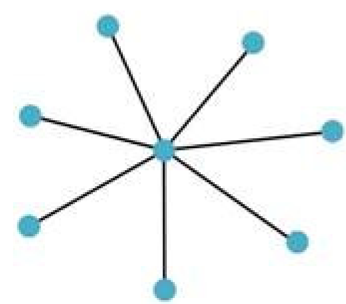


- 7 dots, 7 lines, and 2 spaces?

Impossible! There are a lot of ways to draw 7 dots with 7 lines, but we start to realize that none of them have 2 spaces. We could start with a circle arrangement like in the previous problem, but that only has one space. It's hard to think of a way to make it into two spaces, because in order to have a line to go across the circle we need to remove one of the lines on the edge, so it's still only one space. In fact, no matter how we draw an arrangement with 7 lines and 7 spaces, we always find exactly one space! We give a proof of this below.

- 8 dots, 7 lines, and no spaces?

Possible! This can be done a lot of ways, like this wheel shape:



- 2 dots, 6 lines, and 4 spaces?

Impossible! When we draw 2 dots and 6 lines connecting them, we count 5 spaces, not 4. Any way we arrange the lines, we always count 5 spaces.

In fact, the number of dots and lines *uniquely determines* the number of spaces. This means that if you pick the number of dots and the number of lines ahead of time, every single doodle you draw will have the same number of spaces. In other words, we can write a formula for the number of spaces in terms of lines and dots.

Draw a lot of doodles and make a table with columns for the number of lines, dots, and spaces and a row for each doodle. See if you can come up with a formula. Don't peek onto the next page!

The formula is: spaces = lines – dots + 1.

Let's prove it! We are going to use a type of proof called an invariant proof. We'll show that the formula above is true for the simplest possible doodle. Then we'll show that no matter what changes we make to a doodle, if the formula is true before we make the changes then the formula is true after we make the changes.

To start, we prove the formula is true for the simplest possible doodle: just one dot! In this doodle, dots = 1, lines = 0, and spaces = 0. Since 0 = 0 – 1 + 1, the formula is true for the dot!

What are the most basic changes we can make to a doodle? We can add a line, remove a line (provided the result is connected), add a dot with a line to another dot, or if a dot has only one line connected to it we can remove the dot and its line. We can't just add or remove dots because then those dots wouldn't be connected to the doodle: they'd be part of a separate, new doodle!

With these four basic changes, we can turn any doodle into any other doodle. To see this, we'll first show that any doodle can be turned into a dot using the basic moves. If the doodle has no lines, it's already a dot. So if it's not already a dot, look for a dot attached to a single line. If there is such a dot, use a basic move to remove it together with its line. If there isn't, look for a line that begins and ends at the same dot. If there is such a line, use a basic move to remove it. If no such line exists, then all dots must be connected with at least two lines. This means that there must be a loop in the doodle, because if you pick any dot, you can travel to another dot and keep going always leaving the next dot along a different line until you come to a dot that you've already visited, which means that you've gone in a loop. Use a basic move to remove one of the lines in this loop. It's important to pick a line that belongs to a loop to ensure that the resulting doodle remains connected. Now, keep going until the doodle is reduced to a single dot. Since each basic move is paired with a basic move that undoes it, this argument shows that you can also build up any doodle from a dot. So if you have two doodles, you can get from one to the other by breaking one down to a dot and then building the other one up from a dot. This reduces the problem to showing that the formula is invariant for just the four basic moves.

Let's start with the case of adding a line. Assume that before we add the line, we have a doodle with spaces = S, lines = L, and dots = D. We assume that our formula holds for this doodle, that is $S = L - D + 1$. After we add the line, lines = $L + 1$, since we have one more line than before. The number of dots is still equal to D since we didn't change it. What is the number of spaces after the change? Try adding a line to an existing doodle: You either cut an existing space into two spaces, or you create a new lump on the side that is a new space. So after the change, the number of spaces has increased by 1 so it is $S + 1$. Now we need to see if the formula is true: Does $(S + 1) = (L + 1) - D + 1$? If we simplify, we see this is equivalent to $S = L - D + 1$, which we know is true by our assumption! So we just proved that if the formula is true for a doodle, then the formula remains true if we add a line.

The other cases are very similar. Try to work them out yourself! How do the variables S, L, and D change when you remove a line (leaving something connected), add a dot with a line to another dot, or remove a dot attached to only one line? Verify that the formula remains true in all of the remaining cases. If you did, you just completed an invariant proof!

Summer Fun!

Prime Factorizations

Solutions (1729)

1. $16 = 2^4$; $25 = 5^2$; $360 = 2^3 3^2 5$; $400 = 2^4 5^2$; $2011 = 2011$ (it's a prime number!).

2. $10! = 2^8 3^4 5^2 7$; $20! = 2^{18} 3^8 5^4 7^2 11 \cdot 13 \cdot 17 \cdot 19$.

3. The largest power of 10 that divides 100! is 20 + 4 = 24.

In the following solutions, the prime factorization of n is $p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$ and the prime factorization of m is $p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_k^{m_k}$. (It may be that some n_i and some m_i are zero.)

4. The number n has $(n_1 + 1)(n_2 + 1)(n_3 + 1) \cdots (n_k + 1)$ divisors.

5. If m is a perfect cube, then all the exponents m_i are divisible by 3.

6. The prime factorization of the product nm is $p_1^{n_1+m_1} p_2^{n_2+m_2} p_3^{n_3+m_3} \cdots p_k^{n_k+m_k}$.

7. The number m divides into n if and only if $m_i \le n_i$ for all $1 \le i \le k$.

8. Let g_i be the smaller of n_i and m_i (if $n_i = m_i$, take either one). The prime factorization of the greatest common factor of n and m is $p_1^{g_1} p_2^{g_2} p_3^{g_3} \cdots p_k^{g_k}$.

9. Let h_i be the greater of n_i and m_i (if $n_i = m_i$, take either one). The prime factorization of the greatest common factor of n and m is $p_1^{h_1} p_2^{h_2} p_3^{h_3} \cdots p_k^{h_k}$.

10. Suppose the prime number p divides both n and $n + 1$. Then p must divide their difference. But their difference is 1, which means p divides 1…a contradiction!

11. We didn't receive any answers for this one…so the deal is, we'll provide the solution as soon as we get 3 answers!

12. 648.

A number is said to be **square free** if none of its factors are perfect squares except for 1.

13. 256.

14. We didn't get an answer for this one either…so we'll wait until we get 3 answers before providing a solution!

15. The sum of all numbers of the form $2^x 3^y$, where both x and y are less than 10, is equal to 30,203,052.

Summer Fun!

Calendar

Session 9: (all dates in 2011)

September	8	Start of the ninth session!
	15	
	22	
	29	Start of Rosh Hashanah – No meet
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	
	8	

Author Index to Volume 4

Key: n.pp = number n, page pp

Special Announcement: Girls' Angle will be offering a new Math Contest Prep course starting this fall. Check our website www.girlsangle.org for details.

Here are answers to the *Errorbusters!* problems on page 17.

1. x = 10 and x = -6
2. x = 3 and x = -9
3. x = 3 and x = 5
4. x = 2 and x = -7
5. x = 3 (yes, this one only has one solution, a so-called "double root")
6. x = 1 and x = 2

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club and receive a printed copy of the Girls' Angle Bulletin for the duration of the membership. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin and a 25% discount for any club meet attended. Remote members may email us math questions (although we won't do people's homework!).

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

☐ Enclosed is a check for (indicate one) (prorate as necessary)
- ☐ $216 for a one session Membership
- ☐ $108 for a one year Remote Membership
- ☐ I am making a tax free charitable donation.

☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls
Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ______________________________________

Girls' Angle

A Math Club for Girls