Girls' Angle Bulletin

October 2011 • Volume 5 • Number 1

To Foster and Nurture Girls' Interest in Mathematics

From the Founder

Two big announcements:

First, over the last few months, we have transitioned to a new printer for the Girls' Angle Bulletin: the American Mathematical Society. This transition was brought about through the efforts of Girls' Angle Advisor Bianca Viray. The main reason for this switch was that it saves us some money. Indeed, we thank our friends at Ambit Press, especially Jarrett Brimmer, for their wonderful work on the first ten printed issues.

Second, we're teaming up with MIT's Undergraduate Society of Women in Mathematics to bring the traditional end-of-session Treasure Hunt out of Girls' Angle and to the general public. Eventually, we aim to make this a new kind of math intensive event on a national scale, but the first will be local and take place on January 21…check soon for details on our website!

- Ken Fan, President and Founder

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Girls' Angle Bulletin
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Girls' Angle welcomes submissions that pertain to mathematics.

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Clementine Pyramid* by Toshia McCabe. In this issue, Anna seeks a general formula for the number of clementines in such pyramids. See page 12.

An Interview with Sophie Morel, Part 1

Dr. Sophie Morel is Professor of Mathematics at Harvard University and a Research Fellow at the Clay Mathematics Institute. She is a native of France.

Ken: Hi Professor Morel. Thank you for agreeing to do this interview with Girls' Angle! I guess I'll start with an obvious question: How did you become interested in mathematics?

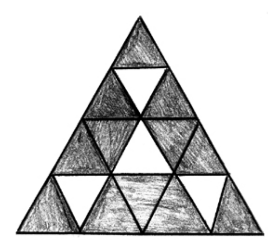
Sophie: Everybody asks that question. I don't know. As far as I remember, I was always interested, although I was by no means a mathematical wunderkind. There are no mathematicians or scientists in my family (at least in my parents' and grandparents' generations), so I was not exposed to much mathematics at an early age, but my parents left me free to explore whatever subjects I liked and were always willing to provide me with the books I asked for. (They are both French teachers.)

When I was in the 9th grade, my mother brought home from her high school a maths magazine for high-school students called "Tangente." I liked it very much and immediately subscribed. It had articles about mathematical concepts that were not taught in school, book recommendations, and maths challenges (I won two HP calculators thanks to them). That's also where I heard about the mathematical summer camp of the FFJM (the French Federation of Mathematical Games), and I convinced my parents to send me there. Later (from 11th grade), I asked my parents to buy me college-level and grad school-level textbooks and read them in my free time. I didn't always understand everything, but it was fun.

Ken: You work on mathematics that takes years of study before one can begin to even understand the questions of interest. But I am hoping to understand something about how you think about math. To that end, would you please describe some relatively elementary piece of mathematics that you find interesting?

Sophie: For about 2 years (last year of high school, first year of higher education - the French system is a bit different and I did not really go to college, but that's another story), I was trying to solve an elementary geometric problem, and here it is:

(a) What are the positive integers n such that there exists a triangle that admits a decomposition into n isometric[1] triangles? ("Decomposition" means that the bigger triangle is the union of the smaller triangles, and the interiors of the smaller triangles don't overlap. At right is an example with $n = 16$.)



Drawing by Dr. Randall Pyke. Reprinted with permission from Randall Pyke.

(b) What are the positive integers n such that there exists a triangle that admits a decomposition into n isometric triangles that are also similar to the bigger triangle? The picture above is actually an example of this. For an example of (a) that is not an example of (b), take an isosceles triangle (that does not have a right angle) and cut it in 2 along its axis of symmetry.

[1] "Isometric" means "congruent."

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I know that $n = 3$, any [perfect] square, and any sum of two [perfect] squares work for (b). For (a), there's also $n = 6$. I don't know, even today, if these are all the possibilities. I once came across a reference to an article that claimed to have proved that there are no other solutions to (b), but I was unable to find the article; I wrote to the author, but he answered that he didn't know where to find a copy of the article and couldn't remember the argument.

By the way, I came across this problem during a problem-solving evening at maths summer camp.

When I think a statement might be true, I try to see all the consequences of that statement, until I arrive at obviously false things; if I can't do that, then I will try to prove my statement.

Ken: Can you please elaborate on why you regard this piece of mathematics as interesting? What makes it interesting?

Sophie: The immediate appeal of it is that it's very easy to grasp and you can experiment to try to guess what the answer is. What makes it difficult to solve is that you don't know what kind of mathematics you can apply to it. But there are other reasons I chose to tell you about this problem, here they are:

(i) It was my introduction to research. Here is a problem, anybody can understand it, nobody knows the answer, and you can play with it for hours (or days, or months). I should note that there are no "Research Experience for Undergraduates" programs in France, as far as I know. There was no maths club in my high school. There was also no official possibility to take more advanced classes; we all had the same classes, and that was it. (Not totally true, because once you get to a certain level and if you're lucky, you can do something about it, but I was not at that level yet.)

(ii) It made me learn new mathematics. While I was trying to solve it, I started asking myself questions like "what are the rational numbers x such that the angle $x\pi$ is constructible with a ruler and a compass?" and other questions about constructibility. I asked my maths teacher and he directed me to an introductory book on Galois theory and constructibility. This is how I first learned about extensions of fields and Galois theory.

(iii) It made me interested in "cut, move the pieces around and reassemble" problems. For example, I read a book about the Banach-Tarski paradox[2] (that contained a full proof, plus lots of set-theoretic considerations that I didn't really understand at the time).

I also researched Hilbert's third problem. We had to write a report about a maths (or physics) subject that we found interesting at the end of what you would call our sophomore year, and I chose Hilbert's third problem. Here is the statement: "Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?" The analogous problem in dimension 2 was known to have a positive answer, and the problem in dimension 3 was solved very quickly by a student of Hilbert called Max Dehn. (The answer is "no.") I read up on the proofs, in dimensions 2 and 3, and this

[2] The Banach-Tarski paradox says that you can partition a unit sphere into a finite number of pieces and rearrange the pieces to form a sphere with radius two.

was my first contact with research articles. (Actually, you can show that the answer to the analogous question in any dimension $n > 2$ is "no," using the same kind of methods. But the answer is less satisfying in dimension > 3.) I had to learn some algebra before I could understand the proofs, and I also read about related problems, such as the same problem when you only allow yourself to move the pieces of the decomposition by a subgroup of the group of isometries (for example, the group of translations; in that case, the question has a full answer in any dimension).

I learned about an important mathematical concept, which is the concept of an invariant of an object (here, a polyhedron) under certain transformations (here, cut, move the pieces around and reassemble). Volume is such an invariant, and Dehn's solution to Hilbert's third problem was to construct another invariant, then to exhibit two polyhedra that had the same volume but different values for this other invariant. That was great. Then you can go further and ask yourself if there are any other invariants that you can't get from volume and the Dehn invariant, and the answer is, in dimension 3, "no." But in dimension > 3, we don't know a complete family of invariants (i.e. a basis of the space of invariants), or at least we didn't last time I checked, which was in 1999 so don't take my word for it.

I didn't always understand everything, but it was fun.

It also made me wonder about other things, such as the notion of area or volume. Until now I had only learnt formulas to calculate the area/volume of certain geometric shapes, but I started to wonder what "volume" actually meant, how it was defined in general. For example, for polygons in the plane, you can define area as follows: (1) define the area of a triangle by the usual formula; (2) define the area of a polygon by cutting it into triangles and adding the areas of the triangles; (3) show that the sum in (2) does not depend on the decomposition you choose. This is possible and elementary, though a bit messy of course. Then you can define the area of more complicated shapes by drawing a little square grid over them, looking at how many squares your shape contains (or intersects), and taking grids with smaller and smaller squares. That will work for "reasonable" shapes, and generalizes to dimension n. Eventually, I read about measure theory and Lebesgue measure on $\mathbf{R}^n$, which is the most general notion of volume that I know about. But my point was, the notion of area is by no means obvious. Do people never wonder about that?

Ken: What process do you employ to gain mathematical insight?

Sophie: Well, anything I can think about? I try to work out examples, I make calculations. I actually really enjoy calculations, but sometimes I'm bad about examples (I try to see the general case directly and it's a dangerous thing to do). Or I try to generalize the problem. I try to compare it with other problems I know about, I read up on things I think are related to see if it will give me an idea. When I think a statement might be true, I try to see all the consequences of that statement, until I arrive at obviously false things; if I can't do that, then I will try to prove my statement.

Ken: How do you think about mathematics? Do you think geometrically? Algebraically?

To be continued…

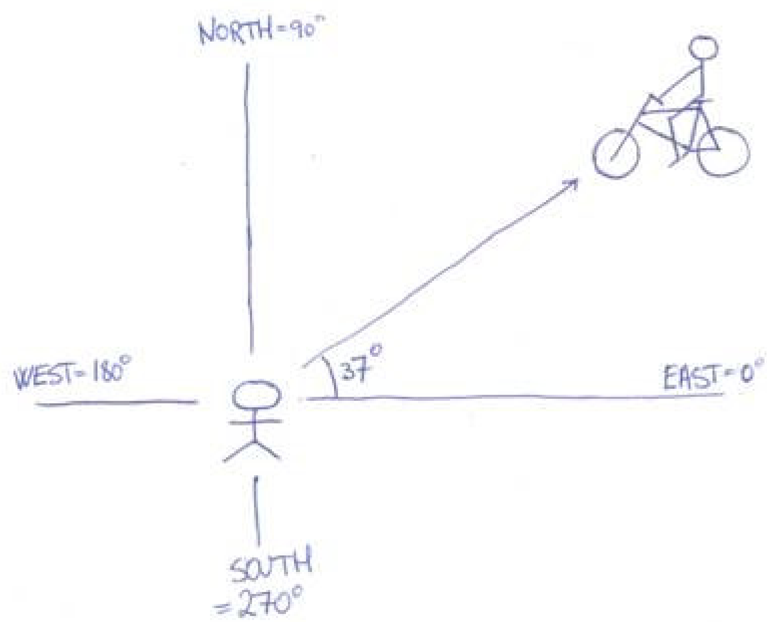
Winding Numbers I

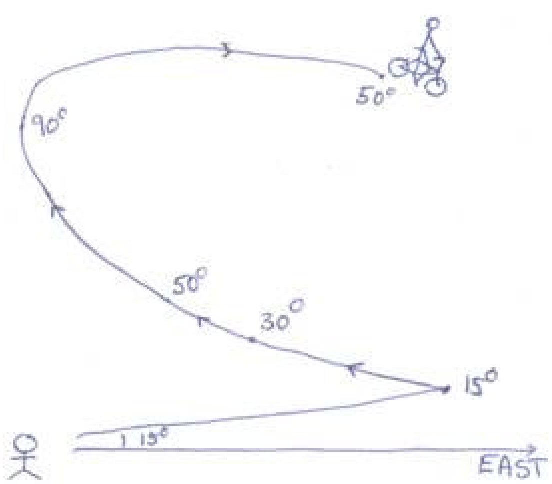
written and illustrated by Søren Galatius

Angles

Chris is riding his bicycle on the soccer field. His friend Hanna is standing in the middle, watching him. As he bikes around her, she turns so that she can always see him. She starts wondering if there is a way to keep track of how much she turns back and forth while watching him. She has been to the soccer field many times, and she knows what direction is north, south, east and west, but she is wondering if there is a better way to describe what direction she's turning to watch Chris. Sometimes she's looking due east, and sometimes due north, but there are a lot of different directions in between, and she's not very happy with just "between east and north". It's not very accurate—a bit like saying the temperature is in the 50's, where a good thermometer might say something precise, like 57. Is there a precise way of measuring what direction she's looking? Chris is still happily biking around while Hanna ponders this question and watches where he is. Suddenly, she remembers that she learned about *angles* in school, and that angles can be used as a precise measure of direction. Then all the directions between north and east can be described by a *number*, namely the angle between that direction and east.

In this way, east is 0°, north is 90°, but in-between directions can be described very precisely now. For example 37° and 51° (without angles she could only describe both as "between east and north"). She also decides that west is 180° and south is 270°, and in this way she can describe any direction on the compass by a number. (A normal compass usually has north being 0° and east being 90°, but we'll use Hanna's system.)





Chris is still biking around happily, and as Hanna keeps watching him, she thinks to herself "now he's at 15°, now he's at 30°, 50°, 90° (just north of me!), now he's back at 50°".

As Chris bikes back and forth (mostly between 0° and 90°), Hanna becomes a little bored with watching him, and almost starts thinking about something else, but then he suddenly starts going around her in a big circle. "30°, 60°, 90°, 120°, 150°, 180°—that's due west of me!" she thinks.

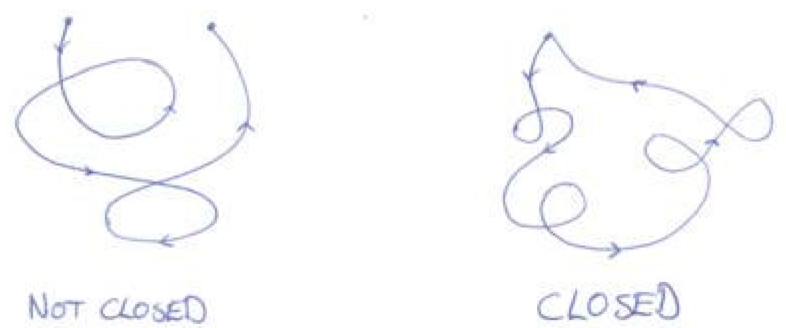
Chris speeds up, and she counts through bigger and bigger angles, passing 270° (due south), and 360°. When she gets to 375°, she realizes that Chris is back to where he started, but then she gets a little confused, because he started at 15° and ended at 375°, and yet he's back where he started? How could that happen? Are angles not so useful after all?

The answer is of course (as she quickly realizes) that a direction corresponds to an angle between 0° and 360°, and that *360° is the same as 0°*. In the same way, *15° is the same as 375°*. "Ah," she thinks, "I made a mistake—when Chris passed 359°, I should have said 0° instead of 360°, and continued from there. Then he would be at 15° when he got back." Problem solved.

But then she realizes that perhaps her mistake was not so stupid after all, and perhaps it actually contains an interesting idea The number she had first counted when he got back, 375°, tells her something about *the way that he got there*. Namely, it tells her that Chris went around her exactly once. If she had counted to 735° instead, he would also be back where he started, but she would know that he had gone around her twice.

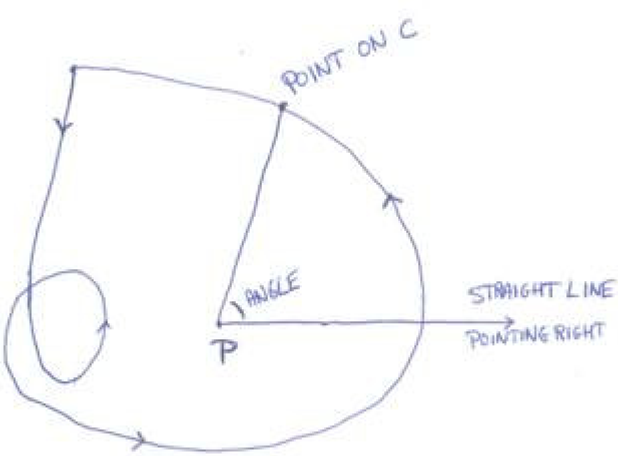
"Hmm, that's interesting," she thinks, and wonders if there's some underlying principle. It seems to her that if Chris starts somewhere on the soccer field, bikes around, and gets back to where he started, and if she watches him and keeps track of the direction by counting degrees (without replacing 360° by 0°), then the number of degrees she ends up with, minus the number of degrees she started with, divided by 360, is the *number of times Chris went around her*.

Winding numbers



Chris' bicycling is modeled mathematically by a *closed curve in the plane*, and I'll first explain what that means. A *curve in the plane* is what you get when you draw on a piece of paper without lifting the pen from the paper. The curve has a starting point and an end point, and a curve is *closed* if it starts and ends at the same point. If for example you draw a circle or a square, then you've drawn a closed curve, but you could of course draw something much more complicated

If you also draw an extra point somewhere on the paper, but not directly on the curve, we can define the *winding number* of the curve around the point, inspired by Hanna's observations on the soccer field. Let's say the curve is called *C* and the point is called *P*. For each point on *C*, we can imagine drawing a line from the point to *P*, and another line from *P* that goes straight to the right, and measure the angle between these two lines.



If the point on the curve is right above *P*, then the angle will be 90°, if it's straight to the left the angle will be 180°, if it's right below *P* the angle will be 270°, and if it's straight to the right the angle will be 0° (which is the same as 360°). Of course, the angle depends on which point on *C* we pick,

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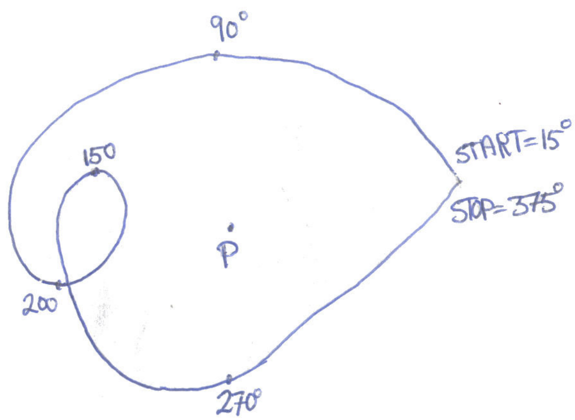
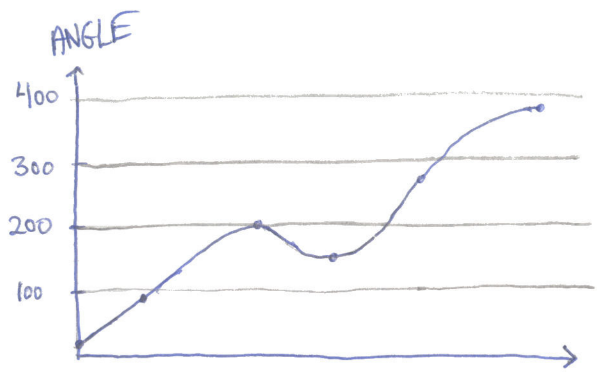
but we can observe how it changes as we move along the curve. Inspired by Hanna's observations on the soccer field, we don't change to 0° when we pass 360°, but keep counting. We can plot the changing angle in a diagram.

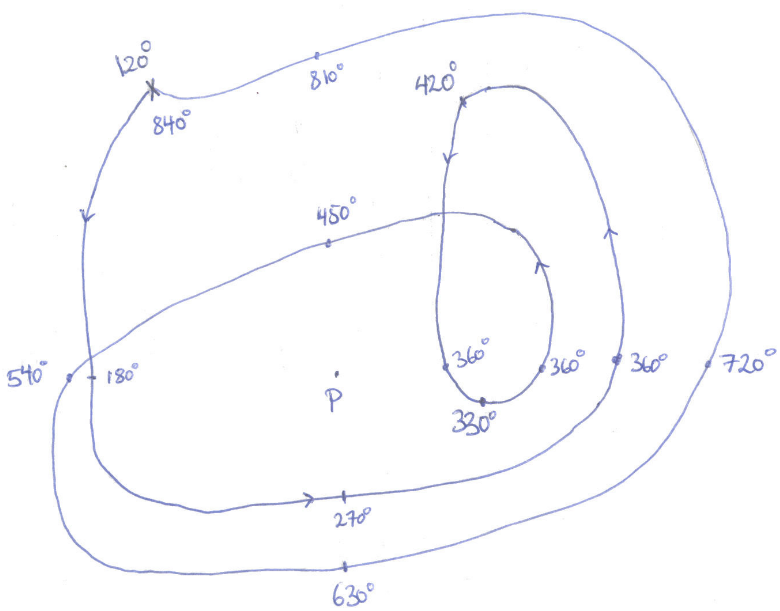




Definition: The winding number of C around P is the angle when the curve ends minus the angle when the curve begins, divided by 360°. If we write $w(C, P)$ for the winding number, we can write the formula

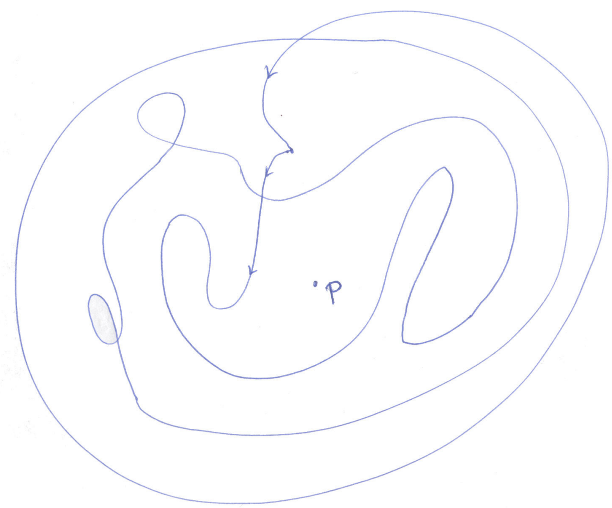
$$w(C,P) = \frac{\text{end angle} - \text{begin angle}}{360^\circ} \tag{1}$$

Let's look at an example. In the following picture, the curve C has winding number 2 around the point P.



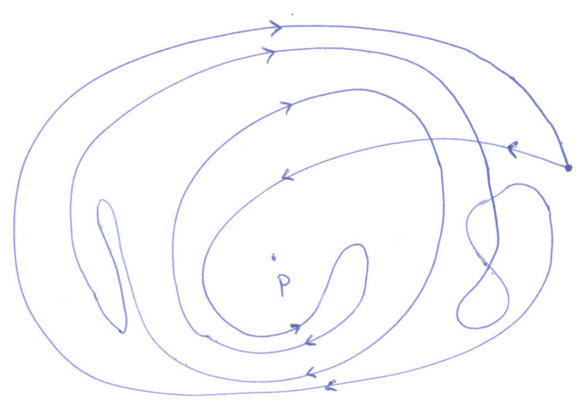
Why? At the point on the curve marked X, the angle is 120°. If we follow the curve in the direction of the arrow, the angle first increases up to 420° (passing 360° on the way), then decreases down to 330° (passing 360° again, but backwards), then increases up to 840° (passing both 360° and 720° on the way). Then we calculate (840 – 120)/360 = 2.

Exercise: Can you calculate the winding number of the curve in the next picture?



If you're very attentive, you might have noticed that there are a couple of things I haven't told you about yet. The first is that *direction is important*: If you just draw a closed curve on a piece of paper, I cannot look at the picture and tell what direction you moved the pen when you drew the picture. The formula talks about "end" and "begin" angles, and to know which is which you need to tell me what direction the curve moves. That's the reason the curve in the exercise has small arrows on it—they indicate the direction of the curve. The second thing I haven't told you has to do with the way angles are counted—they increase when the curve goes *counter-clockwise* around *P* and decrease when the curve goes *clockwise*. What happens if the angle starts at 15° and the curve then moves clockwise? No problem—the angle first goes down to 0°, then it becomes *negative*. For example, it could start at 30° and end at −330°—in that case the winding number will be −1 (exercise: check this using the formula!). What's happening? As we'll see in the next issue of this *Bulletin*, the winding number is counting "how many times *C* winds around *P*", but it counts in a very specific way: Each time *C* goes once around *P* in the counter-clockwise direction we add one to the winding number, and each time it goes once around clockwise we subtract one. If the curve goes around the point once in the counter-clockwise direction, then turns and goes around the point twice in the clockwise direction, it will have winding number 1 – 2 = −1.

Exercise: Here's a picture of a quite complicated curve. Can you find the winding number?



To be continued…

Anna's Math Journal

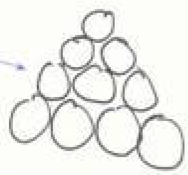
By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna seeks a general formula for the number of clementies in a pyramid as on this issue's cover.

First, I'll make a sketch...

I'm having trouble making a clear 3D sketch.. so instead I'll sketch each layer.



$1 \qquad 1+2 \qquad 1+2+3$

I recognize these layers. The number of clementines in each layer is a triangular number.

nth layer: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Pyramid with n layers:

I'll let $P(n)$ denote the number of clementines in a stack with n layers.

$$P(n) = \frac{1(1+1)}{2} + \frac{2(2+1)}{2} + \frac{3(3+1)}{2} + \cdots + \frac{n(n+1)}{2}$$

$P(n)$ is the sum of consecutive triangular numbers.

Hmmm...I'm going to try to simplify the formula and make it more compact.

$$= \frac{1}{2}\left(1(1+1) + 2(2+1) + 3(3+1) + \cdots + n(n+1)\right)$$

$$= \frac{1}{2}\left(1^2+1 + 2^2+2 + 3^2+3 + \cdots + n^2+n\right)$$

$$= \frac{1}{2}\left(1^2+2^2+3^2+\cdots+n^2 + 1+2+3+\cdots+n\right)$$

$$= \frac{1}{2}\left(\underbrace{1^2+2^2+3^2+\cdots+n^2} + \frac{n(n+1)}{2}\right)$$

$$1^2+2^2+3^2+\cdots+n^2 = \ ? = S(n)$$

If I can figure out a formula for the sum of the first n squares, then I could substitute into my expression for $P(n)$.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

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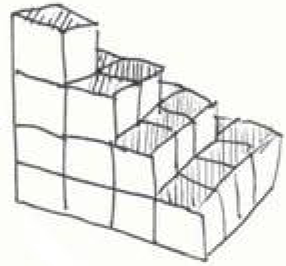
Maybe I can find the sum of the first *n* squares by using geometry, since it's the volume of this stack of unit cubes.



I'll let S(*n*) be the sum of the first *n* squares. Maybe, since it can be interpreted as a volume of a pyramid-like shape, it is equal to a cubic polynomial in *n*.

$$S(n) = an^3 + bn^2 + cn + d \ ?$$

$$S(n+1) - S(n) = (n+1)^2$$
$$= n^2 + 2n + 1$$

The function must satisfy this...

$$a(n+1)^3 + b(n+1)^2 + c(n+1) + d$$
$$- an^3 - bn^2 - cn - d$$
$$= a(3n^2+3n+1) + b(2n+1) + c = n^2+2n+1$$

...and when I force that equation to hold, I can get specific values for three of the coefficients.

$$3a = 1 \Rightarrow a = 1/3$$
$$3a + 2b = 2 \Rightarrow b = 1/2$$
$$a + b + c = 1 \Rightarrow c = 1 - \tfrac{1}{3} - \tfrac{1}{2} = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{4}{6} - \tfrac{3}{6} = \tfrac{1}{6}$$

$$\Rightarrow S(n) = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n + d$$

To get the missing coefficient, *d*, I'll use the fact that S(1) = 1.

$$S(1) = 1$$
$$\Rightarrow \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{6} + d = 1$$
$$1 + d = 1$$
$$d = 0$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n$$

This is great! It works! This formula must be valid because it was chosen to work for *n* = 1 and the equation above is just what you'd need to get a proof by induction to work!

Now I can substitute back into my expression for P(*n*)...

$$P(n) = \tfrac{1}{2}\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n + \tfrac{n(n+1)}{2}\right)$$
$$= \tfrac{1}{2}\left(\tfrac{1}{3}n^3 + n^2 + \tfrac{2}{3}n\right)$$
$$= \tfrac{1}{6}\left(n^3 + 3n^2 + 2n\right)$$
$$= \tfrac{1}{6}n(n^2+3n+2) = \frac{n(n+1)(n+2)}{6}$$

So, here's a nice compact formula for the number of clementines in a triangularly stacked pyramid of clementines!

Hmmm...when I list the formulas for the 1D, 2D, and 3D cases, there's a curious pattern...

000000

4D

$$n \qquad \frac{n(n+1)}{2} \qquad \frac{n(n+1)(n+2)}{2\cdot 3} \qquad \frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4} \ ?$$

Does $\frac{1(1+1)(1+2)}{2\cdot 3} + \frac{2(2+1)(2+2)}{2\cdot 3} + \frac{3(3+1)(3+2)}{2\cdot 3} + \cdots + \frac{n(n+1)(n+2)}{2\cdot 3}$

$$= \frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4} \ ?$$

I wonder if this is true. I'll have to think about this later...

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

For another take on this topic, see pages 23-25 of Volume 4, Number 3 of this Bulletin.

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Music, Modular Arithmetic, and More

By Katherine Sanden

Are you a musician?

This article is for you!

Are you a non-musician, but like math?

This article is for you, too!

You may have heard people talking about the connection between music and math. Maybe you've seen some of the many online articles and videos about it. (There's even a Wikipedia page![1]) Or maybe you've been exploring the connection on your own. A friend of mine likes to say that music is a form of applied math. It's a fun idea to consider – can you find ways that it seems true for you? Or not true?

In this article we're going to investigate one aspect of the connection between music and math: the relationship between notes of a musical scale and modular arithmetic. If you don't know what modular arithmetic is, don't worry. You will soon.

First, let's observe that, just like numbers, musical notes exist "on their own" – whether or not someone is playing them. For instance, the number "3" exists in mathematics. We could write the number "3" on paper, or hold 3 fingers up, or assemble a group of 3 apples – these are different ways to represent a concept that exists whether or not we have a concrete model. Similarly, a musical note could be whistled, hummed, or played on a piano or another instrument – but whether or not someone actually plays it, the concept of that note still exists.

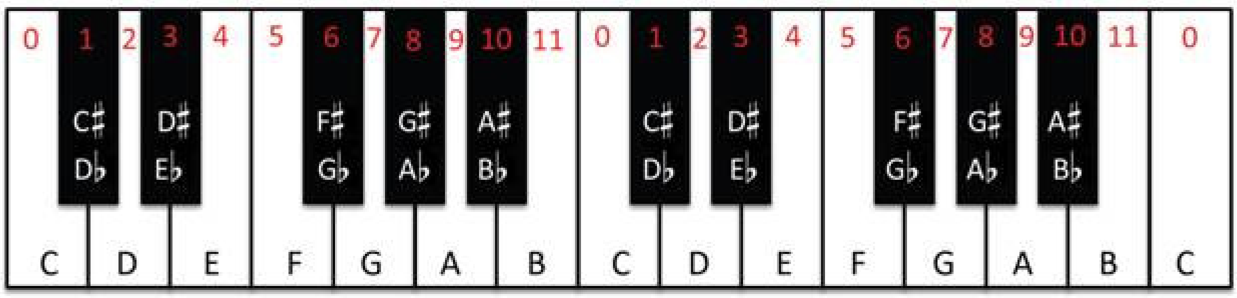
We'll use the keys of the piano to visualize musical notes, just as we often use a number line to visualize the integers. It will help to refer to an actual piano or keyboard as you continue reading. If you don't have access to one, you can use an online keyboard (here's my favorite: www.virtualpiano.net).

Let's now examine a portion of the keyboard (see next page). I've labeled the names of the notes for reference. Each black note has two names written on it, since there are two ways to refer to it. For instance, the black note directly to the right of C is called "C sharp" (denoted by C♯). It can also be called "D flat" (denoted by D♭) since it is directly to the left of D. Both names refer to the same note. If you look closely at a keyboard, you will see that the pattern of white and black keys repeat every 12 notes. Every 12th note is given the same name, even though the actual pitches you'll hear when you play two different Cs on the keyboard will be different. The reason for this is that one of the Cs sounds just like a higher pitched version of the other C. This has to do with the way we perceive sound. I'll say more about this at the end.

One consequence of this repeating note pattern is that if we refer to the notes using numbers instead of letters by calling "C" zero and increasing the number as we move to the right along the keyboard so that C♯ is 1, D is 2, etc., we would have a cycle of 12 numbers that repeats

[1] en.wikipedia.org/wiki/Music_and_mathematics

itself: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0, 1, 2, 3, etc. In the portion shown here, only two repetitions are shown, but on a real piano the pattern would continue many more times.[2]



A section of a piano keyboard.

Can you think of another situation where the numbers 0 through 11 show up again and again? I can: the set of remainders when dividing by 12! Take any portion of the number line – I'll choose the integers from 24 to 43 for illustration right now. If we divide each of these numbers by 12 and write down the remainder, we obtain the same pattern we saw on the keyboard:

…	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	…
…	0	1	2	3	4	5	6	7	8	9	10	11	0	1	2	3	4	5	6	7	…

There's a term for working with remainders in mathematics: **modular arithmetic**. When we say "26 leaves a remainder of 2 when divided by 12," we could also say, "26 is **congruent** to 2 **modulo** 12," which we denote as:

$$26 \equiv 2 \pmod{12}.$$

(Yes, the symbol between the "26" and the "2" is like an equal sign, only it has 3 line segments.)

In the world of modulo-12 arithmetic (i.e. remainders when dividing by 12), we can say things like:

$$11 \times 9 \equiv 3 \pmod{12},$$

since $11 \times 9 = 99$, which leaves a remainder of 3 when divided by 12. We can also write,

$$5 + 7 \equiv 0 \pmod{12},$$

since $5 + 7 = 12$, which is congruent to 0 modulo 12 (i.e. leaves a remainder of 0 when divided by 12).

In general, we write "$a \equiv b \pmod{12}$" if $a - b$ is divisible by 12. So, the statement "$99 \equiv 123 \pmod{12}$" is also valid because 12 divides evenly into $99 - 123 = -24$. Notice that both 99 and 123 are congruent to 3 modulo 12.

Even more generally, we write "$a \equiv b \pmod{m}$" if m divides evenly into $a - b$. Thus, for instance, $51 \equiv 11 \pmod{10}$ and $73 \equiv 1 \pmod{8}$. Modular arithmetic shows up in many areas of

[2] Mini math problem: A standard piano has 88 keys. How many times would this pattern repeat itself on a standard piano?

mathematics, and can be done with any modulus – not just 12. We're going to focus on 12 here because we have just seen that the notes of a musical scale follow the same pattern as the integers modulo 12. This cool connection enables us to use the musical keyboard as a way to illustrate arithmetic modulo 12.

For example, suppose we'd like to compute 3×7 (mod 12). Let's go to the keyboard and start at the C which we've called "0".[3] Next, move up (i.e. right) 3 notes. (In musical terms, we are moving up 3 notes **chromatically**.) You should now be on E♭, the black note directly to the left of E. If we repeat this exactly 7 times and look at the number of the note we end up on, we will have computed 3×7 (mod 12), because $3 \times 7 = 3 + 3 + 3 + 3 + 3 + 3 + 3$. When I do this, I end up on an A, also known as 9. Therefore, $3 \times 7 \equiv 9 \pmod{12}$.

Take it to Your World

- You can do other operations, too – for instance, try using the keyboard to illustrate 10 + 9 (mod 12), 4 + 1 (mod 12), or 4 – 11 (mod 12). Make up some of your own modular 12 arithmetic illustrations on the keyboard.[4]

- In the example I just did, I moved up in increments of 3. I noticed that when I moved up in increments of 3, I ended up hitting the same 4 notes again and again: I started on C, then I landed on E♭, then G♭, then A, then C again. I noticed that if I moved up in increments of 5, I eventually ended up hitting every single note of the scale before returning to another C. When I moved up in increments of 6, however, I landed only on two notes again and again: C and G♭. Can you figure out why? Hint: how do each of these numbers (3, 5, and 6) relate to 12? Finally, can you experiment and find out which other increments enable you to hit every note of the scale, and which ones don't?

- *Musician's note*: If I move up in increments of 7, I will land on every note before returning to the note I started on. This is called the "circle of fifths" in music, since 7 chromatic increments make up a "perfect fifth" in music theory. Notice that there is also a circle of fourths (which corresponds to increments of 5), but no circle of "major thirds" (a major third is made up of 4 chromatic increments). Why? (This is a rephrasing of the question in the previous bullet point.)

- What do these notes and numbers actually have to do with music?! Why are there multiple C notes on the piano? Aren't they technically different notes? I love this question. Because whether or not you're aware of it, your ears know the answer. The interval between a C and the next C to its right or an A and the next A to its right is called an *octave*. Two notes an octave apart tend to sound "the same" – there is a special sound when you play them together, that you don't hear between any other pair of notes. When you hear songs on the radio, there are often several voices singing together at an octave apart. If they changed this interval by even a little bit, you would notice – and you probably wouldn't like it! See if you can recognize the distinctive sound of an octave. Shut your eyes and have someone else play a C, and then a different C. Compare that to the sound of playing C and another note. What do you notice?

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[3] We can pick any C to start at. Can you explain why, no matter which C we call zero, we will obtain the same answer?

[4] Hey, did you get the same answer for 4 + 1 (mod 12) as you did for 4 – 11 (mod 12)? What's up with that?

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

Last time we talked about the importance of remembering negative roots when taking the square root of an equation. There is another common error associated with square roots, something that I call “distributing the square root across a sum.” By this I mean trying to simplify a square root in the following manner:

$$\sqrt{x^2+4} = \sqrt{x^2} + \sqrt{4} = x + 2.$$

Unfortunately, square roots – or any kind of roots (cube roots and so forth) – do **not** distribute across sums. That is to say that if you have two terms being added together, the square root of their total does not necessarily equal the sum of the square roots of the individual terms. Moreover, the above simplification is wrong because the negative root has been forgotten when taking the square root of x^2.

It is true, however, that roots distribute across products. So if you have two or more factors being multiplied together, the root of their product is indeed the product of their roots. For example:

$$\sqrt{4x^2} = \sqrt{4}\sqrt{x^2} = 2|x|.$$

My theory is that the common temptation to distribute roots across sums comes from a misunderstanding about the difference between addition and multiplication. Let me begin with some fundamental definitions.

The pieces of an addition problem that are separated by plus signs are called **terms** (or **summands**). For instance, the quadratic expression $x^2 + 6x + 9$ has three terms: x^2, $6x$, and 9. What you get when you add the terms together is called the **sum**. Meanwhile, the pieces of a multiplication problem are called **factors** (formerly known as **multiplicands** and **multipliers**, but nowadays those words are seldom used). The factors of $6x$ are 6 and x. What you get when you multiply the factors together is called the **product**.

Addition and multiplication are similar in some ways. For both operations, the order of the **operands** (the pieces that are being added or multiplied together) does not matter. It is true that

$$x + 3 = 3 + x$$

and

$$x \cdot 3 = 3 \cdot x = 3x.$$

This property is called **commutativity**. It is also true that you can move parentheses around among the terms of an addition problem, or amongst the factors in a multiplication problem. This is to say that

$$(3 + 4) + 5 = 3 + (4 + 5)$$

and

$$(3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5).$$

Indeed, $3 + 4 = 7$ and $7 + 5 = 12$, just as $4 + 5 = 9$ and $3 + 9 = 12$. Analogously, $3 \cdot 4 = 12$ and $12 \cdot 5 = 60$, while $4 \cdot 5 = 20$ and $3 \cdot 20 = 60$ as well. This property is called **associativity**.

Multiplication and addition are related by the **distributive law** – that is, for instance,

$$3 \cdot (4 + 5) = 3 \cdot 4 + 3 \cdot 5$$

(3 times 9 equals 27, which is the same as 12 plus 15) – but this is where their similarities begin to break down. Multiplication distributes across addition, but addition does **not** distribute across multiplication. Fortunately, I have yet to see anyone make the following error, wherein the distributive law has been applied with multiplication and addition confused:

$$3 + (4 \cdot 5) = (3 + 4) \cdot (3 + 5).$$

We can see that this does not work because $3 + (4 \cdot 5) = 3 + 20 = 23$, whereas $(3 + 4) \cdot (3 + 5) = 7 \cdot 8 = 56$.

The distributive law works when used in the proper manner because, in fact, multiplication is actually repeated addition. This is to say that $5 \cdot 2$ is actually shorthand for writing five 2's and adding them together: $5 \cdot 2 = 2 + 2 + 2 + 2 + 2$. Hence, $3 \cdot 2 + 5 \cdot 2$ means that we add three 2's together and then add the result to that of adding five 2's together, which is the same as adding eight 2's to each other:

$$3 \cdot 2 + 5 \cdot 2 = (2 + 2 + 2) + (2 + 2 + 2 + 2 + 2) = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 8 \cdot 2.$$

In other words, $3 \cdot 2 + 5 \cdot 2 = (3 + 5) \cdot 2$, or $6 + 10 = 16$.

Now, recall from a previous column that division by a nonzero k is the opposite of multiplication by k, just as subtraction by any k is the opposite of addition by k. So distribution still works if subtraction is replaced by addition, or if multiplication is replaced by division. This is to say that, for example,

$$3 \cdot (5 - 4) = 3 \cdot 5 - 3 \cdot 4$$

and

$$(9 + 12) \div 3 = 9 \div 3 + 12 \div 3.$$

(Notice that this second equation can also be written as $\frac{9+12}{3} = \frac{9}{3} + \frac{12}{3}$. Also, it's a good idea to check that these equations are really true by computing each side and comparing the answers!)

It may now seem that we have gone far astray from the original topic: when we can and cannot distribute square roots. But this is not the case, as taking a square root is really raising a number or variable to the one-half power, and exponentiation (the process of taking a number or variable to a power) is actually repeated multiplication, just as multiplication is repeated addition. Moreover, taking an nth root is the opposite of raising something to the nth power, just as division by a nonzero k is the opposite of multiplying by k. Let me give examples of what I stated in the previous sentences. Taking the square root of 4 is the same as raising 4 to the ½ power, that is: $\sqrt{4} = 4^{\frac{1}{2}} = 2$. Similarly, raising 8 to the 1/3 power means to take its cube root: $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$, and so on. Furthermore, taking 4 to the second power just means to multiply two 4's together, or, in other words, $4^2 = 4 \cdot 4 = 16$, while taking 4 to the third power means to multiply three 4's together: $4^3 = 4 \cdot 4 \cdot 4 = 64$, and so forth. Lastly, just as $4^2 = 16$, going in the opposite direction tells us that $\sqrt{16} = 4$.

Therefore, we can distribute powers and roots across multiplication in the same way that multiplication and division can be distributed across addition. For instance,

$$(3 \cdot 4)^2 = 3^2 \cdot 4^2 = 9 \cdot 16 = 144$$

and

$$\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10.$$

The ability to distribute roots across products allows us to make all radicands square-free if we so desire. That is, we can write

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

and

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}.$$

But you **must** resist the temptation to distribute powers and roots across addition or subtraction. It is **not** the case that

$$(3 + 4)^2 = 3^2 + 4^2,$$

as the left-hand side gives us $7^2 = 49$, whereas the right-hand side equals $9 + 16 = 25$. (This example provides a brief preview of what we will treat in the next column: multiplying out products of sums. So check back next time for more on this topic!) It is also **not** true that

$$\sqrt{25+144} = \sqrt{25} + \sqrt{144},$$

as the left side gives us $\sqrt{169} = 13$, while the right side is $5 + 12 = 17$.

One context in which I often see the above error is when I have asked students to find the radius of a circle. For example, a circle with center (1, 2) that passes through the point (4, 6) has radius

$$r = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

It would **not** be correct to say that

$$r = \sqrt{9+16} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

For practice, try simplifying the following expressions. For #6, express the given radical expression using radicals with square-free radicands. If an expression cannot be simplified, simply leave it alone. The answers can be found on page 33.

1. $\sqrt{x^2+9}$
2. $\sqrt{9x^2}$
3. $\sqrt{8^2+15^2}$
4. $\sqrt{12+16}$
5. $(4x)^2$
6. $\sqrt{48}$
7. $\sqrt{16-x^2}$
8 $(2x)^3$

A Letter from Vishakha Apté, Architect

Last spring, some members at Girls' Angle designed a dollhouse. Vishakha Apté, an architect who owns her own company, Vishakha Apté Architects, saw the blueprints and sent this letter.

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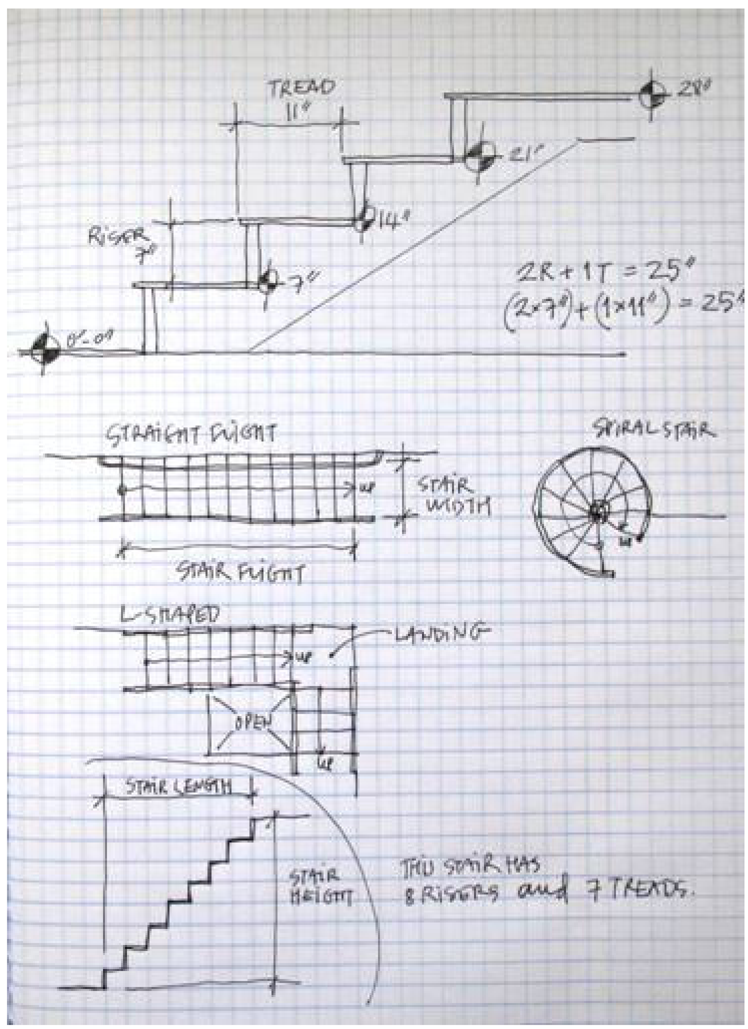
September 28, 2011

Dear Math Enthusiasts,

I am an architect and I wanted to share with you how we use mathematics in design and architecture. I understand that you all worked on putting together a dollhouse recently and gained some valuable experience with regard to designing stairs. This gave me an idea of what to share with you today: how we design stairs. As architects we use math in many ways; stair design calls for understanding basic math and geometry, which along with adherence to building codes, allows us to make safe and beautiful staircases.

Here are some key terms that you will need to know regarding the anatomy of a stair: A **stair** consists of steps. Each **step** has a **riser** (the vertical part of the step) and a **tread** (the horizontal part of the step). Each set of steps, between landings, is known as a **flight**. Most building codes specify that within each flight, steps need to have equal risers and equal treads. The sum of the risers will equal the **height** of the entire stair, while the sum of the treads will equal its **length**. The **pitch** is the angle at which a flight of steps is built.



I've introduced the concept of building codes in the previous paragraph, so here's a small digression about them. First, know that building codes differ from state to state and city to city, so one's design has to follow the codes that apply to your project location. Because building codes vary so much, I won't go into detail about the specifics but know that they exist to make sure our spaces are designed safely for the people who use them. Second, codes apply to more than stair design; they govern how many bathrooms are needed for each space, the number of exits required, the height of the building, and so on.

For the purposes of our stair design exercise, we will set aside specific building codes and instead use what is known as a **rule of thumb**. A rule of thumb is a broad version of a rule that is easy to remember and provides an approximate result. Since stair design is an **iterative process** (a process that we do over and over again, adjusting each time until we get our final result), we will use the 'approximate' stair height or tread as a starting point to get to the exact result we need.

Stairs take us from one level of the building to the next. So the first step (no pun intended) is to know what the height of this level change is. Let's say the height we have to traverse in a house is 9′-0″. *The rule of thumb we will use is this: the sum of two risers and one tread should equal 25″, which is approximately the length of a human stride.* Typically the riser height is somewhere between 7″ and 8 ½", which means that one possibility for the equation is that the risers are 7″ high and that the treads are 11″ long. This particular set of numbers also provides an average pitch for the stairs. Risers higher than 7″ would result in a steeper set of stairs.

Let's now calculate how many steps, i.e. risers, we will need for our stair that goes 9′-0″ high. Because our steps are worked out in inches, we convert the foot-inches measurement to only inches, making 9′-0″ equal to 108″. If our risers are 7″ high, we would need 15 $^3/_7$ steps (that's 108″ divided by 7″). Since we cannot have a fractional number of steps, we usually round up the number of steps and then we work backwards to calculate a new riser height.

So 108″ divided by 16 steps (rounded up from 15.42 steps) would result in risers that are 6 $^3/_4$″ high. But this would also mean that our stair run is 176″ or 14′-8″ long (16 steps multiplied by 11″ tread length). Depending on our house, we would have to see whether a 14′-8″ stair length would fit and in what configuration. Sometimes the house is large and we have space to do one straight run. Other times the stair conditions are tight and would force us to design a stair that returns, which would mean that we would have a landing in the middle. The stair could return back along itself or as an L-shaped or spiral staircase; there are many possibilities.

The lower the riser height, the more comfortable a stair is to climb up. One would typically see small riser heights for grand staircases in a museum or opera house or theatre. In homes, where space is often limited, riser heights tend to be larger than on the opera grand staircases. In our example, if we rounded down the number of steps needed from 15 $^3/_7$ to 15, then our riser height would be approximately 7 1/4″. At this new riser height, we would once again check the stair run to see how it fit in the greater plan layout. With 15 steps, the stair run would be 165″ or 13′-9″ long (15 steps multiplied by 11″ tread length). *This back and forth process of tweaking the number of steps, the riser height, the stair run etc, is the iterative process mentioned earlier*. We could work through the stair calculations once more, this time with 14 steps, which would make the riser height approximately 7 3/4″ and the stair run 154″ or 12′-10″ long (14 steps multiplied by 11″ tread length). All of these various stair designs could work, and our choice would be based on plan layout, the ease of climbing that the client needed, the amount of space we had, etc.

One way to learn more about stairs is to observe the types of stairs that we use everyday in our lives. Measure the riser and tread dimensions for the stair you have at home, subway steps, the exit stair at school, steps in your local museum. As you study these different stairs, consider why one stair's riser is higher or lower than another, or ponder why the treads are deep or short. Are you able to discern why a particular stair was designed the way it was? Is the intent of the stair design evident?

There are many more details to stair design, such as providing stair nosing, different kinds of landings, stair rail dimensional rules, minimum angle restrictions for wedge-shaped stairs, and headroom above the stair etc. And in reality, we never have such clean cut 9′-0″ heights to work with. Existing structures have varied and unusual story to story heights, making stair design all the more involved and complex. But the basic primer outlined here works for all stairs, whether they are simple runs or complicated curved stairs. Once you know the basics, you will understand that stair design uses an iterative process in order to find the best possible riser and tread combination for the project at hand.

I hope you have fun understanding stairs, learning more about stair design and designing stairs for your own projects.

- Vishakha Apté

P.S. Did you know that your hand span is a great tool for measuring stairs, or for measuring anything else for that matter? Find out what your hand span length is and you'll never be lost without a ruler when you need it most. My hand span is 8″ long, and I use it almost daily to measure things like steps, how big a piece of tile is, or the width of a door.

Dear Vishakha,

Thank you for this letter!

- Girls' Angle

Diagrmas by Vishkha Apté
Edited by Jennifer Silva

Even though there are rules and regulations, inspired architecture is still possible as this stunning spiral staircase designed and constructed by Spencer Luckey shows. Photo courtesy of Tom Virant of Virant Design Inc.

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COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction

At first when you were a baby you didn't know where things went when you let go of them. They disappeared. Then one day, perhaps you peered over the edge of your bouncy seat and spotted the item you had just dropped. Fascinating! Eventually, after enough of this, you figured out that things fell *down* when you dropped them. Except for helium balloons. Then it was time to move on to figuring out what happens to all the things that fall down. No doubt you tried to gain some knowledge with food, dishes, liquids, and toys. The grownups in your life probably have some funny stories.

By now it may seem like you've always known about breaking, sticking, and dropping. You don't have to remind yourself that eggs but not rocks break when thrown, that peanut butter sticks, and that milk sloshes. Imagine if you did! You might constantly tell yourself as many of the particular facts as you could cram into your head. Alternatively, you might decide it's futile to attempt to keep track of it all. Instead you would just sit quietly and stare dully at whatever happened to be in front of your eyes. In either case, you would lead an incredibly boring life.

Something similar can happen with fractions. The procedures for dealing with them can seem like a bunch of directions and things to remember that have no connection either to each other or to how we deal with whole numbers.

Wouldn't it be nice if adding fractions were as natural as adding whole numbers? If the procedures seemed sensible, they'd be **much** easier to remember. Forgetting something or making a mistake isn't the end of the world, but it is a shame to miss out on the satisfaction that results from truly understanding what is going on.

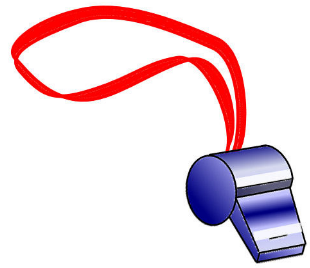
In the case of fractions, getting to know them very well goes a long way in helping us to know, and remember, what to do with them. So let's get some fraction satisfaction!

We're going to do this in three parts. In this column we will talk about what fractions are. Next time we will add and subtract them. Finally, we will investigate multiplying and dividing them.

So, let's get to know a particular fraction, $\frac{3}{7}$. The way we'll do this is to ask $\frac{3}{7}$ some questions and listen closely to the answers. Let me warn you: $\frac{3}{7}$ reminds me of my wacky great-aunt Myrna.

Q: What are you?
$\mathbf{\frac{3}{7}}$**:** I'm a number. My name is $\frac{3}{7}$.

Q: Why do you look so funny?
$\mathbf{\frac{3}{7}}$**:** I am a number that cannot be described by one whole number; two are needed. By the way, missy, I don't look funny, I look elegant.



Q: Why can't one number describe you?

$\mathbf{\frac{3}{7}}$**:** One number does, and that number is $\frac{3}{7}$! It's a perfectly fine number.

Q: Why can't one whole number describe you?

$\mathbf{\frac{3}{7}}$**:** Because my value is between two whole numbers: 0 is too small and 1 is too big.

Q: Do all fractions have values between two whole numbers?

$\mathbf{\frac{3}{7}}$**:** No. Whole numbers can also be written as fractions. My best girlfriend, 3, sometimes goes by $\frac{3}{1}$, sometimes by $\frac{21}{7}$, and sometimes by other names. We'll talk more about this later, but I'll let you in on a little secret ... she doesn't do this just so we can have matching tops, though it often *is* about having matching bottoms.

Q: Please, please tell me now! Why would anyone want to use two whole numbers to describe a number that can be described with one whole number?

$\mathbf{\frac{3}{7}}$**:** Don't beg, dearie; it's unbecoming. When whole numbers are written as fractions, it is because it's useful to think of them that way. You may want to compare the whole number with a number that can only be written as a fraction, for instance. Or you might want to combine a whole number with one or more fractions.

Q: What do you mean by combine?

$\mathbf{\frac{3}{7}}$**:** Add, subtract, multiply, or divide.

Q: What does your 7 mean?

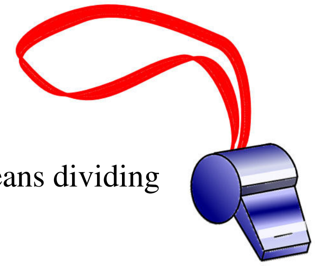
$\mathbf{\frac{3}{7}}$**:** My bottom number, 7, tells people which equal portions of one – or "a whole" – are used to compose me.

Q: Huh?

$\mathbf{\frac{3}{7}}$**:** Remember how I said that 0 was too small to describe me and 1 was too big? Well, that's just another way of saying that, on a number line, I reside between 0 and 1. If you start at 0, and take steps of the same size as the distance between 0 and 1, you would miss visiting me. That would be a crying shame, trust me. My bottom number tells you the size of the smaller steps you would need to take to find me.

Q: But 7 is greater than 1 and you said we needed to take smaller steps, not larger steps!

$\mathbf{\frac{3}{7}}$**:** Hold on a sec, missy, there's no need to get snippy! Remember that this is math. Things *do* make sense. You just have to keep asking questions until you are satisfied. This all can be done calmly. Ask and ask. Persist until you receive an explanation that makes sense to you.



Q: How can 7 be used to describe a number less than 1?

$\mathbf{\frac{3}{7}}$**:** It's all about location, honey. When 7 is the bottom number of a fraction, it means dividing 1 – a whole – into 7 equal pieces.

Q: Sevenths! Right?

$\mathbf{\frac{3}{7}}$**:** Yes dear, the small steps I have been talking about have lengths of one-seventh, or a seventh, for short. I am made up of one-sevenths. If you started at 0 on the number line and took steps of length one-seventh, then you would land on me.

Q: So any fraction with 7 on the bottom is made up of sevenths?

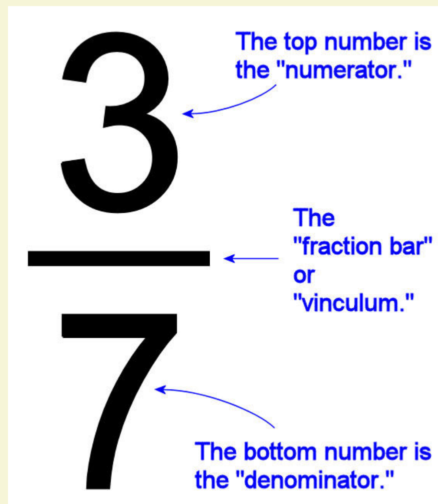
$\mathbf{\frac{3}{7}}$**:** You got it, girly. You see, the 7 tells what type – or denomination, you might say – of fraction I am. That's why some folks use the highfalutin term **denominator** to talk about the bottom number of fractions.

Q: What about your top number?

$\mathbf{\frac{3}{7}}$**:** My top number, 3, tells how many sevenths comprise me. I have 3 sevenths. I am $\frac{3}{7}$. If you want to visit me on the number line, take 3 steps of length one-seventh. (Here I am assuming you are starting at 0, and heading in the direction of 1.)

Q: Doesn't the top number of fractions have a fancy name, too?

$\mathbf{\frac{3}{7}}$**:** Fancy, schmancy! Nobody ever forgets "top number." The hoity-toity term you are searching for is **numerator**.



Anatomy of a Fraction

Q: Well, if the Terminator terminates, I bet it's called numerator because it numerates how many sevenths – in your case – you have. Right?

$\mathbf{\frac{3}{7}}$**:** Darling, darling, darling ... where to start? Unfortunately, your straightforward explanation is hampered by the fact that numerate doesn't mean "to count," it means "to list." Numerate doesn't even get used much as a verb. Generally, enumerate is used as a verb meaning "to list," and numerate is used as an adjective meaning "skilled with numbers." As you no doubt gather, word meanings cannot always be deduced from reasonable starting points. What they seemingly *should* mean and what they actually *do* mean are not guaranteed to be the same. It's all so frightfully messy, inconsistent, and jumbled. That's why I stick with the precision and beauty of math. Oh, and please spare me the mention of that vulgar Terminator fellow.

Notes from the Club

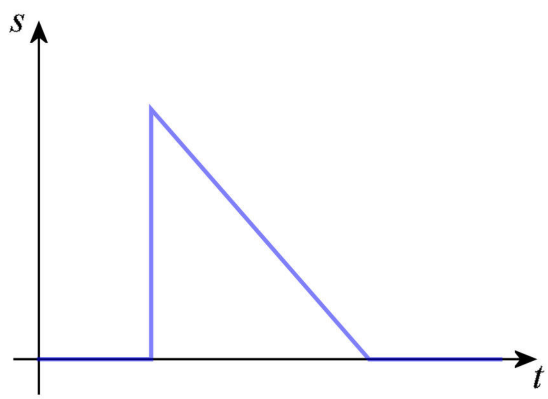
These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 9 – Meet 1 – September 8, 2011

Mentors: Samantha Hagerman, Rediet Tesfaye, Bianca Viray

One activity we did today was "walk out" graphs of speed versus time. That is, we would show the girls a graph of speed versus time. The girls would then have to walk, confined to a line, in such a way that the graph represented their actual speed as a function of time. In effect, the girls were performing an integration.

One girl did an admirable job of "walking out" the following graph:



Another who watched remarked that "that is what happens when you have a sugar high."

Session 9 – Meet 2 – September 15, 2011

Mentors: Samantha Hagerman, Jennifer Melot, Liz Simon

Some members have begun exploring proofs. One of the activities we made for the members consisted of unscrambling scrambled proofs. We gave the member a stack of rectangular slips of paper. Each slip had a mathematical statement on it. The problem was to arrange the statements into a coherent proof.

By giving the girls a scrambled proof, we avoided questions of what could or could not be assumed and were able, instead, to focus on logical coherence.

When the girls mastered these, we increased the difficulty level in two ways. One was to split the statement to be proven in two. The other was to include erroneous and irrelevant statements that had to be discarded. If you'd like to try your hand at one of these unscrambling tasks, we've included a sample on the next page.

Scrambled Proofs

Copy this page and then cut along the dotted lines so that each statement is on its own separate rectangular slip of paper. Arrange the statements so that they form a single coherent proof. It's easy to create more problems like this. Just take any math book that contains proofs and have someone copy the proof with each sentence of the proof on a separate line. Print the proof out and then cut the paper up so that each sentence is on a different slip of paper. Shuffle the slips well and then try to arrange them back into a coherent proof.

We can then write $a = 2n$ for some integer n.
Suppose that a is odd.
Suppose that a is even.
This can be rewritten as $4(n^2 - n - b) = 2$.
Substituting $2n - 1$ for a in the equation yields $(2n - 1)^2 - 4b = 3$.
Therefore, a cannot be odd.
Subtracting 1 from both sides yields $4n^2 - 4n - 4b = 2$.
Since $n^2 - b$ is an integer, $4(n^2 - b)$ is a multiple of 4.
Since 2 is not a multiple of 4, the equation $4(n^2 - n - b) = 2$ has no integer solutions.
Therefore, a cannot be even.
Substituting $2n$ for a in the equation yields $(2n)^2 - 4b = 3$ or $4n^2 - 4b = 3$.
Since 3 is not a multiple of 4, the equation $4(n^2 - b) = 3$ has no integer solutions.
We can rewrite this as $4(n^2 - b) = 3$.
Since a can neither be even nor odd, the proposition is true.
Proposition. There are no solutions in integers to the equation $a^2 - 4b = 3$.
Since $n^2 - n - b$ is an integer, $4(n^2 - n - b)$ is a multiple of 4.
We can then write $a = 2n - 1$ for some integer n.
Expanding out, this becomes $4n^2 - 4n + 1 - 4b = 3$.

Session 9 – Meet 3 – September 22, 2011

Mentors: Samantha Hagerman, Ryan Heffrin, Bianca Viray

Special Visitor: Ally Hartzell, Pixtronix

Ally Hartzell talked about Micro Electrical Mechanical Systems (MEMS). MEMS are very tiny machines. A common thread in her presentation was the use of exponentials. Many graphs had scales that were logarithmic instead of linear. This led us to explore exponential growth in a variety of ways. You can read about some of them on the Girls' Angle blog. Ally also showed us some of ways that she uses probability to study the failure rate of MEMS.

Session 9 – Meet 4 – October 6, 2011

Mentors: Jennifer Balakrishnan, Jennifer Melot, Bianca Viray

Some members completed a proper scale graph of the function $f(x) = x^2$ over the range of values $0 \le x \le 30$ using one-quarter inch as the unit length. If you haven't done this before, it's worth doing at least once in your life. You might be surprised at how long your paper has to be to fully contain this graph!

Session 9 – Meet 5 – October 13, 2011

Mentors: Jennifer Balakrishnan, Samantha Hagerman, Rediet Tesfaye

Special Visitor: Diana Hubbard, Boston College

Diana Hubbard discussed tic-tac-toe on a donut. Last month, we posted a WIM video featuring Diana explaining this topic. She described how to modify a rectangle to make a flat model of the surface of a donut. Then she introduced tic-tac-toe on that surface and had the girls analyze features of this game. In the end, the girls determined that tic-tac-toe on a donut, unlike regular tic-tac-toe, is a win for the first player.

After her visit, some girls made models of the Platonic solids using a method I learned at the recent Math Prize for Girls event that took place at MIT. The method is inexpensive and effective: Roll up paper to make cylinders using tape to keep the shape. String up these cylinders to form the edges of the polyhedra. While making these, **Ratties Are Cute** began to wonder what is the minimum number of "cylinder strings" required in order to make each of the Platonic solids. A "cylinder string" is a sequence of cylinders where each one touches the end of the next.

Another way of describing the problem is as follows: Consider the vertices and edges of the polyhedron. If you start at a vertex and walk along the edges to another vertex (which could be back to your starting vertex) keeping track of where you've gone, you will have traced out some path. Let's say the path is "clean" if you don't cross over an edge more than once. **Ratties Are Cute** wanted to know what is the minimum number of clean paths necessary to cover all the edges of the polyhedron. This is a **graph theory** question and is related to **Eulerian cycles**. An **Eulerian cycle** is a single clean path that comes back to the starting vertex and traverses every edge in the graph. According to Melody Chan, a graduate student at UC Berkeley, "clean paths" can also be called "trails" or "walks using each edge at most once." (For more on graph theory, check out Katherine Sanden's *Math In Your World* columns in the previous two issues of this *Bulletin*.)

After thinking carefully about the tetrahedron and the cube, **Ratties Are Cute** formulated this conjecture:

> *The minimum number of clean paths needed to cover the edges of a Platonic solid is equal to half the number of vertices.*

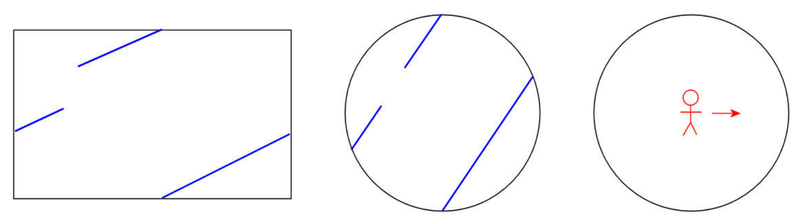
Interesting! Being able to make a clear statement represents real progress. There's more on this story in Meet 7.

<u>Session 9 – Meet 6 – October 20, 2011</u>

Mentors: Jennifer Balakrishnan, Liz Simon

We began by following up on Diana's visit by revisiting life on a torus (the surface of a donut). A torus can be modeled by taking a rectangle and saying that points on the border that are horizontally across from each other or vertically across from each other are to be considered the same point.

We then moved to another world. In the new world, we live on a circle but declare that diametrically opposed points on the boundary are to be considered the same point. Think about this world. What happens when a person starts walking to the right and just keeps on going?



Although it looks like there are 3 separate blue segments in the rectangle (left) and circle (middle), the blue segments actually represent a single, continuous path in their respective worlds. What happens when the stick figure person in the modified circle world keeps moving to the right?

<u>Session 9 – Meet 7 – October 27, 2011</u>

Mentors: Jennifer Balakrishnan, Jennifer Melot, Rediet Tesfaye

Ratties Are Cute and **billy-bob-joe-bob-jim** continued to consider **Ratties Are Cute**'s conjecture from Meet 5. They considered the octahedron and eventually came to the following modification of the original conjecture:

> *The minimum number of clean paths needed to cover the edges of a simple graph is equal to half the number of* ***odd degree*** *vertices.*

Beautiful! They also observed that for polyhedra where the degree of every vertex is constant (such as for the Platonic solids or the truncated Platonic solids), the number of edges is equal to half the product of the number of vertices and the degree. For a highly relevant article, see *Facebook and Graph Theory*, Katherine Sanden's *Math In Your World* column in Volume 4, Number 5 of this *Bulletin*.

Calendar

Session 9: (all dates in 2011)

September	8	Start of the ninth session!
	15	
	22	Ally Hartzell, Pixtronix
	29	Start of Rosh Hashanah – No meet
October	6	
	13	Diana Hubbard, Boston College
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	
	8	

Session 10: (all dates in 2012)

January	26	Start of the tenth session!
February	2	
	9	
	16	
	23	No meet
March	1	
	8	
	15	
	22	
	29	No meet
April	5	
	12	
	19	No meet
	26	
May	3	

Special Announcement: If you like solving math contest problems, sign up for Girls' Angle's Math Contest Prep course. Check our website www.girlsangle.org for details.

Here are answers to the *Errorbusters!* problems on page 20.

1. can't be simplified
2. $3x$
3. 17
4. $2\sqrt{7}$
5. $16x^2$
6. $4\sqrt{3}$
7. can't be simplified
8. $8x^3$

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club and receive a printed copy of the Girls' Angle Bulletin for the duration of the membership. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin and a 25% discount for any club meet attended. Remote members may email us math questions (although we won't do people's homework!).

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership
  - ☐ $108 for a one year Remote Membership
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

Girls' Angle

A Math Club for Girls