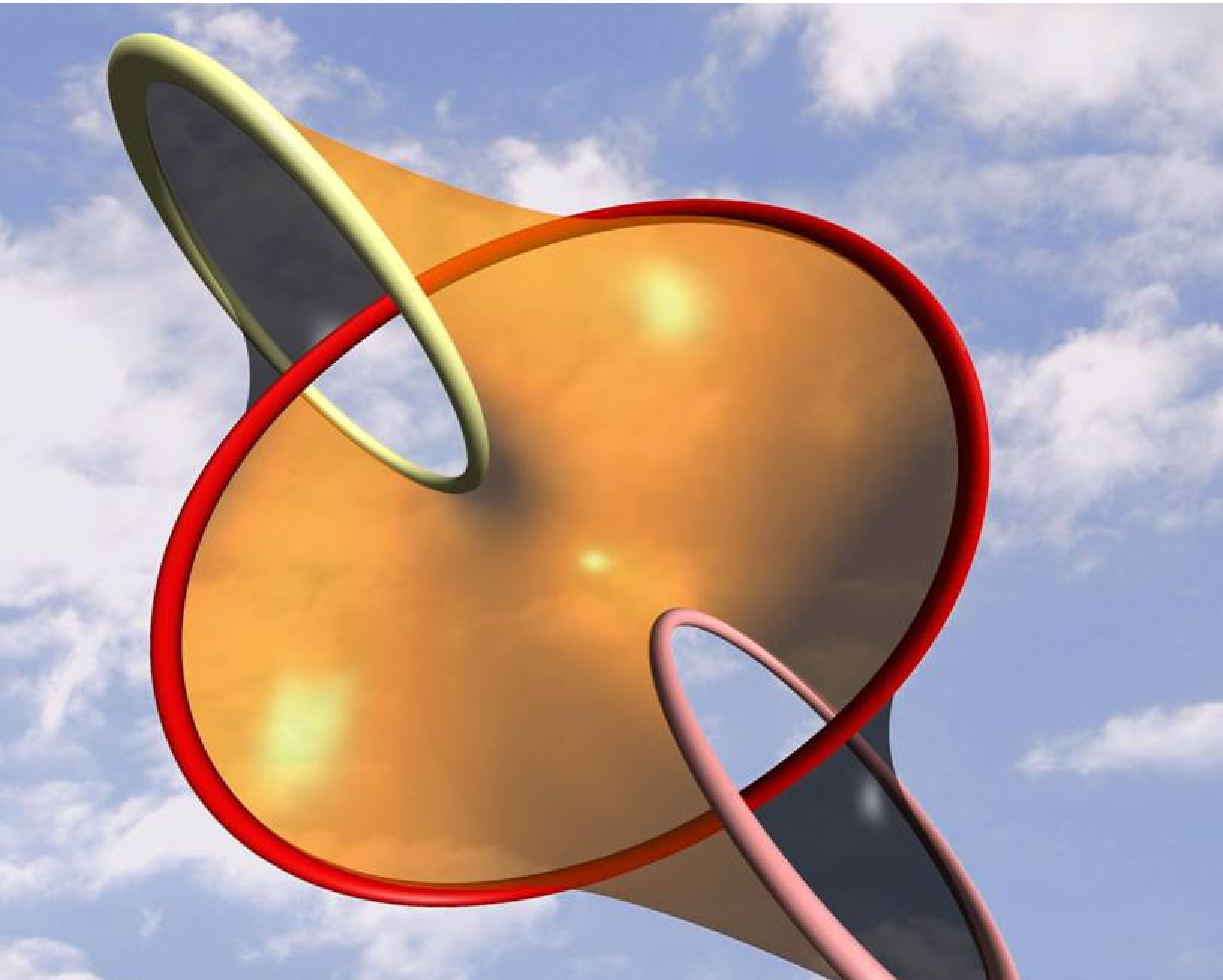
Girls' Angle Bulletin

February 2012 • Volume 5 • Number 3

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

First, one more huge Congratulations to participants at SUMiT 2012. It simply cannot be overstated how wonderful all of you did! There were many extremely challenging problems and you succeeded well beyond all of our expectations.

Second, a big Thank You to Ryan Dembroski and the team at About Face Media for making the video vignette of Girls' Angle Support Network member Elissa Ozanne and to Jan Rimmel for organizing that project. Please help us spread the word about that video. We hope to raise funds so we can create a whole series of video vignettes of the extraordinary women in the Girls' Angle Support Network.

Finally, if there's any topic you'd like us to address in the *Bulletin*, do let us know. We'd love to hear from you!

- Ken Fan, President and Founder

Girls' Angle Donors

Girls' Angle thanks the following for their generous contribution:

Individuals

Marta Bergamaschi
Doreen Kelly-Carney
Robert Carney
Lauren Cipicchio
Merit Cudkowicz
Ingrid Daubechies
Anda Degeratu
Eleanor Duckworth
Vanessa Gould
Andrea Hawksley
Julee Kim
Brian and Darlene Matthews
Toshia McCabe
Alison Miller
Mary O'Keefe
Heather O'Leary
Beth O'Sullivan
Elissa Ozanne
Craig and Sally Savelle
Eugene Sorets
Patsy Wang-Iverson
Mary and Frank Zeven
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathematical Sciences Research Institute

Corporate Donors

Big George Ventures
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
MathWorks, Inc.
Microsoft
Nature America, Inc.
Oracle
Science House
State Street

For Bulletin Sponsors, please visit our website.

Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls Electronic Version (ISSN 2151-5743)

girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: A Seifert surface made using SeifertView, written by Jarke J. van Wijk, Technische Universiteit Eindhoven. SeifertView can be downloaded from the web for free: Google "SeifertView".

An Interview with Dana Pascovici

Dana Pascovici is a biostatician who works at the Australian Proteome Analysis Facility at Macquarie University in Australia. She received her doctoral degree in mathematics from the Massachusetts Institute of Technology.

Ken: Hi Dana, Thank you so much for doing this interview! My first question is: When did you realize that you wanted to be a mathematician? What got you interested in math?

Dana: Surely I am starting the wrong way, because I don't think I ever wanted to be a mathematician! I have liked maths for a long time though, and maybe that started with the (often inspiring and frequently quirky) people involved in national Maths Olympiads back home, in Romania, . . .

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

We will make the rest of this interview available here at some time in the future. But what we hope is that you consider the value of interviews with women like Dr. Dana Pascovici and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider sponsoring the Bulletin. Currently, such sponsorships cost $50/year, and as the number of sponsors increases, this cost will drop. As a sponsor, you will receive a printed copy of this Bulletin with complete contents and no advertisements.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

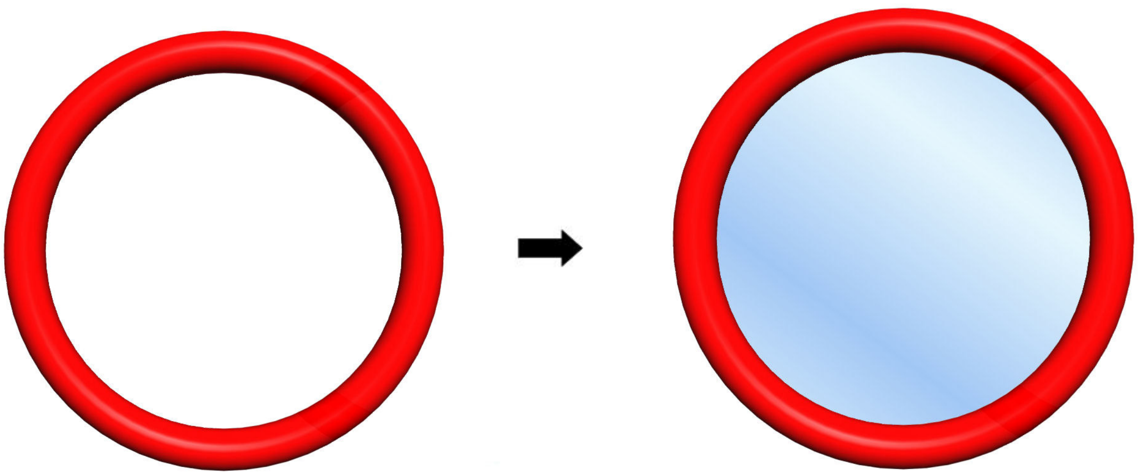
Mathematical Buffet

Seifert Surfaces by Berit Singer and Ken Fan

During Dr. Akveld's visit to Girls' Angle, she explained that every knot can be realized as the edge of an orientable surface. Orientable means that the surface has two sides, unlike the Möbius band (see page 13 of Volume 1, Number 3 of this *Bulletin*). The same is true when you have multiple knots (a.k.a. **links**). For an example, see the cover. The mathematician Herbert Seifert invented a way to construct such surfaces, so these surfaces are known as Seifert surfaces.

The images in this *Mathematical Buffet* were made by Berit Singer and Ken Fan using *SeifertView* a computer program written by Jarke J. van Wijk, Technische Universiteit Eindhoven. You can download this program for free: Google "SeifertView".

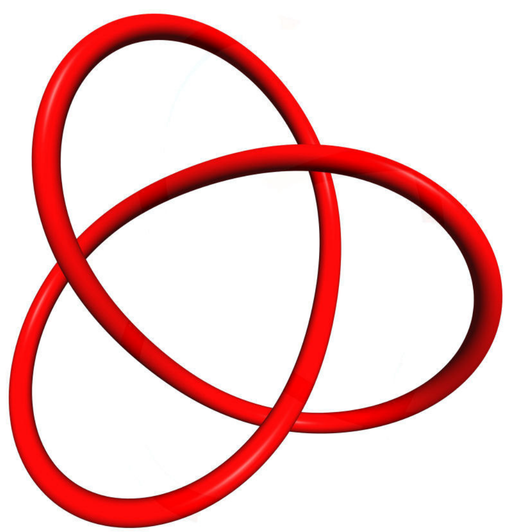
For some knots, like the unknot, it is easy to imagine what a corresponding Seifert surface is:



But it can be challenging to find an oriented surface whose boundary is a given knot. Can you make an oriented surface whose boundary is the trefoil knot shown at right?

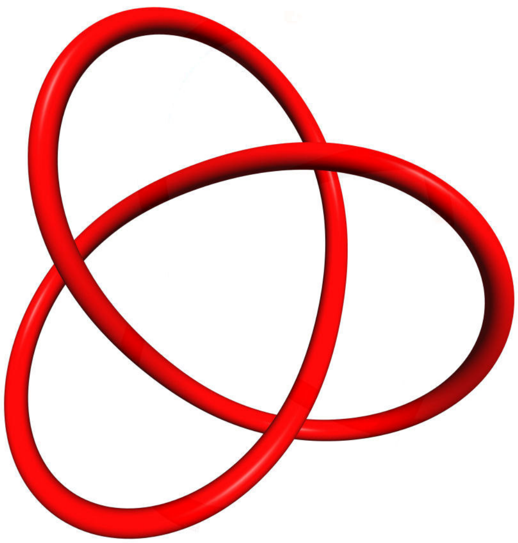
Be careful not to count non-oriented surfaces, such as a Möbius type band with 3 twists.

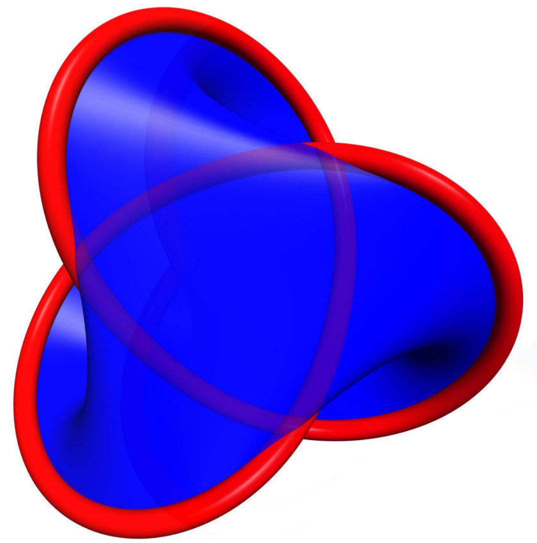
Turn the page to see an answer.

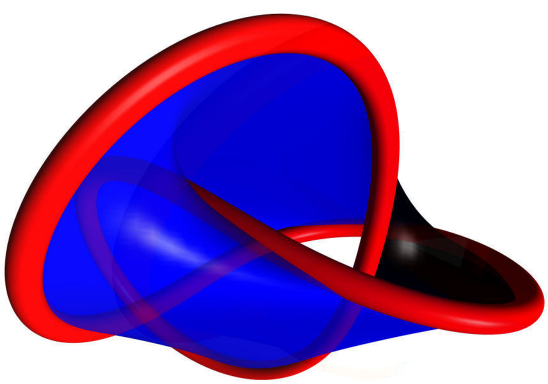


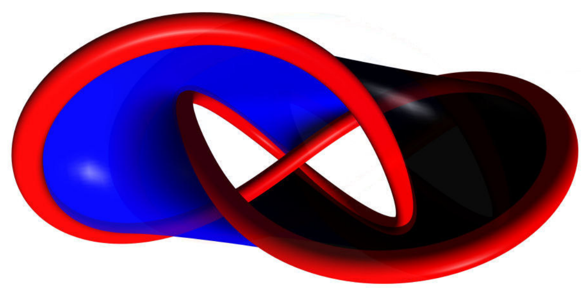
Images on this page by Ken Fan made with SeifertView

The Trefoil Knot

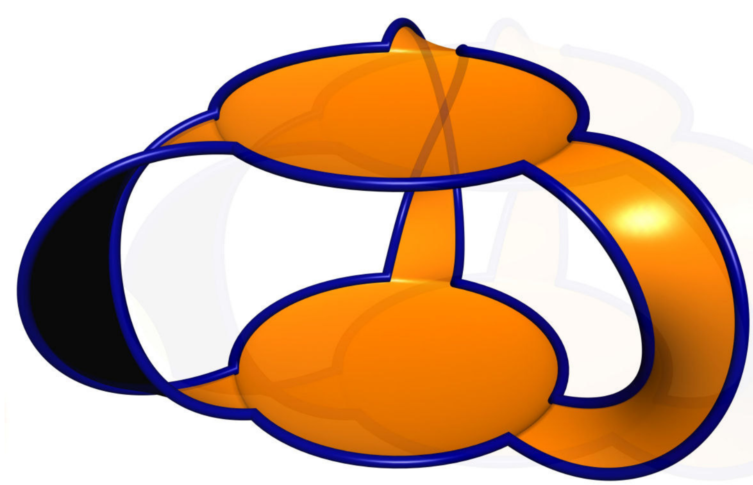




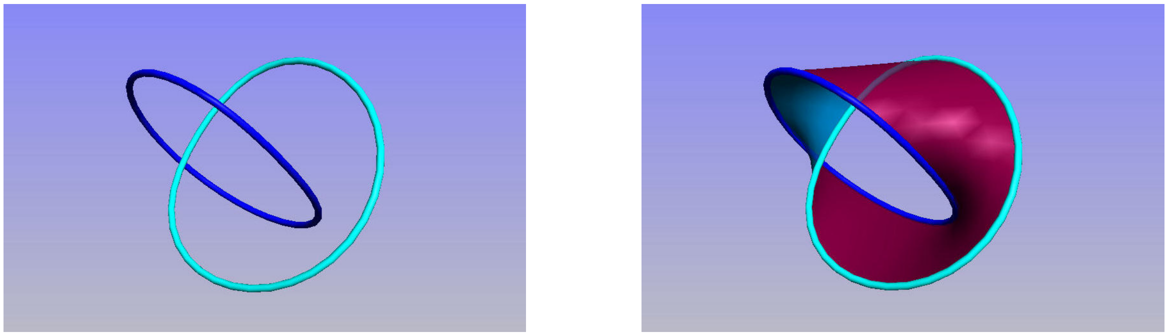




With some paper, scissors, and tape, you can build a model of this Seifert surface. The surface can be molded into the shape shown at right. Cut out 2 circles and 3 paper strips. Use the 3 strips to connect the 2 circles together. Make sure to put a half twist in each strip. If you trace around the edge, you'll find a single loop: the trefoil knot!



Images on this page by Ken Fan made with SeifertView



Hopf Ring

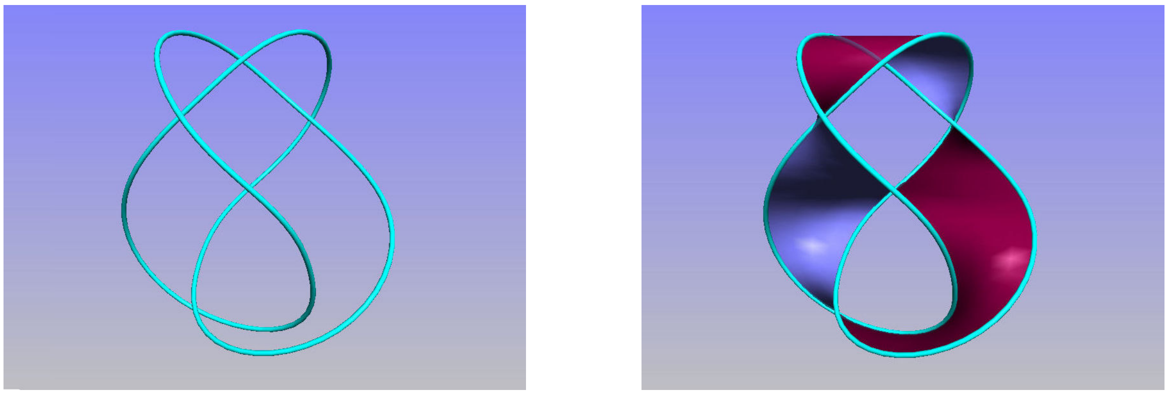
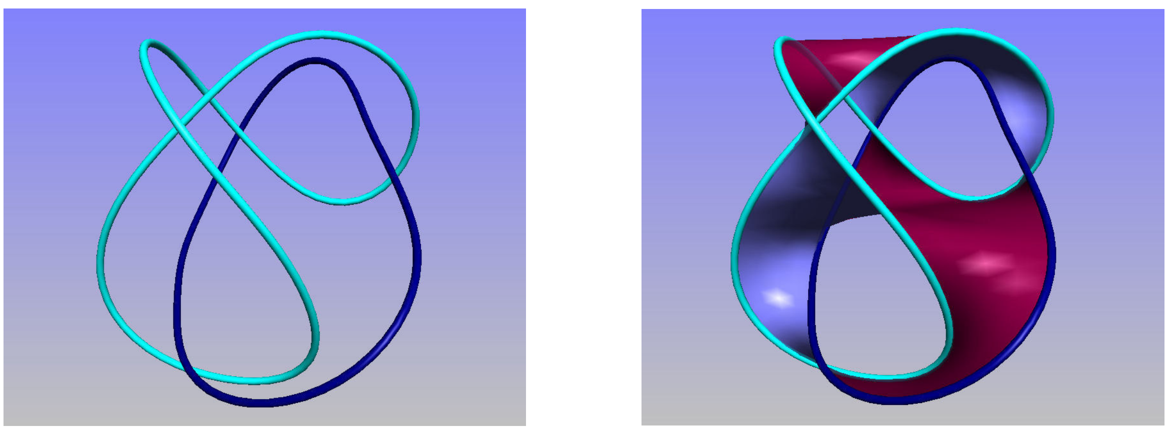
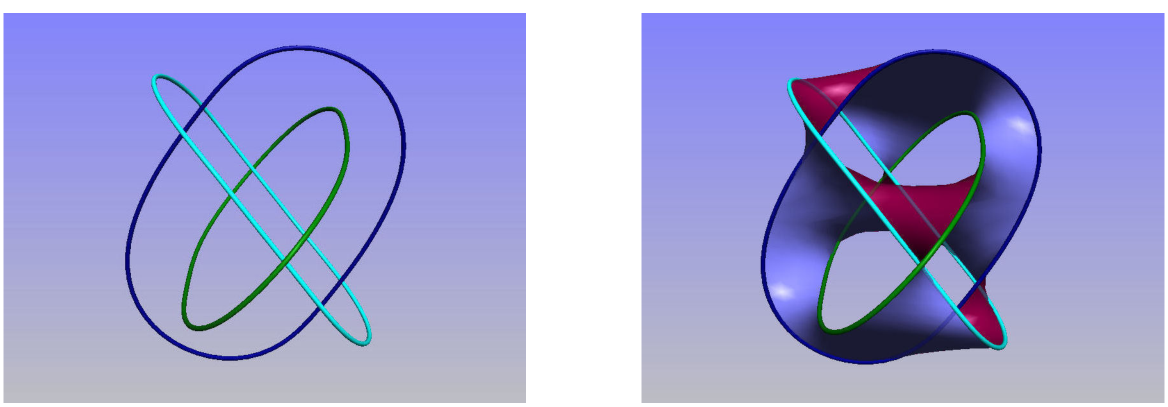


Figure 8



Whitehead Link



Borromean Rings

Images on this page by Berit Singer made with SeifertView

Counting

by Shravas Rao | edited by Jennifer Silva

You might think, "Counting? I learned how to count in first grade!" But there's a lot more to counting than you may imagine. We are often able to describe which things we want to count, but we have to do a bit more work to find out exactly how many of these things there are.

Let's start with an example. We have 3 balls, each a different color – red, green, and blue. There are many ways to arrange these balls in a line. For example, we could put the red ball on the left, the green ball in the middle, and the blue ball on the right. We could also put the green ball on the left, the blue ball in the middle, and the red ball on the right. The problem we want to solve, then, is how many different such arrangements of balls there are. To answer this, we can just list out all of the possibilities (as done in the box at right).

red green blue
red blue green
green red blue
green blue red
blue red green
blue green red

There are no other ways to arrange the balls, so this gives a total of 6 different arrangements. That was easy enough. But what if we add 2 more colors, yellow and orange? How many different arrangements are there now? One approach would be to list out all of the possibilities, like we did before. But this turns out to be rather tedious. In fact, if we have 5 differently-colored balls, there are over a hundred different ways to arrange them in a line. Listing out all of the possibilities would be a lot of work. Maybe there's a different way to approach this problem.

Let's place the balls one by one. This way, we can just look at one ball at a time. We'll start with the leftmost position. Any of the 5 balls can be placed here, so there are 5 ways to place just the first ball. Once we've placed one ball, we can put another ball to its right. Now we only have 4 choices – each of the balls except for the first one (you can't put a ball in two different locations!). So in total, we have $5 \cdot 4 = 20$ ways to place the first two balls. We can continue on in this way to get 3 choices for the next position, then 2, and then 1 for the last, rightmost position. This gives a grand total of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ arrangements. That was a lot easier than listing out all of the possibilities!

If you paid close attention, you might have noticed that our answer for the first example – in which there were only 3 balls – was 6, which is equal to $3 \cdot 2 \cdot 1$. This is not a coincidence. In general, if we have n different objects, the number of ways to arrange them in a row is

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$$

We can reason this out like we did before. For the leftmost position we have n possible objects to place, then $n - 1$ for the second position, and so on, until we only have 1 object for the rightmost position. The product of the first n numbers comes up so often that there's even a special way to notate $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. We write "$n!$" to represent this product. When we read this, we say "n **factorial**." Additionally, as a matter of convention, we declare that $0!$ must be equal to 1. In our context, we interpret this to mean that there is only one way to arrange 0 objects.

Now that we've figured out how to calculate the number of ways to arrange n objects in a line, we can consider a similar type of problem such as this: How many ways can we arrange 3 books side by side on our bookshelf if we can pick these books from a pool of 5 different books? Although this problem may seem completely different, we've actually already solved it. When we wanted to arrange 5 balls, we started with the first ball, then went on to the second, and so on, until we had placed the last ball. This time we have 5 books with room for only 3, so we can do the same thing but stop once we're done with the third book. In this instance, we have just $5 \cdot 4 \cdot 3 = 60$ different possibilities.

Your
Ad
Here

Girls' Angle is now selling advertising space in the electronic version of the Bulletin.

The print version is ad free.

We can generalize this idea to selecting r objects, in order, from a pool of n objects. As before, we can go back to the way we figured out how to arrange all n objects, but we stop once we've selected the rth object. So overall, we have a total of $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ different possibilities (notice that once we've placed $r-1$ balls, there are $n-r+1$ choices to place the rth ball). There is a special name for such products: **permutation**. We typically write $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ as $n\mathrm{P}r$.

You might be interested in the fact that there's a way to express $n\mathrm{P}r$ using factorials: it is $n!/(n-r)!$. To see why this is the case, carefully expand out the factorials on the top and the bottom and then simplify the fraction. You'll find that you get the aforementioned product expression for $n\mathrm{P}r$.

Finally, we will consider one last type of problem. What if there is a group of 5 kids, and we want to choose 3 of them to be on a basketball team together? This may seem pretty similar to the problem with the books, but this time we don't care about what order we choose the kids for our team. For example, if we choose Sally, Anne, and Michelle, we'll end up with the same team as if we choose Anne, Michelle, and Sally. When we count the number of possible teams, we want to make sure we aren't overcounting.

Although we don't want to use permutations, that might be a good place to start since it's fairly close (conceptually) to what we want. Given that we have 5 kids but only want to choose 3 for our basketball team, we could start by pretending that order does matter, thus overcounting the number of teams. Then using our formula for permutations, we have $5!/(5-3)!$.

Now we just have to figure out how to change our overcount into the right answer. For example, the team of Sally, Anne, and Michelle gets counted again as Sally, Michelle, and Anne, and as Michelle, Anne, and Sally, among other ways. In fact, the number of ways a team gets overcounted is the number of ways you can arrange the team in a line. So in our case, we've counted the team with Sally, Anne, and Michelle 3! times, since there are 3! different ways to arrange these 3 girls on the team. Because we overcounted by just this factor for all of the teams, we can divide $5!/(5-3)!$ by 3! to find the correct answer of $\frac{5!}{3!(5-3)!}$.

Let's generalize this fact and figure out how many ways we can choose r objects from a pool of n objects. If we have n objects and want to choose r of them, but we don't care about how we arrange them, then we can start by calculating the permutation $n\mathrm{P}r$. But for each group of r objects, we overcount this group by the number of ways we can arrange these r objects, or by $r!$. So the answer is $\frac{n!}{r!(n-r)!}$. There is a special name for this answer. It is called a **combination**. We typically write this number as $n\mathrm{C}r$ or $\binom{n}{r}$.

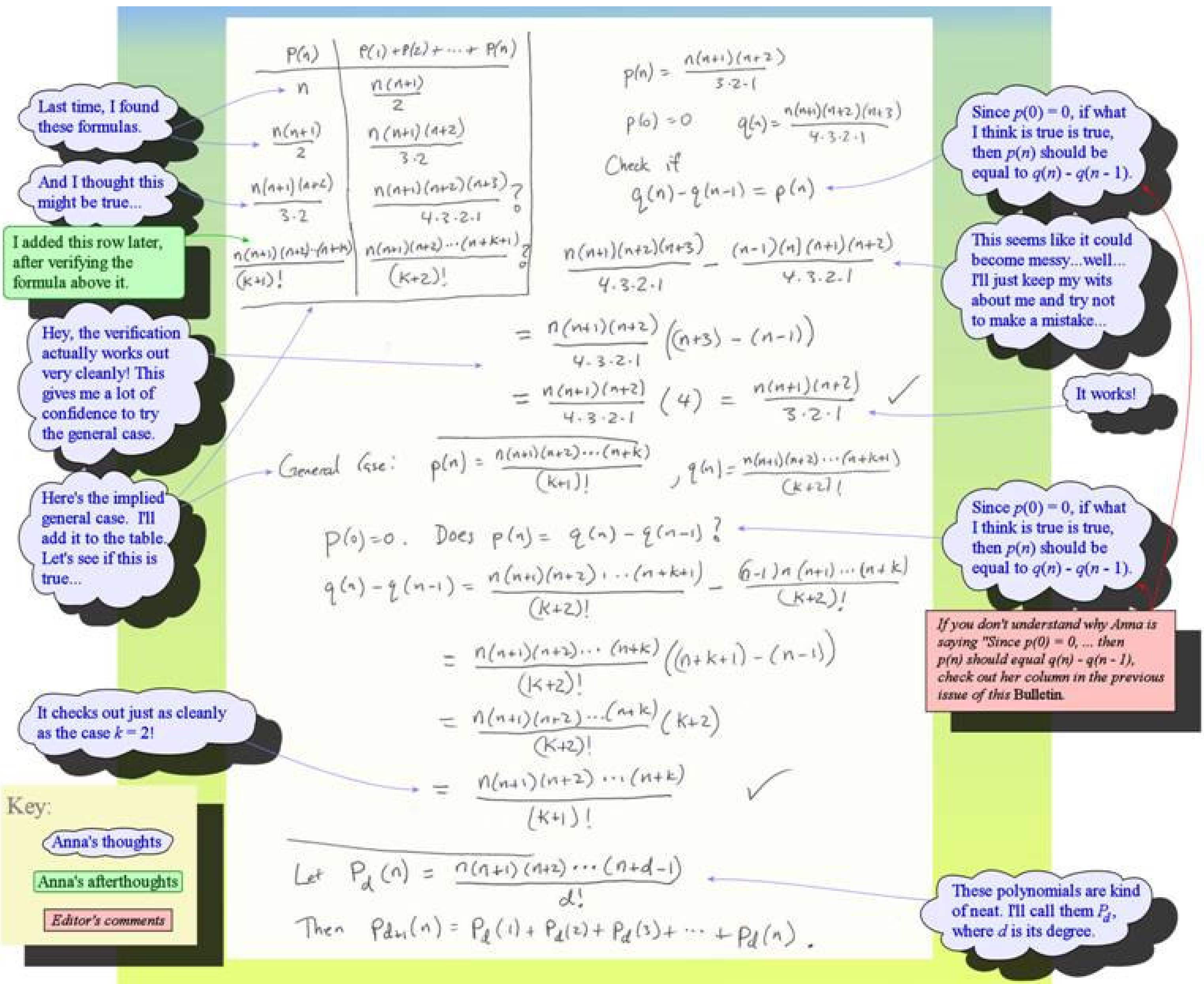
Although permutations and combinations may seem easy to mix up, there is a simple way to tell them apart. If you have a group of people and you want to give out prizes for first place, second place, etc., then you would want to use the formula for permutations to figure out how many ways you can give out prizes. You can think of this as arranging some of the people in a line, then giving the leftmost person first place, the next person second place, and so on. On the other hand, if you want to choose some people to be on a committee, where everyone is equal and two committees are the same as long they have the same people (even if they were chosen in a different order), then you want to use a combination to figure out how many possibilities there are. This is really similar to when we chose people for our basketball teams. Note that the words "permutation" and "prize" start with the same letter, and so do the words "combination" and "committee!"

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna continues her exploration of sums of polynomials.



Your
Ad
Here

Girls' Angle is now selling advertising space in the electronic version of the Bulletin.

The print version is ad free.

I think I'll record some of the nice properties of the polynomials P_d.

$P_d(n)$ is a degree d polynomial —
roots are $0, -1, -2, -3, \ldots, -(d-1)$ —
lead coefficient is $\frac{1}{d!}$

Since $P_1(n) = n$, makes sense to define $P_0(n) = 1$.

But what about powers of n? Shouldn't these be easier because powers of n seem simpler than the polynomials P_d.

What about n^k? What is $1^k + 2^k + 3^k + \cdots + n^k$?

Here are the ones I already know...

$p(n)$	$p(1)+p(2)+p(3)+\cdots+p(n)$
1	n
n	$\frac{n(n+1)}{2}$
n^2	$\frac{n(n+1)(2n+1)}{6}$
n^3	$\left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$
n^4	

I don't know the fourth power of n yet.

$$(n-1)^2 = n^2 - 2n + 1$$
$$(n-1)^3 = n^3 - 3n^2 + 3n - 1$$
$$(n-1)^4 = n^4 - 4n^3 + 6n^2 - 4n + 1$$
$$(n-1)^5 = n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1$$

I put these computations here later, just before I started comparing the coefficients below.

I don't detect any pattern, so I'll go ahead and use the same method I used to find the sum of the first n cubes to find the sum of the first n fourth powers.

$$n^4 = a_5 n^5 + a_4 n^4 + a_3(n^3) + a_2 n^2 + a_1 n - (a_5(n-1)^5 + a_4(n-1)^4 + a_3(n-1)^3 + a_2(n-1) + a_1(n-1))$$

To do this, I need to know the expansion of $(n - 1)$ to various powers, so I'll work those out first.

Coeff of n^4: $1 = 5a_5 \Rightarrow a_5 = 1/5$

n^3: $0 = -10a_5 + 4a_4 \Rightarrow a_4 = \frac{10}{4} \cdot \frac{1}{5} = \frac{1}{2}$

n^2: $0 = 10a_5 - 6a_4 + 3a_3 \Rightarrow a_3 = \frac{1}{3}$

n^1: $0 = -5a_5 + 4a_4 - 3a_3 + 2a_2 \Rightarrow a_2 = 0$
$\quad -1 + 2 - 1$

n^0: $0 = a_5 - a_4 + a_3 - a_2 + a_1 \Rightarrow a_1 = \frac{1}{2} - \frac{1}{5} - \frac{1}{3}$
$\quad 1/5 - 1/2 + 1/3 \qquad = \frac{1}{2} - \frac{8}{15} = -\frac{1}{30}$

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

Here's the result. Hmmm...I guess I'll try to factor it since the earlier results all factored nicely.

$$= \frac{n(6n^4 + 15n^3 + 10n^2 - 1)}{30}$$

$$= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} \qquad -\frac{6}{8} + \frac{9}{4} - \frac{1}{2} - 1 = \frac{12}{8} - \frac{1}{2} - 1 = 0$$

$$= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} \qquad \frac{-3 \pm \sqrt{9 + 12}}{6} = \frac{-3 \pm \sqrt{21}}{6}$$

Curious that n, $n + 1$, and $2n + 1$ are all factors, just as for the sum of the first n squares...hmmm... Well, I don't see any other patterns and I don't feel like doing the fifth powers right now...I'm going to give this a rest and come back later.

ABB 2.27.12

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Can you find a general pattern for the sum of the first n kth powers? Do you see any other patterns? Can you prove them?

MATH IN YOUR WORLD

Logo Design by Hana Kitasei

Strength in Triangles

Written and Illustrated by Katherine Sanden
Edited by Jennifer Silva

Have you ever noticed that many bridges (as well as other structures) are full of triangles? My dad used to point them out to me along the George Washington Bridge as we crossed the Hudson River in New York City.

The photo below captures the triangles present in the towers that support the George Washington Bridge. You can also find triangles in pictures of the Eiffel Tower in France and the Akashi-Kaikyo Bridge in Japan, one of the longest suspension bridges in the world.

Or, look closer to home – I notice triangles in my local train overpass, as well as in cranes at construction sites. See if you can find triangles used as support structures in your own house or school.

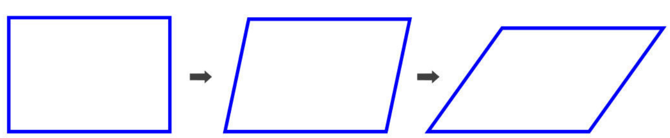
Adapted from a photo courtesy of the Historic American Engineering Record

The George Washington Bridge in New York

And what's with all the triangles? I remember being told, "triangles are stronger than rectangles (and other polygons)." But why? The answer lies in some geometrical observations made over 2,000 years ago by the Greek mathematician Euclid.

Near the beginning of his famous 13-book *Elements*,[1] Euclid noted that if two triangles have the same three side lengths, then they also share the same three angles. In other words, the triangles are *congruent*. In geometry class, this theorem is often referred to as "SSS congruency." The "SSS" is short for "side-side-side."

SSS congruency tells us that once 3 side lengths are determined in a triangle, the angles are also determined. In an engineer's terms, triangles are "rigid." Quadrilaterals, on the other hand, are not. If I take 4 sticks and arrange them into a rectangle, I could easily "squish" the rectangle into a parallelogram. For quadrilaterals, even if all 4 lengths are specified, the angles are still flexible:



These 3 parallelograms have the same side lengths.

[1] Learn more: en.wikipedia.org/wiki/Euclid's_Elements

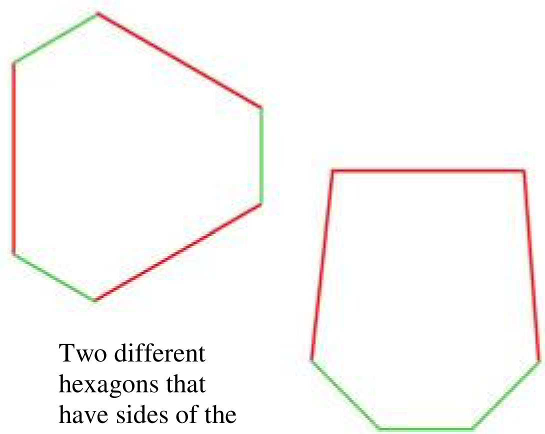
These observations have interesting mathematical implications. SSS congruency tells us that once a triangle's 3 side lengths are specified, the triangle is determined up to congruency. Think about that. That means that every property of a triangle, such as its perimeter or its area, is a function of its 3 side lengths. Indeed, if we let *a*, *b*, and *c* be the side lengths of a triangle, the perimeter is $a + b + c$ and the area is given by Heron's formula, $\sqrt{s(s-a)(s-b)(s-c)}$, where *s* is the **semiperimeter**, $(a + b + c)/2$. You might like to challenge yourself to see if you can find formulas for other quantities associated with triangles in terms of *a*, *b*, and *c*. For example, can you express the radius of the inscribed and circumscribed circles in terms of *a*, *b*, and *c*? How about the lengths of the altitudes? What about the angles?

Now consider this question: Given the 4 side lengths of a quadrilateral, *a*, *b*, *c*, and *d*, what is the area of the quadrilateral? Stop and think about this question for a moment before reading on.

—

It's a bit of a trick question, for *no such formula exists* for general quadrilaterals. That's because, as we noted earlier, quadrilaterals are not uniquely determined by their side lengths. You can fix 4 side lengths and create many different quadrilaterals with different areas. Therefore, there cannot possibly be a formula for the area of a quadrilateral in terms of the lengths of its sides! Remember this argument because it can spare you from spending a lot of effort in the search for a formula that doesn't exist.

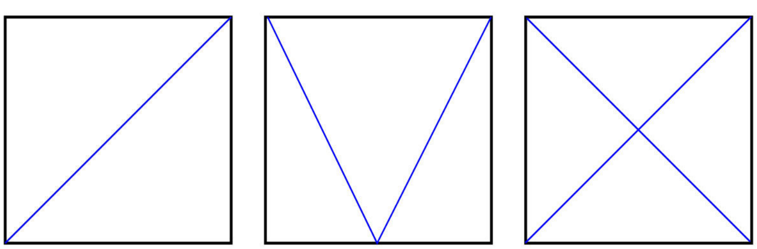
Take it to Your World



Two different hexagons that have sides of the same length.

The lack of rigidity for quadrilaterals is true of other polygons as well. As an exercise, gather four or more pencils together (they could be of equal length or varying lengths). How many different shapes can you make with the same set of pencils? Notice that since the pencils are fixed in length, the *angles* and the *ordering* of the side lengths are what change to produce different shapes.

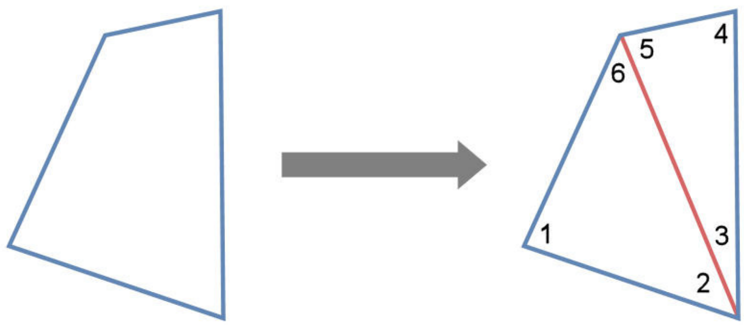
Although triangles can in some ways be thought of as the "strongest" shapes, rectangles are quite useful as well. They allow us to build upward and maintain a level surface, such as a road or floor. Most buildings are filled with them. Revisiting the photograph of the George Washington Bridge on the previous page, I can find plenty of rectangles – what I see is a series of rectangles that are strengthened by the insertion of diagonals, forming triangles inside them. This process of splitting a planar region into triangles is known as *triangulation*. Note that there are many different ways to triangulate a given square.



Three ways to triangulate a square.

Imagine that you are constructing a bridge whose design involves breaking a bunch of squares into triangles. How would you like to break them up? Which way do you think would be strongest? Why?

Triangulation has other applications within mathematics. For instance, it enables us to better understand polygons and their angles. Suppose we know that the sum of the interior angles of a triangle is 180 degrees.[2] We can then take any polygon and strategically triangulate it in such a way that we are able to determine the sum of the polygon's interior angles. Let's apply this first to a quadrilateral:

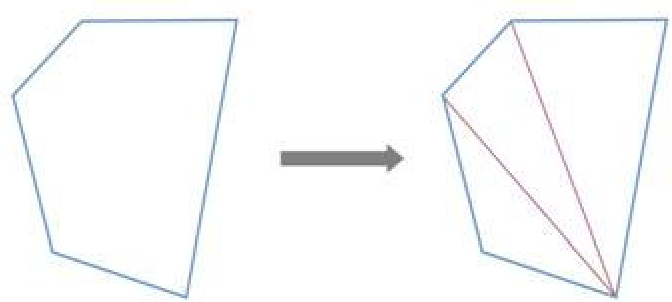


By drawing the red diagonal, I've triangulated the quadrilateral by splitting it into two triangles. I've labeled the angles formed by the numbers 1 through 6. Such a triangulation is possible for any quadrilateral. (Check this for yourself!) The sum of the interior angles is then

$$\begin{aligned}\angle 1 + (\angle 2 + \angle 3) + \angle 4 + (\angle 5 + \angle 6) &= (\angle 1 + \angle 2 + \angle 3) + (\angle 4 + \angle 5 + \angle 6) \\ &= (\angle 1 + \angle 2 + \angle 6) + (\angle 3 + \angle 4 + \angle 5) \\ &= (180 \text{ degrees}) + (180 \text{ degrees}) \\ &= 360 \text{ degrees.}\end{aligned}$$

So the sum of the interior angles of any quadrilateral must be 360 degrees.

Similarly, we could break a pentagon into three triangles (as shown below), and find that the sum of the interior angles of a pentagon is equal to three times the sum of the interior angles of a triangle, or $3 \times 180 = 540$ degrees. Notice that in both the quadrilateral and the pentagon, we were careful to use a triangulation whose constituent triangles had vertices that were also vertices of the original polygon. Why do you think that's important?



Try breaking a hexagon into triangles and find the sum of its interior angles. How about a heptagon (a seven-sided polygon)? Can you derive a general formula? That is, what is the sum of the interior angles of an *n*-sided polygon?

Challenge

Even though there is no formula for the area of a general quadrilateral in terms of its four side lengths, it is conceivable that there could be such a formula for the area of a restricted class of quadrilaterals. Can you find a formula for the area of a **cyclic** quadrilateral in terms of the lengths of its sides? A cyclic quadrilateral is a quadrilateral whose vertices sit on the circumference of some circle.

[2] Do you know this? Can you prove it?

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

As promised in my most recent column, this time we will discuss **function composition**. Last time I mentioned that I had noticed several errors related to a question I posed on my precalculus final exam. The question asked students to find the composition $f \circ g$ of the functions $f(x) = x^2 + 3x$ and $g(x) = x + 3$. In the last issue, we looked at the errors made by those who composed the functions correctly but then simplified incorrectly. Now we'll address the errors made by those who did not understand function composition in the first place.

The most common error that I saw was multiplying instead of composing: a few students found the "composition" of f with g by simply multiplying the functions together, getting something incorrect like this:

$$f \circ g\,(x) = (x^2 + 3x)(x + 3).$$

Function composition is *not* multiplication and must be computed differently.

To properly understand function composition, we must first understand what a function itself is. In order to define functions, we need the concept of sets, which we shall simply take to be collections of mathematical objects. (This naïve definition of a set can lead to paradoxes, but for now it will suit our purposes.) Examples of sets include the set of natural numbers {0, 1, 2, 3, ...} (denoted by **N**), the set of integers {0, ±1, ±2, ±3, ...} (denoted by **Z**), the set of rational numbers (that is, all numbers that can be expressed as the ratio of two integers, denoted by **Q**), and the set of real numbers (which are modeled by points on the number line, denoted by **R**). Sets don't have to be collections of numbers. For example we can consider the set of points in a plane.

To define a function, we need two sets A and B. It's OK if $A = B$. A function from A to B is something that associates to each element of A an element of B. We can think of the elements of A as the set of inputs to the function. The set A is also called the **domain** of the function. The elements of B can be thought of as the set of possible outputs, although not every element of B must be associated with some element of A. The set B is also known as the **range** of the function. For each input – that is, for each element of A – there is exactly one output – that is, exactly one element of B – paired up with it. We can't put in the same input twice and get two different outputs. If f is a function from A to B and x is an element of A, the element of B that the function f pairs up with x is denoted by "$f(x)$," which is stated "f of x."

Some basic examples of functions are the identity function, which is a function whose range and domain are equal and which pairs each element in the domain with itself, (that is, $f(x) = x$), and the constant functions that map every element in the domain to the same element in the range.

When a function is defined, you have to specify the domain, the range, and the pairing.[3] When defining the pairing, you must make sure that you are truly specifying a function. For instance, if we were to attempt to defined a function from the rational numbers, **Q**, to the integers, **Z**, we would be committing an error if we declared that $h(p/q) = p + q$. The reason is that this rule does not assign a unique number to each rational number. For example, I can

Mathematician Timothy Gowers wrote a nice blog post on functions. See gowers.wordpress.com/2011/10/13/domains-codomains-ranges-images-preimages-inverse-images/

[3] In high school, it is often assumed that all functions map from a subset of the real numbers to the set of real numbers and the domain is assumed to be the largest subset of the real numbers where the function rule makes sense. These definitions differ from those used by professional mathematicians, and they overly restrict the concept.

express 0 as both 0/1 and 0/2, and this rule would assign 1 to the first expression and 2 to the second. So we can see that this rule is ill-defined and does not define a function.

The notion of function is important to nearly all of mathematics. You may have already encountered a variety of functions, such as polynomial, rational, exponential, logarithmic, and trigonometric functions.

Now that we know what a function is, let's turn our attention to function **composition**. Suppose we have three sets called A, B, and C, and two functions f and g. Assume that the domain of f is B and its range is C, and the domain of g is A and its range is B. In other words, f maps from set B to set C while g maps from set A to set B. In this setup, one can imagine creating a new function whose domain is A and whose range is C by pairing each element in A with an element C in the following way: we first map the element of A to an element of B using the function g, and then map this element of B to an element of C using the function f. The resulting function is what is known as the composition of the functions f and g and is written like this: $f \circ g$. We state this as "f of g" or "f composed with g." So, $f \circ g$ is a single function whose domain is A and whose range is C. It is the function that you get by "first doing g and then doing f." In other words, the value $f \circ g\ (x)$ is what you get when you plug the output of g (at the value x) in as the input of f; that is, $f \circ g\ (x) = f(g(x))$.

Function Shorthand

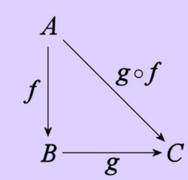
Let f be a function whose domain is the set A and whose range is the set B. Mathematicians will often notate this situation by writing:

$$f : A \to B,$$

or

$$A \xrightarrow{f} B\ .$$

This latter notation is particularly nice for function composition. We can make diagrams like this:



Notice that we can only compose two functions if the range of one is a subset of the domain of the other.

Let's look at an example of function composition. Consider the functions f and g from **R** to **R** given by $f(x) = x^2 + 3x$ and $g(x) = x + 3$. The composition of f with g is given by

$$f \circ g\ (x) = f(g(x)) = f(x + 3) = (x + 3)^2 + 3(x + 3) = x^2 + 6x + 9 + 3x + 9 = x^2 + 9x + 18.$$

Another way to see this is to write

$$f \circ g\ (x) = f(g(x)) = (g(x))^2 + 3(g(x)) = (x + 3)^2 + 3(x + 3) = x^2 + 9x + 18.$$

When composing two or more functions, pay close attention to which function is evaluated first, especially when the domains and ranges of both functions are all equal to the same set: $f \circ g\ (x) = f(g(x))$ means to first evaluate g at x, and then evaluate f at $g(x)$. This is very important because with function composition, the order of the functions usually matters. For instance, if we compute $g \circ f$ for the same functions as in the last paragraph (which makes sense since the range of f is equal to the domain of g), we find that

$$g \circ f\ (x) = g(f(x)) = g(x^2 + 3x) = (x^2 + 3x) + 3 = x^2 + 3x + 3.$$

So in this case, $f \circ g$ is a completely different function from $g \circ f$.

Let's compute another example. This time, suppose that f maps from the set of nonzero real numbers to the set of nonzero real numbers and is given by the function rule $f(x) = 1/x$. Let g also map from the set of nonzero real numbers to the set of nonzero real numbers according to the rule $g(x) = 4x^2$. (You may have noticed that the domain of g could be extended to all real numbers, but we are restricting to nonzero real numbers so that the domains and ranges match up and it makes sense to compute both $f \circ g$ and $g \circ f$. Please verify for yourself that both f and g are valid functions when their domains and ranges are restricted to the set of nonzero real numbers.) Let's compute $f \circ g$ and $g \circ f$. We find

$$f \circ g\ (x) = f(g(x)) = f(4x^2) = 1/(4x^2)$$

and

$$g \circ f\ (x) = g(f(x)) = g(1/x) = 4(1/x)^2 = 4/x^2.$$

Again, we see that $f \circ g$ is not the same function as $g \circ f$.

So why would one compute the incorrect composition

$$f \circ g\ (x) = (1/x)(4x^2) = 4x?$$

My best guess is that composition is being confused for multiplication of functions. When we want to express the product of two functions, we write "$f(x)g(x)$." Sometimes you will also see the notation "$f \cdot g$," which looks an awful lot like function composition! So you have to take care and be aware of the context at all times. Composition of two functions and multiplication of two functions are very different notions. Notice, for instance, that multiplication of functions *only makes sense* when both functions have the same domain *and* multiplication makes sense between elements in their respective ranges. If, for instance, the ranges were the set of triangles in a plane, then multiplication would not make sense (unless you're able to make sense of the notion of multiplying two triangles together). However, it might still make sense to *compose* functions whose domains and ranges were the set of triangles in a plane. When composition is called for, be sure not to multiply!

For each problem, write "N/A" under the column $f \circ g\ (x)$ if $f \circ g$ does not make sense; otherwise, state the domain and range of $f \circ g$ and give an expression for $f \circ g\ (x)$. Then do a similar thing for $g \circ f$. Here, P is the set of points in the xy-coordinate plane. (Recall that if A and B are sets, then "$A \backslash B$" denotes the set consisting of all elements in A that are not in B. It is read "A set minus B.") Answers are on page 31.

	f		g		$f \circ g\ (x)$	$g \circ f\ (x)$
1.	$f : \mathbf{R} \to \mathbf{R}$	$f(x) = 2x$	$g : \mathbf{R} \to \mathbf{R}$	$g(x) = x + 5$		
2.	$f : \mathbf{R} \to \mathbf{R}$	$f(x) = x^2 - 4$	$g : \mathbf{R} \to \mathbf{R}$	$g(x) = x + 2$		
3.	$f : \mathbf{R} \to \mathbf{R}$	$f(x) = x - 3$	$g : \mathbf{R} \to \mathbf{R}$	$g(x) = x + 3$		
4.	$f : \mathbf{R}\backslash\{-1\} \to \mathbf{R}$	$f(x) = 2x/(x + 1)$	$g : \mathbf{R} \to \mathbf{R}$	$g(x) = x^2 - 1$		
5.	$f : \mathbf{Z} \to \mathbf{Z}$	$f(x) = x + 1$	$g : \mathbf{Z} \to \mathbf{R}$	$g(x) = x/2$		
6.	$f : \mathrm{P} \to \mathrm{P}$	$f(x)$ = the reflection of x in the x-axis	$g : \mathrm{P} \to \mathrm{P}$	$g(x)$ = the reflection of x in the line $y = x$		

COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 3

Great news! Just as you are considering pulling out all of your hair because your know-it-all frenemy is driving you nuts with her endless bragging (and her parents aren't due to collect her anytime soon), here comes your devoted friend, 3/7, to your rescue.

$\frac{3}{7}$: Hi dearies. Let's talk multiplication … with fractions!

Frenemy: Actually, I know that already. It's easy. In fact it's my favorite thing to do with fractions because you just do the obvious thing – multiply the two top numbers and put them on top of a new fraction, and multiply the two bottom numbers and put them on the bottom of the new fraction.

$\frac{3}{7}$: Why do you do that?

Frenemy: I do it because it's the right thing to do. I know it.

$\frac{3}{7}$: Well dearie, if you don't know why you are doing it, you don't really know it.

Frenemy: I do too know it! I get 100s on all my worksheets.

$\frac{3}{7}$: Let's change the subject a bit from you and your worksheets, shall we? In fact let's change the subject from multiplication of fractions for a moment and just focus on multiplication with whole numbers.

Frenemy: Okay, fine. I'm good at that. Wanna hear my times tables?

$\frac{3}{7}$: Some other time, darling, some other time. What I want to know is what IS multiplication?

Frenemy: Oh, that's easy. It's just how many of a number to add up. Say, 3×5. You can add up three 5's or five 3's. You get the same answer either way.

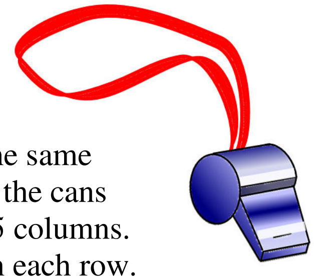
$\frac{3}{7}$: Why do you get the same number either way?

Frenemy: It just works that way. Wanna hear about my grades?

$\frac{3}{7}$: Some other time, darling, some other time. I do notice your quiet little friend here is arranging 15 soda cans. Sweetums, tell us why, please.

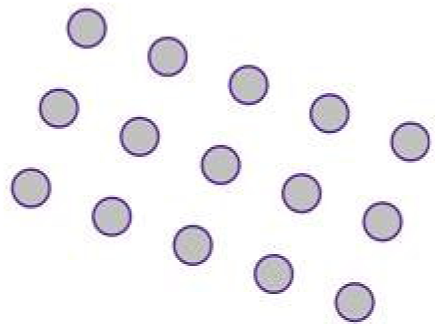
You: Well, I was just thinking about a good way to show why order doesn't matter in multiplication.

$\frac{3}{7}$: Wonderful! Do tell, darling.



You: Well, sometimes you can arrange a group of things into rows that each have the same number of things. You can do that with 15. I made 3 rows of 5 cans. If you line up the cans neatly, you can make columns also, with one can from each row. So here we have 5 columns. Well, the total number of things is the number of rows times the number of things in each row. So we see $15 = 3 \times 5$. But then you can walk to another side of the rectangle that you made and the columns turn into rows and the rows turn into columns. Of course, the number of things is still the same – 15 – and it is still the number of rows times the number of things in each row. So you can see $15 = 5 \times 3$, also.

$\mathbf{\frac{3}{7}}$: That's simply lovely, darling. Your explanation is poetry to my ears and art to my eyes.



You: Thank you, ma'am.

$\mathbf{\frac{3}{7}}$: Not that it matters terribly much, but do you remember the names we give to amounts we can put into rectangular arrangements? Of course, I'm not talking about silly rectangles with only one row or column.

You: Composite numbers – those are the kind we can make into non-silly rectangles.

Frenemy: Nooo! Composite numbers are numbers that are not prime. I know that. I know what prime numbers are too. They are whole numbers with exactly two different divisors.

$\mathbf{\frac{3}{7}}$: Math definitions are not bullets to shoot at people; they are meant to convey meaning. Although I must say I do delight in employing the bloated English language to lob an obscure word at people just to befuddle them. But that's sport; this is math, where everything makes sense. You with the cans, dear – can you try to enlighten your little friend? I do find your demonstrations charming.

You: Well … when you arrange a number of cans into a rectangle, you are showing that number as a product of the number of rows times the number of columns (or the number of columns times the number of rows). Of course, any number of cans can be arranged into a silly rectangle with a single row (or a single column), which is the same as saying that every number is divisible by 1 and itself.

$\mathbf{\frac{3}{7}}$: Very nice. Carry on, dumpling. Carry on.

You: Then not-prime, or composite, is exactly when you can arrange a number into a non-silly rectangle, because a non-silly rectangle has more than one row and more than one column; this shows that the number has divisors other than one and the number itself. So the official definitions and the soda can ones do say the same thing.

Frenemy: Hey! I know the phrase for the order of the two factors not mattering!

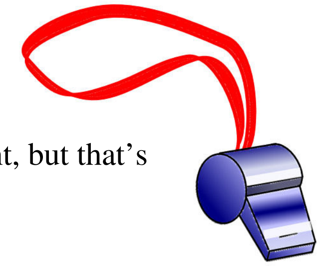
$\mathbf{\frac{3}{7}}$: Hay is for horses, snookums. Do share the phrase, though.

Frenemy: It's the Commutative Property of Multiplication!

Your
Ad
Here

Girls' Angle is now selling advertising space in the electronic version of the Bulletin.

The print version is ad free.



$\frac{3}{7}$: Indeed it is, darling. I like to say, “multiplication commutes,” just to be different, but that’s just me.

Frenemy: Ooooh. I’m going to say it that way, too.

$\frac{3}{7}$: Although I generally do know practically everything, I must admit I am stumped as to why you are in possession of such an extraordinary number of soda cans.

You: She drinks lots of soda.

Frenemy: I drink 15 cans a day!

$\frac{3}{7}$: Horrors! You will rot your teeth!

Frenemy: Oh don’t worry, it’s diet.

$\frac{3}{7}$: Even worse! Aaah! I better not get started. Here, have this nice glass of water and tell me all you know about how multiplication of fractions relates to multiplication of whole numbers.

Frenemy: What are you talking about? They have nothing to do with each other. We learn them in different grades. Hey, why are you falling over?

$\frac{3}{7}$: Shock, darling. Shock and despair. Please say more; I will try to bear it.

Frenemy: Well, remember how I said for 3×5 you can add up three 5’s or five 3’s?

$\frac{3}{7}$: Yes, dearie. That’s certainly true and making me feel a bit better to hear you say so.

Frenemy: Well when you’re multiplying, say 3/5 and 3/8, you cannot add 3/5ths 3/8ths times, nor can you add 3/8ths 3/5ths of a time. It makes no sense!

$\frac{3}{7}$: Math always makes sense!

You: Calm down. I think I can help you.

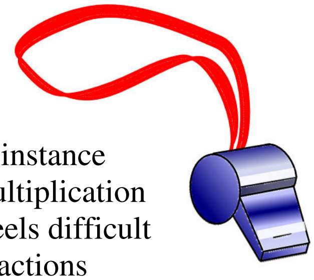
$\frac{3}{7}$: Oh dear, I have broken my vow to Keep Calm and Carry On. I just become incensed whenever someone says … oh dear … what she said.

You: The problem isn’t that multiplying fractions doesn’t make sense.

$\frac{3}{7}$: That’s right, it’s math. It makes sense. The people who designed multiplication chose a sensible way to proceed.

You: The problem is that you’re thinking about multiplication in a way that only seems sensible for whole numbers. The way you picture multiplication in your head – your “model of multiplication” – is not necessarily identical to what multiplication actually is.

Frenemy: What? So what is it?



$\frac{\mathbf{3}}{\mathbf{7}}$: Manners, darling. As we enlarge the types of factors that we're multiplying, for instance from whole numbers to fractions, it can help to have a more general model of multiplication that makes sense for all the numbers. If counting partial rows of partial soda cans feels difficult to imagine, try to dream up a different model of multiplication where multiplying fractions doesn't cause you difficulty.

Frenemy: Just tell me how multiplying fractions and multiplying whole numbers can be thought of as the same thing! Now!

$\frac{\mathbf{3}}{\mathbf{7}}$: Hrmph! Listen here, you little...

You: Excuse me. The way I picture multiplication so it's applicable to both fractions and whole numbers is to think about the area of a rectangle.

$\frac{\mathbf{3}}{\mathbf{7}}$: Oh, me too, darling! I am, however, too weak to explain it to your trying little friend. Might you give it a go?

You: Sure. See, distances don't just come in whole numbers. Sides of a rectangle may have lengths in between two whole numbers. But all rectangles have an area, and that area is the product of the length and the width whether the length and width are whole numbers or not. So I make one of the factors the length of a rectangle and the other its width.

Frenemy: Area of a rectangle equals length times width. Area of a triangle...

$\frac{\mathbf{3}}{\mathbf{7}}$: Some other time, darling, some other time. The area model of multiplication is so clarifying. I do love it so. Go on, sweetheart.

You: With the two factors corresponding to the length and width, the product corresponds to the area of the rectangle. The unit with which you measure the answer – product, I mean – is a square with sides of length one.

$\frac{\mathbf{3}}{\mathbf{7}}$: I call it a unit square.

Frenemy: What's the point?

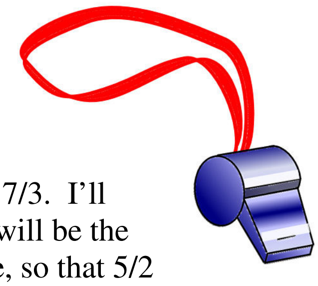
$\frac{\mathbf{3}}{\mathbf{7}}$: Oh, darling, it may be too much to ask, but could you use the lovely area model to "see" the multiplication formula for fractions?

You: Certainly. I'll explain by giving an example, and I'll leave it to you two to see how it relates to the formula since there is nothing special about the particular numbers I use.

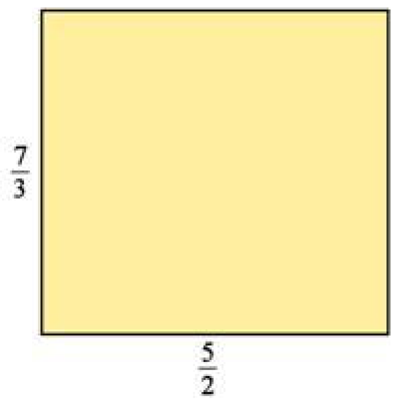
Frenemy: What do ya mean, nothing special?

You: I mean that my description doesn't require the use of the particular numbers I'm choosing.

$\frac{\mathbf{3}}{\mathbf{7}}$: The human mind is remarkably able to generalize from an example. Too able, I'd say, because it is possible to overgeneralize.

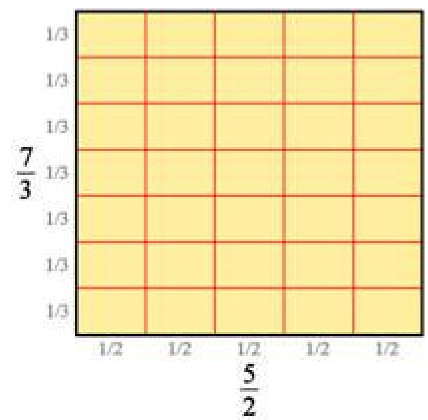


Frenemy: So are we ever going to hear this eighth wonder of the world?

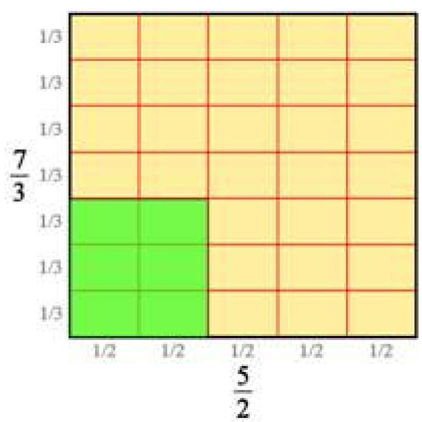


You: I'll ignore that. Say we want to multiply 5/2 and 7/3. I'll make a rectangle with those two side lengths. Its area will be the answer we want. Then I'll draw a grid on the rectangle, so that 5/2 is split into halves and 7/3 is split into thirds.

Frenemy: Halves and thirds are unit fractions!



You: Anyway, that grid splits up the rectangle into many smaller rectangles. Since there are 5 halves in 5/2 and 7 thirds in 7/3, we get 5 × 7 or 35 of the small rectangles altogether. Notice that we are multiplying the two numerators to get this. It's just like the soda cans.



Next, we find the area of a small rectangle. What I do is first shade in a unit square with side length one. I made it green here. It has area of one. Then I see that the little rectangles fit in exactly – two on one side and three on the other, so 2 × 3 or 6 altogether. Notice here that we are multiplying the two denominators of the original factors together. Since 6 identical little rectangles fit in the unit square exactly, each one has area of 1/6.

Finally, since the product we want is the area of 35 of the small rectangles that each have area 1/6, we know that the product is 35 × 1/6 or 35/6. That's one way to "see" that $a/b \times c/d = (ac)/(bd)$.

$\frac{3}{7}$: Yes, darling. I did love that clear view of the area model of multiplication. It reminds me of some of my other favorite ways to envision multiplication … the stretching model, for instance …

You: Oooh, stretching. I'll have to consider that way of thinking about it.

$\frac{3}{7}$: Oh yes, dear, you must. One of my very favorite activities is to mull over the different models of multiplication. Consider under what circumstances each is appropriate, useful, easiest … oh I could go on and on!

Frenemy: Well that's all fine, but now I want to tell you the rule for dividing fractions. I always get those right, too.

$\frac{3}{7}$: Some other time, darling, some other time. I did so much want to talk division, but that will have to wait for another time. It has been a pleasure. Too-da-loo!

You: Good-Bye!

Frenemy: Bye! Oh hi, Mom and Dad. Boy, do I have a lot to tell you!

Your
Ad
Here

Girls' Angle is now selling advertising space in the electronic version of the Bulletin.

The print version is ad free.

Fraction Worksheet

This worksheet contains problems about fractions of varying levels of difficulty. What *fraction* of the problems can you get?

Let's start with fraction arithmetic. Evaluate the following expressions and express your answer as a fraction in lowest terms.

1. $\frac{1}{2}+\frac{1}{3}$
2. $\frac{3}{4}+\frac{2}{3}$
3. $3+\frac{7}{6}+\frac{2}{3}$
4. $\frac{5}{6}+\frac{3}{10}+\frac{4}{15}$
5. $\frac{1}{2}-\frac{1}{3}$
6. $\frac{2}{5}+\frac{1}{9}-\frac{1}{3}$
7. $\frac{10}{11}+\frac{11}{12}-\frac{12}{13}$
8. $\frac{1}{7}-\frac{1}{4}$
9. $\frac{1}{2}\cdot\frac{1}{3}$
10. $\frac{5}{2}\cdot\frac{2}{5}$
11. $\frac{10}{3}\cdot\frac{9}{5}$
12. $\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{6}{5}$
13. $\frac{1}{2}\div\frac{1}{3}$
14. $\frac{5}{2}\div\frac{2}{5}$
15. $\frac{1}{2}\cdot\frac{5}{3}\div\frac{5}{6}$
16. $\frac{3}{4}\div(\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{7})$

17. Show that $\frac{1}{n-1}-\frac{1}{n+1}=\frac{2}{n^2-1}$.

18. Use the previous problem to simplify $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+\frac{1}{143}+\frac{1}{195}+\frac{1}{255}+\frac{1}{323}+\frac{1}{399}$.

19. After Johnnie did this fraction worksheet, he claimed to have gotten $\frac{7}{8}$ of the problems correct. How can you tell that Johnnie is lying?

20. For every positive integer n, show that $\frac{n(n+1)}{2}$ is an integer.

21. For every positive integer n, show that $\frac{n(n+1)(n+2)}{6}$ is an integer.

22. More generally, can you show that the product of k consecutive integers is always divisible by $k!$ (that's k factorial)?

23. Give an example of four numbers a, b, c, and d where $a/b \neq (a + c)/(b + d)$.

24. Suppose that $a/b = c/d$ and $b + d \neq 0$. Show that $a/b = c/d = (a + c)/(b + d)$.

25. Assume that a, b, c, and d are all positive and $a/b \leq c/d$. Is it true or false that

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}?$$

Notes from the Club

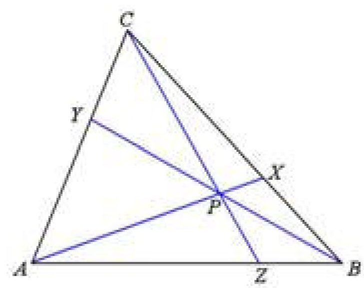
These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

<u>Session 10 – Meet 1 – January 26, 2012</u>

Mentors: Jennifer Balakrishnan, Connie Liu, Liz Simon

Members studied intersections between geometric objects, least common multiples, greatest common factors, and Ceva's theorem all in the context of a treasure hunt.

Ceva's theorem is about line segments that join a vertex of a triangle to a point on the side opposite that vertex. Such line segments are known as **Cevians**.



Ceva's theorem says that if three Cevians AX, BY, and CZ of triangle ABC intersect in a common point P (as in the figure at right), then

$$\frac{AZ}{ZB}\frac{BX}{XC}\frac{CY}{YA} = 1.$$

The converse of this statement is also true and provides a quick way of seeing that the medians of a triangle intersect in a point.

Try to prove this. If you run into trouble, there are many proofs on the web that you can study. Here, we'll draw attention to the trigonometric form of Ceva's theorem. Using the labeling in the figure above, Ceva's theorem is equivalent to: If three Cevians AX, BY, and CZ of triangle ABC intersect in a common point P, then

$$\sin \angle ABP \cdot \sin \angle BCP \cdot \sin \angle CAP = \sin \angle ACP \cdot \sin \angle CBP \cdot \sin \angle BAP.$$

Try to prove this too! One way to proceed is to apply the formula $\frac{1}{2}ab \sin \theta$ for the area of a triangle (that has an angle of measure θ sandwiched between two sides of lengths a and b) to the six triangles into which the Cevians split the original triangle.

Laura observed a nifty fact about tables of greatest common divisors. She noticed that if you multiply every number in the table by 10 (including the row and column headings), the result will also be a valid table of greatest common divisors. Let (a, b) denote the greatest common divisor of a and b. Another way of putting what **Laura** observed is that $(10a, 10b) = 10(a, b)$. More generally, because you can multiply by 10 again, or as many times as you wish, it's also true that $(10^n a, 10^n b) = 10^n(a, b)$. That's a very nice observation and begs the question: is there something special about the number 10 that makes this work? What happens if the 10 is replaced by another number, like 2 or 3, for instance? Or, more generally, is it true that $(Na, Nb) = N(a, b)$?

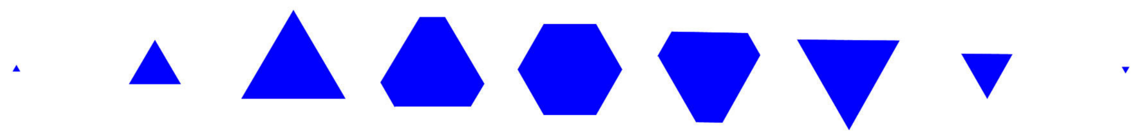
Another noteworthy fact that arose at this meet concerns fractions. Let a/b and c/d be two fractions. Usually, it is *not* the case that $a/b = (a + c)/(b + d)$. However, if $a/b = c/d$ and $b + d \neq 0$, then it *is* the case that $a/b = c/d = (a + c)/(b + d)$. Why?

Session 10 – Meet 2 – February 2, 2012

Mentors: Jennifer Balakrishnan, Connie Liu, Jennifer Melot, Liz Simon

We began a series of meets whose ultimate goal was to free a (stuffed animal) dog from a cage. Aside from the Euclidean algorithm, Pappus's centroid theorems, and properties of medians, one important message of the meet was the following: If you want to learn something, don't worry about where you are relative to others. Just concentrate on where you are with respect to what you want to learn and focus on mastering the next step. Don't let other people's progress or abilities affect your ability to learn what you wish to learn. Don't chase people; chase ideas.

Here's a fun game that will help you develop your geometric intuition. It's a game you can play wherever you are. Think of a geometric object such as a cube, a donut, a coffee cup, a guinea pig …any object will do. Now imagine the object fixed in some orientation. Then imagine passing the object through a plane and try to visualize the animated movie sequence in the plane created by the various cross sections of the object as it passes through. Here's some of what you get if you dangle a cube from one of its vertices and pass it through a horizontal plane:



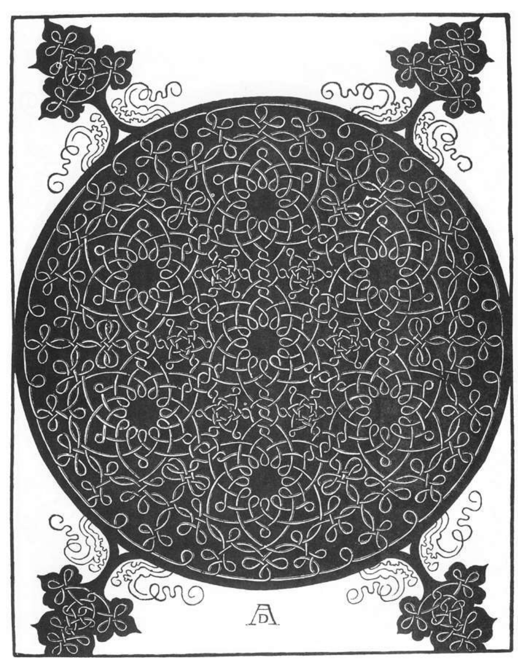
Passing a cube dangling by a vertex through a plane.

Session 10 – Meet 3 – February 9, 2012

Mentors: Connie Liu, Jennifer Melot, Rediet Tesfaye, Fan Wei

Special Guest: Meike Akveld, ETH Zürich

Meike Akveld presented on knots. In mathematics, a knot is a closed loop (in 3D space). It is remarkable how so many interesting concepts arise from the simple idea of a closed loop, but that is often how mathematics goes. Meike showed us many places where knots have been used, such as in sailing, rock climbing, glacier hiking, shoelace tying, and decorating (such as in the woodcut by Albrecht Dürer at right). In fact, in Switzerland, where Meike lives, the government requires that papers be bundled together and knotted up with a string for the purpose of recycling. Serious study of knots was instigated by Lord Kelvin when he hypothesized that atoms were "knotted vortices." That's when Peter Guthrie Tait began to systematically classify knots. Meike showed us a picture of a whole variety of knots (Google "knot zoo") and posed two



Knots in a woodcut by Albrecht Dürer.

basic questions that fuel knot theory research: first, how can you tell if a knot can be untangled, and second, how can you tell if two knots are the same? Meike pointed out that just fiddling with a knot to untangle it isn't satisfactory because you "can't try forever, so you need to find something better." Meike observed that the complete classification of knots with 15 crossings remains unsolved. You can read more about knots (and learn what a "crossing" is) in a very nice book that Meike wrote with co-author Andrew Jobbings called *Knots Unravelled*. Also, check out Allison Henrich's 3-part knot series in this *Bulletin*, Vol. 2, Nos. 5 and 6 and Vol. 3, No. 1.

For more pictures of knots, see the *Mathematical Buffet* on page 5.

Session 10 – Meet 4 – February 16, 2012

Mentors: Samantha Hagerman, Jennifer Melot, Fan Wei

Members freed the dog! Congratulations! It's not every dog that demands learning about Ceva's theorem, Pappus's centroid theorems, medians of triangles, least common multiples, greatest common divisors, Fibonacci numbers, and cross sections before coming out of its cage.

Pappus's centroid theorems concern solids of revolution. A solid of revolution is a solid that is created by taking a two-dimensional region in a plane and rotating it about some line in that plane. The line becomes an axis of revolution. By their construction, solids of revolution have all the symmetries of a circle. In fact, it is a common theme in mathematics to create an object with desired symmetries by somehow adding up or taking the union of an object together with all of its images under a collection of symmetries. The donut, cone, cylinder, and sphere are all examples of solids that can be regarded as solids of revolution.

Let d be the distance of the centroid of the planar region from the axis of revolution and let d' be the distance of the centroid of the planar regions' perimeter from the axis of revolution. (Note that d and d' are generally different.) The centroid can be thought of as the balancing point, if the object were made of a uniform material.

Pappus's centroid theorems state that:

1. The volume of the solid of revolution is equal to $2\pi d$ times the area of the planar region
2. The surface area of the solid of revolution is equal to $2\pi d'$ times the perimeter of the planar region.

For example, suppose the planar region is a rectangle of dimensions r by h. Let's rotate this rectangle about one of its sides of length h. The resulting solid of revolution is a cylinder with base radius r and height h. The centroids of both the rectangle and the rectangle's perimeter are located where the diagonals intersect (by symmetry). The distance of this center from our axis of revolution is $r/2$, so the centroid travels a distance of πr about the axis of revolution. Since the area of our rectangle is rh and its perimeter is $2(r + h)$, we can apply Pappus's theorems to find that the volume of the resulting cylinder is $\pi r^2 h$ and its surface area is $2\pi r(r + h)$.

Here's a problem: Knowing that the volume of a sphere is $4\pi r^3/3$, where is the centroid of a semicircular region?

Calendar

Session 9: (all dates in 2011)

September	8	Start of the ninth session!
	15	
	22	Ally Hartzell, Pixtronix
	29	Start of Rosh Hashanah – No meet
October	6	
	13	Diana Hubbard, Boston College
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	Alison Malcolm, MIT
	8	

Session 10: (all dates in 2012)

January	26	Start of the tenth session!
February	2	
	9	Meike Akveld, ETH Zürich
	16	
	23	No meet
March	1	
	8	Julie Yoo, Kyruus
	15	
	22	
	29	No meet
April	5	Beth Kanell, Author
	12	Sarah Spence Adams, Olin College
	19	No meet
	26	
May	3	

Here are answers to the *Errorbusters!* problems on page 19.

	$f \circ g\ (x)$	$g \circ f\ (x)$
1.	$\mathbf{R} \to \mathbf{R}$, $2x + 10$	$\mathbf{R} \to \mathbf{R}$, $2x + 5$
2.	$\mathbf{R} \to \mathbf{R}$, $x^2 + 4x$	$\mathbf{R} \to \mathbf{R}$, $x^2 - 2$
3.	$\mathbf{R} \to \mathbf{R}$, x	$\mathbf{R} \to \mathbf{R}$, x
4.	N/A	$\mathbf{R}\backslash\{-1\} \to \mathbf{R}$, $4x^2/(x + 1)^2 - 1$
5.	N/A	$\mathbf{Z} \to \mathbf{R}$, $(x + 1)/2$
6.	P → P, 90 degree clockwise rotation about the origin	P → P, 90 degree counterclockwise rotation about the origin

Girls’ Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls’ Angle? Girls’ Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls’ interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls’ Angle mentors, the Girls’ Angle Support Network, the Girls’ Angle Bulletin and Community Outreach.

Who are the Girls’ Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls’ Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls’ Angle Bulletin? The Girls’ Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls’ Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members’ efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club and receive a printed copy of the Girls’ Angle Bulletin for the duration of the membership. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of “catching up with the group” doesn’t apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin and a 25% discount for any club meet attended. Remote members may email us math questions (although we won’t do people’s homework!).

Where is Girls’ Angle located? Girls’ Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________
(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership
  - ☐ $108 for a one year Remote Membership
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

Girls' Angle

A Math Club for Girls