Girls' Angle Bulletin

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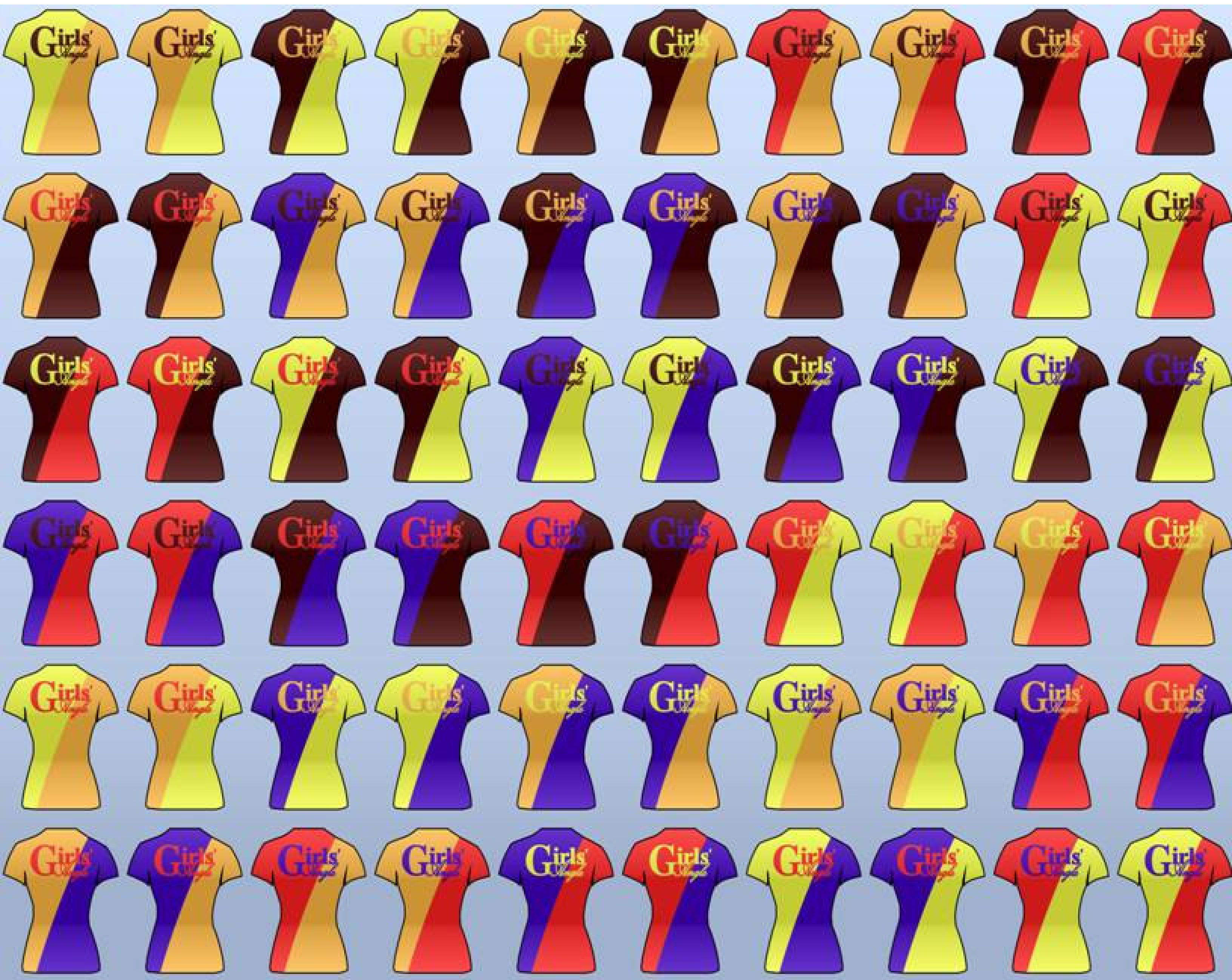
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

First, Thank You to MathWorks for their generous community grant in support of the Girls' Angle Bulletin. We'll use those funds to ensure that we continue to bring you quality math content.

We've begun taking the math treasure hunts that we do at our club out of the club in various ways. There was SUMiT 2012 at MIT, and since then, we've visited 3 schools: the Buckingham, Browne, and Nichols school, Concord-Carlisle High School, and the Graham and Parks Alternative School. If you're interested in having one of these math treasure hunts at your school, please contact us! They are mathematically immersive, fully collaborative, fun, and challenging events that we can tailor to different group sizes and ability levels.

- Ken Fan, President and Founder

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Girls' Angle Bulletin
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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Three From Five: A showcase of every way to color the 3 components of these T-shirts from a palette of 5 colors. For more, see Shravas Rao's Summer Fun Problem Set. Which T-shirt is your favorite?

An Interview with Jean Pedersen, Part 2[1]

Ken: When did you realize that you wanted to be a mathematician? Was it difficult to become a mathematician?

There is something ethereal about symmetry and how it manifests itself in different settings... The interconnectedness of the paper-folding, geometry, combinatorics, group theory and number theory still amazes me.

Jean: Mathematics was always easy for me and I enjoyed doing homework where I could verify that what I had done was correct. So I always took mathematics classes whenever I had elective courses. During my senior year at Brigham Young University (BYU) high school I was allowed to take classes at BYU since I needed only 2 high school classes to graduate. I had decided I would major in home economics (involving sewing, cooking, and household administration, as it was called in those days). This choice involved some classes in physics and chemistry. However, rather than taking the courses required of home economics majors I took the courses designed for engineers and biology majors and filled in all my electives with mathematics classes.

At the end of my junior year at BYU I had a 3-hour foods lab in which we hardboiled an egg three different ways and then tested them for appearance and flavor. I was bored silly and it wasn't surprising at all to me that the method my mother taught me turned out to be the best.[2] It then occurred to me that although I enjoyed cooking and sewing I wasn't going to enjoy teaching others to do it. I went home and looked up the possible majors I could do with the courses I had already taken. I discovered that if I took a 12-hour course in organic chemistry over the summer I could still graduate the next spring (3 years after my high school graduation) with a teaching credential in combined mathematics, chemistry and physics. My parents and friends were a bit mystified about my decision, but they didn't try to stop me.

Then, during my senior year at BYU I was asked by the chair of the mathematics department to teach one elementary algebra course (making me the first woman to teach a mathematics class at BYU). My future husband, Kent, took a course from me during the winter term and we began dating in the spring quarter. By the next Christmas we were engaged; I had graduated and was teaching mathematics at BYU high school. When our engagement announcement appeared in the paper the chair of the mathematics department at BYU made a point of talking with Kent and suggested that we should leave BYU and go to the University of Utah (U of U) where Kent could finish his engineering degree and I could pursue a master's degree in mathematics — and he kindly offered to write the chairman of the mathematics department at the U of U recommending me for a fellowship. We got married in May of 1956 and moved to Salt Lake City. Two years later, in 1958, I finished my master's degree in mathematics and the next year Kent got his electrical engineering degree at the U of U while I taught mathematics at Olympus High School in south Salt Lake City. I then returned to the

[1] Please refer to the previous issue for references.

[2] Just in case you're interested, all you do is put the eggs in a pan, cover with cold water, add a pinch of salt, bring the water to a boil on high, and turn the heat off. Cover the pan and leave it on the stove for 15 minutes. Then pour off the water, add ice cubes and cool water. Wait about 10 minutes. The eggs will be beautifully hard-boiled.

U of U mathematics department as an instructor and Kent went to work at Sperry Engineering Lab. We lived quite happily in Salt Lake City and had both of our children there.

In the summer of 1965 we moved to San Jose where my husband had obtained a better job at IBM. After finding a new house and getting our family settled — Chris was four and Jennifer was one — I landed a part-time teaching position at Santa Clara University (SCU) where I discovered I was the first woman to teach mathematics there. Later that year Alice Chamberlain was hired as the first full-time woman in the mathematics department.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

We will make the rest of this interview with Prof. Jean Pedersen available here at some time in the future. But what we hope is that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
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Searching for Amelia: Geometry's Role

Written by Katherine Sanden and Ken Fan
Edited by Jennifer Silva

At 7:52 am on July 2, 1937, the famous aviator Amelia Earhart broadcast from the cockpit of her Lockheed Electra, "We must be on you, but cannot see you – but gas is running low. Have been unable to reach you by radio. We are flying at 1,000 feet." She was nearing the end of an historic, round-the-world flight, but an hour later, she and her navigator, Fred Noonan, vanished. To this day, their fate remains a mystery.

Almost exactly 75 years later, on the morning of July 3, 2012, a research crew from *The International Group for Historic Aircraft Recovery* (TIGHAR) will board the research vessel Ka'imikai-o-Kanaloa and set sail from the shores of Hawaii with the goal of learning Earhart's fate. Many have tried to figure out what happened to Earhart before, so what makes this new expedition any more likely to succeed?

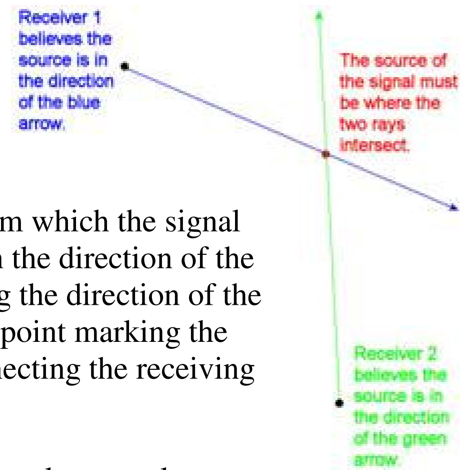
After Earhart disappeared, a search and rescue operation was launched by the U.S. Navy and Coast Guard. There were numerous reports of distress signals from Earhart. Planes were sent aloft to follow-up on these reports, but to no avail. Earhart and Noonan were never seen again, and eventually the search was abandoned. According to TIGHAR, "The U.S. Navy and Coast Guard dismissed the widely publicized receptions by amateur radio listeners as hoaxes or misunderstood interceptions of the searchers' attempts to contact the plane." Most suspected that Earhart ran out of fuel and crashed somewhere in the vastness of the Pacific Ocean.

However, Ric Gillespie, leader of the TIGHAR expedition, decided to take a closer look at these so-called "post-loss radio signals." Perhaps some of them were genuine distress calls sent by Earhart. After all, there were hundreds of such signals. Could they really all be hoaxes or errors? Even more intriguing, as Mr. Gillespie writes, "The location of essential radio components aboard the aircraft meant that transmissions were not possible if the plane was afloat on the ocean. If only one alleged post-loss radio signal from Earhart was genuine, the Electra had to have been on land and the official U.S. government verdict that the Electra crashed and sank at sea could not be correct."

Thus began a systematic study of the post-loss signals. Using some interesting logical reasoning (which you can read about on the TIGHAR website[3]), they determined which signals were most likely genuine. People who observed signals not only recorded their location and the time of signal reception, but they also recorded the likely direction to the signal source. Using **radio direction finder** technology, radio operators could get a good estimate of the direction to the source of a received signal.

[3] tighar.org/Projects/Earhart/AEdescr.html

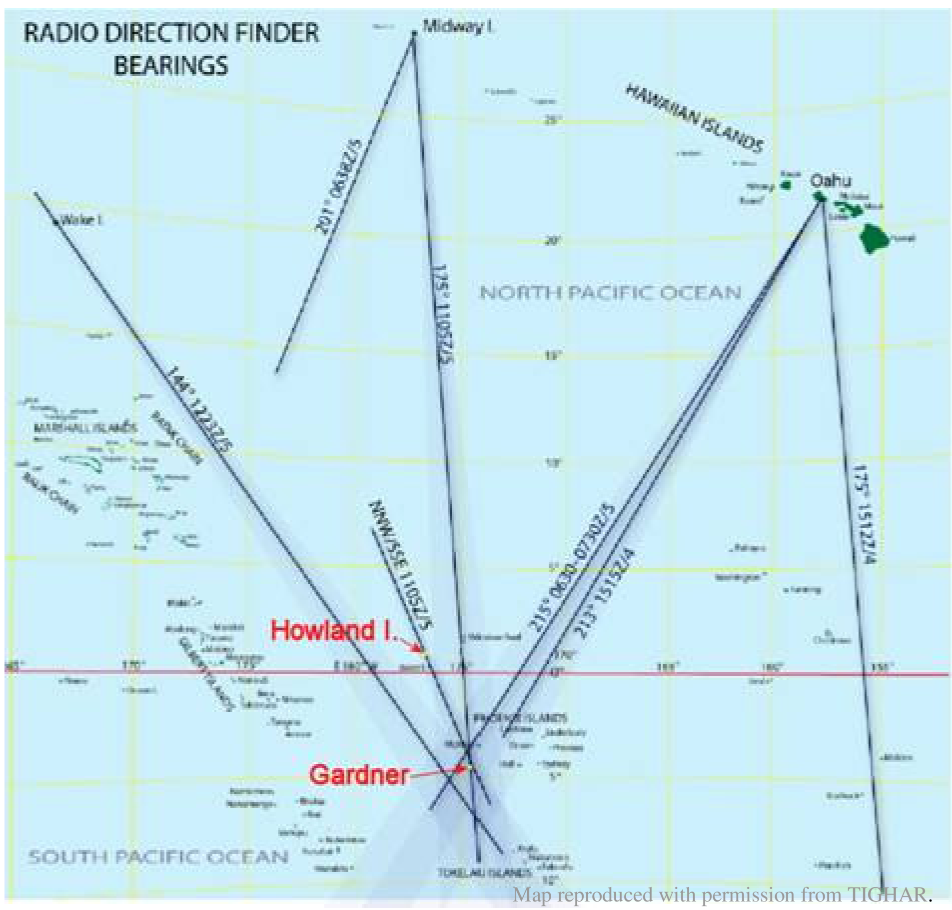
Intersecting Lines



This directional information enables the use of an age-old geometric tool known as **triangulation** to locate the source of the distress calls. Imagine that a signal is broadcast and picked up by two different receivers. Each receiver uses radio direction finder technology to determine the direction from which the signal came. On a map, rays are drawn from each receiving station in the direction of the signal source. As long as the direction to the signal is not along the direction of the other receiving station, these two rays will intersect in a single point marking the location of the source. The two rays and the line segment connecting the receiving stations will form a triangle, hence the name "triangulation."

Astute readers may be concerned that the above discussion takes place on planar maps, but the real world is like a sphere. Rays drawn from a point on a sphere will travel around the sphere and come back to the starting point forming great circles instead of lines, and great circles intersect in diametrically opposed points. However, because of the nature of radio transmissions, the points on the other side of the world are highly unlikely to be the sources of the signals. Also, the inherent error in the measurement of the radio directions renders local corrections in the paths of these lines due to the curvature of the earth less important.

TIGHAR researchers plotted the rays associated with the signals that seemed most authentic, shown on the map at right. Radio Direction Finder technology is not perfect. Sometimes there are spurious results, and there is always some error in the measurement of the direction. Earhart's planned destination was Howland Island, but the results of TIGHAR's analysis suggest that Earhart missed Howland and landed on an island in the vicinity of the Phoenix Islands, perhaps McKean or Gardner Island.



This map, from the TIGHAR website, has been slightly modified to bring out the location of Earhart's intended destination, Howland Island, and Gardner Island, which is where TIGHAR believes Earhart and Noonan lived as castaways. To model error in the radio directions, rays have been widened into sectors.

Amelia Earhart[2]

When this geometric evidence is combined with a wealth of other bits and pieces of information, TIGHAR concluded that it was highly likely that Earhart landed on Gardner Island, which is now known as Nikumaroro. For example, shards of glass were found on Nikumaroro that fit together to form a jar that looks very much like jars marketed as "Dr. Berry's Freckle Ointment." It is documented that Earhart had freckles and disliked them. You can read about all of the other evidence on the TIGHAR website[5] or in Mr. Gillespie's book, *Finding Amelia*.

For triangulation to work, at least two observing posts are required because it takes two lines to determine a point. If you also know the distance between the observing posts, then you can use the geometric concept of similarity and the tools of trigonometry to determine the distances from each observer to the source. For more on this topic, see Volume 4, Numbers 1-3 of this Bulletin.

We wish TIGHAR a successful journey of discovery. Pomaika`i!

Take it to Your World

Triangulation is a very practical concept. Try it for yourself and see!

It may help to borrow a friend for this. Go to a park and pick out a tree in the distance. You and your friend should take up positions with known locations on a map of the park that are some distance apart, so that the two of you and the tree form the vertices of a triangle. We'll call the line passing between you and your friend the **baseline**. Now, both of you measure the angle between the baseline and the line of sight to the tree.

On a map of the park, mark the locations of your friend and yourself and draw in the baseline. Using a protractor, take the angles you measured and draw in rays with those angles measured from the baseline. The two rays will intersect at the point on the map that represents the location of the tree.

If you know the distance separating you and your friend, you have all of the information you need to figure out the distance between you and the tree. Can you see how? Can you figure out a way to use triangulation to measure the height of buildings or mountains, or the distance across a chasm or through buildings?

If you don't know where you and your friend are located on the map, you can use triangulation to known landmarks to figure that out. Can you explain how?

Why is it harder to use triangulation if the distance between the observing posts is small relative to the distance of the source?

Where else might you use triangulation?

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4 Source: www.americaslibrary.gov/assets/aa/earhart/aa_earhart_learns_2_e.jpg

5 tighar.org/

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna continues her exploration of sums of the first *n* *k*th powers and unveils a mystery!

Today I'm going to investigate the sequence c_p that I defined last time.

I need to reorient myself, so I'll start by summarizing what I did.

Last time: Set $S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$.

$S_k(n)$ is a polynomial of degree $k+1$:

$$S_k(n) = a_{k,0} + a_{k,1} n + a_{k,2} n^2 + \cdots + a_{k,k+1} n^{k+1}$$

$a_{k,k+1} = \frac{1}{k+1}$

Defined c_p by $a_{k,k-p} = c_p \binom{k}{p}$ for $p \geq 0$.

Found: $c_0 = \frac{1}{2}$, $c_1 = \frac{1}{12}$, $c_2 = 0$, $c_3 = -\frac{1}{120}$

By comparing coefficients, I should be able to get a recursive formula for c_p.

By comparing coefficients in the equation (of $n^{k-(p+1)}$)

$$n^k = S_k(n) - S_k(n-1),$$

we get

$$0 = (k-p)\, a_{k,k-p} - \binom{k-p+1}{2} a_{k,k-p+1} + \binom{k-p+2}{3} a_{k,k-p+2} - \cdots + (-1)^{p+1} \binom{k+1}{p+2} a_{k,k+1}$$

$$= (k-p)\, c_p \binom{k}{p} - (k-p)\, c_{p-1} \frac{p!}{2!\,(p-1)!} \binom{k}{p} + (k-p)\, c_{p-2} \frac{p!}{3!\,(p-2)!} \binom{k}{p} - \cdots + (-1)^m (k-p)\, c_{p-m} \frac{p!}{(m+1)!\,(p-m)!} \binom{k}{p} + \cdots$$
$$+ (-1)^p (k-p)\, c_0 \frac{p!}{(p+1)!\,0!} \binom{k}{p} + (-1)^{p+1} \binom{k+1}{p+2} \frac{1}{k+1}.$$

I better be careful with all this algebra! Slow and steady...

I need to write this last term as something independent of k multiplied by $k - p$ times k choose p.

$$\binom{k+1}{p+2} \frac{1}{k+1} = \frac{(k+1)!}{(p+2)!\,(k-p-1)!} \frac{1}{k+1} = \frac{k!}{p!\,(k-p)!} \frac{k-p}{(p+1)(p+2)} = (k-p) \cdot \frac{1}{(p+1)(p+2)} \binom{k}{p}$$

There, that does the trick...

If you don't understand what Anna is doing, please read the last few installments of Anna's Math Journal. This is a continuation of her investigation into sums of kth powers.

In this installment, Anna unveils a mystery... It seems that half the numbers in the sequence c_p are zero.

Is it really true? Can you explain it?

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Now that every term has $k - p$ and k choose p in it, I can cancel them all, and that leaves c_p all by itself!

A lot of the fractions are binomial coefficients. I'll put those in next.

So, $$0 = c_p - c_{p-1}\frac{p!}{2!(p-1)!} + c_{p-2}\frac{p!}{3!(p-2)!} - \cdots + (-1)^m c_{p-m}\frac{p!}{(m+1)!(p-m)!} + \cdots + (-1)^p c_0 \frac{p!}{(p+1)!0!} + (-1)^{p+1}\frac{1}{(p+1)(p+2)}$$

$$= c_p - \frac{c_{p-1}}{p+1}\binom{p+1}{2} + \frac{c_{p-2}}{p+1}\binom{p+1}{3} - \cdots + (-1)^m \frac{c_{p-m}}{p+1}\binom{p+1}{m+1} + \cdots + (-1)^p\frac{c_0}{p+1}\binom{p+1}{p+1} + (-1)^{p+1}\frac{1}{(p+1)(p+2)} .$$

OK, now I'll isolate c_p.

$$\Rightarrow \boxed{c_p = \frac{1}{p+1}\left(c_{p-1}\binom{p+1}{2} - c_{p-2}\binom{p+1}{3} + \cdots - (-1)^m c_{p-m}\binom{p+1}{m+1} + \cdots - (-1)^p c_0\binom{p+1}{p+1} - (-1)^{p+1}\frac{1}{p+2}\right)}$$

All these $(p + 1)$'s and minus signs make me wonder if it would be slightly cleaner to shift the indexing of c_p by one and build in an alternating sign factor. Maybe I'll do that later.

Just for fun, I'm going to compute c_p for the first few values of p, starting with $p = 0$. I want to get a feel for this recursion.

$$c_0 = \frac{1}{2}$$

$$c_1 = \frac{1}{2}\left(\frac{1}{2}(1) - \frac{1}{3}\right) = \frac{1}{2}\left(\frac{1}{6}\right) = \frac{1}{12}$$

$$c_2 = \frac{1}{3}\left(\frac{1}{12}\binom{3}{2} - \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right) = 0$$

$$c_3 = \frac{1}{4}\left(-\frac{1}{12}\binom{4}{3} + \frac{1}{2} - \frac{1}{5}\right) = \frac{1}{4}\left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{5}\right) = -\frac{1}{120}$$

$$c_4 = \frac{1}{5}\left(-\frac{1}{120}\binom{5}{2} + \frac{1}{12}\binom{5}{4} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{5}\left(-\frac{1}{12} + \frac{5}{12} - \frac{1}{2} + \frac{1}{6}\right) = 0$$

$$c_5 = \frac{1}{6}\left(+\frac{1}{120}\binom{6}{3} - \frac{1}{12}\binom{6}{5} + \frac{1}{2} - \frac{1}{7}\right) = \frac{1}{6}\left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2} - \frac{1}{7}\right) = \frac{1}{252}$$

$$c_6 = \frac{1}{7}\left(\frac{1}{252}\binom{7}{2} - \frac{1}{120}\binom{7}{4} + \frac{1}{12}\binom{7}{6} - \frac{1}{2} + \frac{1}{8}\right)$$
$$= \frac{1}{7}\left(\frac{1}{12} - \frac{7}{24} + \frac{7}{12} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

Coach Barb remarks, "Talk about fractions!"

$$c_7 = \frac{1}{8}\left(-\frac{1}{252}\binom{8}{3} + \frac{1}{120}\binom{8}{5} - \frac{1}{12}\binom{8}{7} + \frac{1}{2} - \frac{1}{9}\right)$$
$$= \frac{1}{8}\left(-\frac{2}{9} + \frac{7}{15} - \frac{2}{3} + \frac{1}{2} - \frac{1}{9}\right) = \frac{1}{8}\left(-\frac{1}{30}\right) = -\frac{1}{240}$$

$$c_8 = \frac{1}{9}\left(-\frac{1}{240}\binom{9}{2} + \frac{1}{252}\binom{9}{4} - \frac{1}{120}\binom{9}{6} + \frac{1}{12}\binom{9}{8} - \frac{1}{2} + \frac{1}{10}\right)$$
$$= \frac{1}{9}\left(-\frac{3}{20} + \frac{1}{2} - \frac{7}{10} + \frac{3}{4} - \frac{1}{2} + \frac{1}{10}\right) = \frac{1}{9}\left(\frac{-3+10-14+15-10+2}{20}\right) = 0$$

Strange how all even terms beyond the first are equal to zero! It feels like more than a coincidence. I'll put it down as a conjecture! Hmmm...what's the meaning? Why should every other one be zero?

$$\boxed{\text{Conjecture: } c_{2m} = 0 \text{ for } m \geq 1}$$

I'll think about the conjecture later. But for now, I feel like writing down the polynomial expression for the sum of the first n perfect eighth powers!

$$1^8 + 2^8 + 3^8 + \cdots + n^8 = c_8\binom{8}{8} + c_7\binom{8}{7}n + c_6\binom{8}{6}n^2 + \cdots + c_0\binom{8}{0}n^8 + \frac{1}{9}n^9$$
$$= -\frac{1}{30}n + \frac{2}{9}n^3 - \frac{7}{15}n^5 + \frac{2}{3}n^7 + \frac{1}{2}n^8 + \frac{1}{9}n^9$$

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Key:
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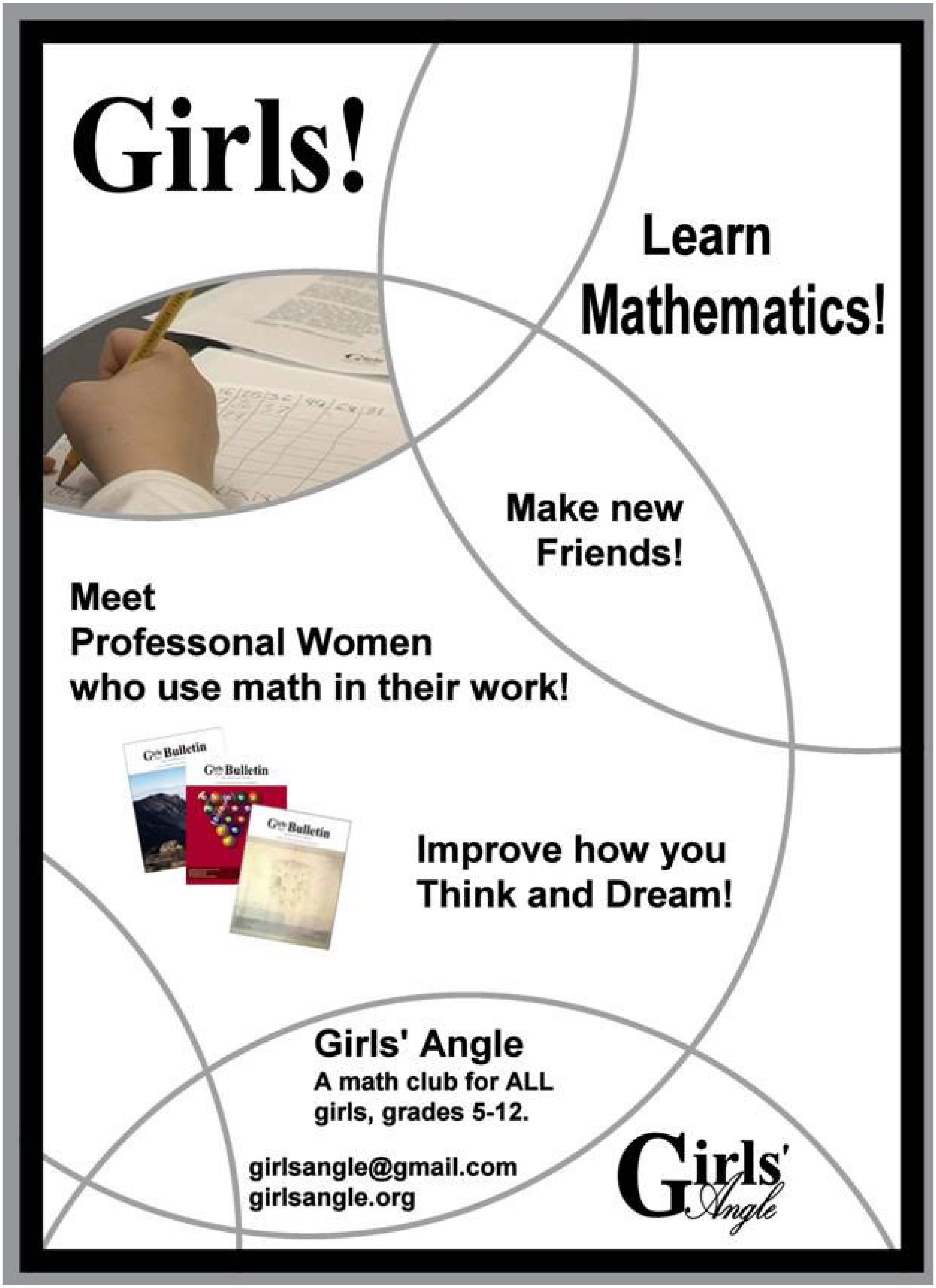
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Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

In the last issue, we discussed inverse functions. Now let's ask, what is the inverse of the **exponential** function $f(x) = 4^x$? Let's think through what this exponential function does: it takes as input the variable x and gives as output 4^x, or 4 raised to the exponent x. Recall that an inverse function is the function that "undoes" the original function. To "undo" an exponential function, we need a function that "peels off" the exponent, so to speak. That is, we need a function $g(x)$ such that $g(4^x) = x$. We can define a function to do just that. Such a function is called a **logarithm**. In this case, we write $g(z) = \log_4 z$, which is read "the logarithm, base 4, of z."

We can compute the logarithm, base 4, by rewriting our input as a power of 4 and then "peeling off" the exponent. For instance, since $16 = 4^2$, we compute that $\log_4 16 = \log_4 4^2 = 2$. Similarly, we can compute that $\log_4 64 = \log_4 4^3 = 3$, since $64 = 4^3$.

If our input is not a perfect power of 4, then we'll end up with an answer that isn't an integer:

$$\log_4 32 = \log_4 2^5 = \log_4 (\sqrt{4})^5 = \log_4 (4^{1/2})^5 = \log_4 4^{5/2} = 5/2 = 2.5.$$

Recall that $\sqrt{4} = 4^{1/2}$ and that $(b^x)^y = b^{xy}$.

Of course, we can take logarithms with bases other than 4. The base of a logarithm can be any positive number. One particularly common case is the logarithm base 10, which is also known as the **common logarithm**. Sometimes, people will simply write "log x" for the common logarithm (as opposed to $\log_{10} x$). There's also the logarithm base e, where e is the special constant that is approximately 2.718281828459045235… (see box at right). This logarithm is known as the **natural logarithm**. Sometimes people denote the natural logarithm as "ln x," though mathematicians often will write "log x" for the natural logarithm. (Yes, that *is* the same as what some people will use to denote the common logarithm! You just have to make sure you know what the author means whenever you see a "log" without a subscript.) To properly explain the constant e and where it comes from, we would need calculus.

The logarithm has many special properties, but one mistake I have seen students make is to say that

$$\log (a + b) = \log a + \log b.$$

Let's bust this error now! I call it "erroneously distributing the logarithm." Just like the square root, logarithms do **not** distribute over addition! For instance, we have seen that $\log_4 (16 + 16) = \log_4 32 = 2.5$, yet $\log_4 16 + \log_4 16 = 2 + 2 = 4 \neq 2.5$.

The Base of the Natural Logarithm

The constant e can be defined in many ways. Here are two:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

As we know from calculus, the constant e arises naturally when one studies the simplest differential equation:

$$f(x) = \frac{df(x)}{dx}.$$

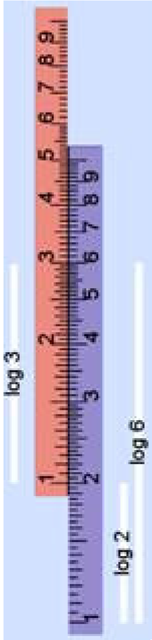
The most general solution to this differential equation is $f(x) = Ce^x$, where C is a constant.

Slide Rules

The fundamental property

$$\log ab = \log a + \log b$$

explains how slide rules work. A slide rule is a tool that allows one to multiply by simply sliding one specially marked ruler along another. Because the logarithm "turns multiplication into addition," by marking the ruler using a logarithmic number line, products are revealed when lengths are added together. This is illustrated using the product 2 × 3 = 6 at left.



It is true, however, that

$$\log ab = \log a + \log b$$

for any positive numbers a and b. In other words, logarithms "turn products into sums." This makes sense when we remember that logarithms give exponents as outputs, and when we recall that products of powers with the same base become sums in the exponents:

$$3^x 3^y = 3^{x+y}.$$

Now $\log_3 3^x = x$ and $\log_3 3^y = y$, while $\log_3 (3^x 3^y) = \log_3 3^{x+y} = x + y$. Hence,

$$\log_3 (3^x 3^y) = \log_3 3^{x+y} = \log_3 3^x + \log_3 3^y.$$

But 3^x and 3^y can be any positive numbers a and b, so indeed

$$\log_3 ab = \log_3 a + \log_3 b.$$

The base 3 can be replaced throughout with another positive number base.

Let's illustrate this property with an example. Note that $3^1 = 3$ and $3^3 = 27$. So

$$\log_3 81 = \log_3 (3 \cdot 27) = \log_3 3 + \log_3 27 = \log_3 3^1 + \log_3 3^3 = 1 + 3 = 4.$$

Thus, since $3^4 = 81$, we know that $\log_3 81 = 4$.

Another special property of the logarithm is that

$$\log a^x = x \log a.$$

In other words, powers become products, with the exponent coming down to multiply with the logarithm. This is true because raising a power to another power multiplies the exponents: $(b^y)^x = b^{yx}$. Suppose that $a = b^y$, which is the same as saying that $\log_b a = y$. Then

$$\log_b a^x = \log_b (b^y)^x = \log_b b^{yx} = yx = (\log_b a)x = x \log_b a.$$

Imagine, for instance, that we're computing $\log_4 32$ again. Since $32 = 2^5$, we see that

$$\log_4 32 = \log_4 2^5 = 5 \log_4 2 = 5 \log_4 4^{1/2} = 5(1/2) = 2.5.$$

Note in addition that $\log a/b = \log a - \log b$. Quotients become differences. This can be seen using the two previous rules that we've established:

$$\log a/b = \log (a \cdot \frac{1}{b}) = \log a(b^{-1}) = \log a + \log b^{-1} = \log a + (-1)\log b = \log a - \log b.$$

Another useful formula for logarithms is called the "change of base formula." This can come in handy when you want to take a logarithm to a base that your calculator doesn't offer. (Some scientific calculators only have the common logarithm and the natural logarithm.) It is true that

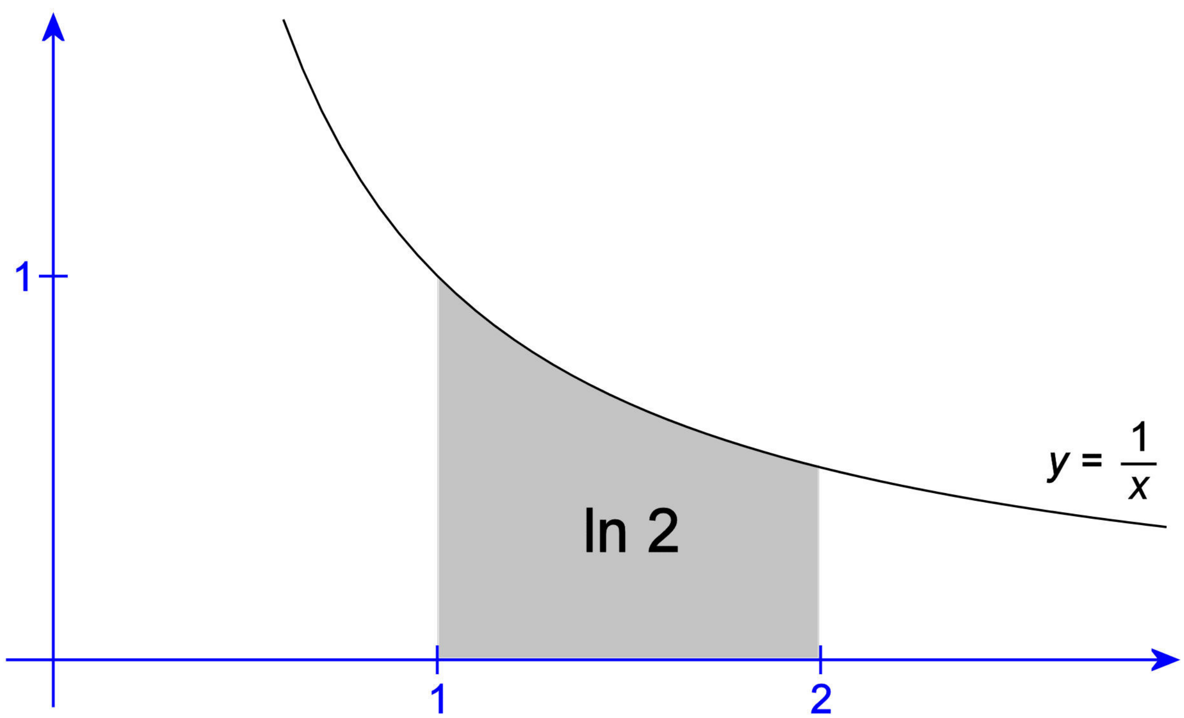
$$\log_b a = \frac{\log_c a}{\log_c b},$$

which means you can change a base b to any positive number c. To derive this formula, let's say that $\log_b a = x$ and $\log_c b = y$. Changing to exponential notation, we see that $b^x = a$ and $c^y = b$. Therefore $a = b^x = (c^y)^x = c^{yx}$ so $\log_c a = yx = (\log_c b)(\log_b a)$. Dividing both sides by $\log_c b$ yields the change of base formula.

For practice, try to find the following logarithms without using a calculator. You may need to use the special properties of logarithms! The answers can be found on page 29.

1. $\log_5 125$
2. $\log_{10} 0.0001$
3. $\log_6 9 + \log_6 4$
4. $\log_{10} 180 - \log_{10} 18$
5. $\log_{1000} 10{,}000$
6. $\ln e^{-5}$
7. $\ln 1$
8. $\log_{10} 2 + \log_{10} 300 - \log_{10} 6$
9. $\log_{3.5} 3.5^4$

Area Interpretation of the Natural Logarithm



The natural logarithm of a can also be interpreted as the area between the x-axis and the graph of $y = 1/x$ in the xy-coordinate plane between the vertical lines $x = 1$ and $x = a$. For example, the shaded region shown above has an area of ln 2. (If $0 < a < 1$, then one takes the negative of the area as the value for ln a.)

COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 5

Here comes that venerable lady, 3/7. She is headed right for you and looks like she has important business on her mind.

$\mathbf{\frac{3}{7}}$: Hi dearie. I've been looking forward to talking about dividing by fractions with you for ever so long. I am thrilled that, at last, the time has come!

You: Uh … hi … I'm not sure what to say. I do know how to divide by fractions.

$\mathbf{\frac{3}{7}}$: Remind me then, darling.

You: Well, if you're dividing one fraction by a second fraction, the answer is a new fraction. The top number of the answer is the top of the first fraction times the bottom of the second. The bottom number of the answer is the bottom number of the first fraction times the top number of the second.

$\mathbf{\frac{3}{7}}$: My, my, you are generous with words. You must be kind-hearted. As for me, I am stingy – I mean, frugal – and one of the many things I love about math is the potential to convey big ideas with only a few words.

You: Oh … then let me say it in a way that you'll like better.

$\mathbf{\frac{3}{7}}$: Delightful! I do so appreciate generosity in others.

You: Suppose you have two fractions $\frac{a}{b}$ and $\frac{c}{d}$. Then $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

$\mathbf{\frac{3}{7}}$: Lovely!

You: Want to hear another way of saying it that you'll really hate? It's a song about dividing fractions.

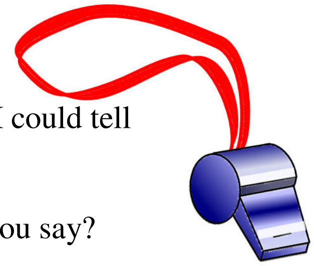
$\mathbf{\frac{3}{7}}$: "Want" isn't the right word, but I am intrigued. Lay it on me, honey.

You: Ours not to reason why, Just flip and multiply!

$\mathbf{\frac{3}{7}}$: Aaaaargh! What a horrid ditty. Where did you pick that up?

You: Oh, it's something my dad used to sing when he saw me doing fraction homework.

$\mathbf{\frac{3}{7}}$: Dreadful! What a beastly man! Run away from home! Immediately!



You: It was really sort of a joke. Once when I rolled my eyes, he said he'd stop if I could tell him the reason why "flip-and-multiply" is the thing to do.

$\frac{3}{7}$: Oh, forget what I said before; he sounds like a marvelous fellow. So what did you say?

You: Well, nothing at first. I had to think for a while about why it is a reasonable procedure, and then I had to think some more about how to explain it.

$\frac{3}{7}$: Thinking before you speak is an admirable quality. It's also highly useful when you're trying to convey mathematical ideas. The more work you put into being clear, the less work the listener has to do to understand.

You: Yeah, and in the case of my younger sister, she'll just give up if I'm at all confusing. Then she'll complain and whine and it's really annoying. As if it's my fault her homework isn't done.

$\frac{3}{7}$: As if … Back to your explanation of dividing fractions. I would so love to hear what you came up with.

You: Well, I started with the idea that division is actually defined with multiplication.

$\frac{3}{7}$: Ooooh, I like this way of thinking about it. Big picture. Your little sister may require a few more words, however. How is division defined using multiplication?

You: Okay. Say you're asked to divide 48 by 6. Another way to put the question is to ask, what do I have to multiply 6 by to get 48? Since 6 times 8 equals 48, we know that 48 divided by 6 is 8. In other words, a divided by b is just the number you need to multiply with b in order to get a. I guess you could say that division by b "undoes" multiplication by b: if you start with a number, then multiply by b, then divide by b, you get back the number you started with.

$\frac{3}{7}$: Nice. Now back to fractions.

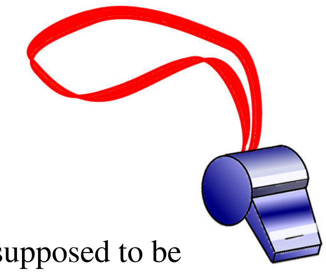
You: I started with the multiplication formula for fractions. It's $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.
It's just a shorter way of saying the following: *If you're multiplying two fractions, the answer is a new fraction. The answer's top number is the product of the two top numbers from the two fractions you're multiplying. The answer's bottom number is the product of the two bottom numbers from the two fractions you're multiplying.*

$\frac{3}{7}$: Much shorter. Simple yet elegant, I'd say.

You: Yes. Well, from the multiplication formula, as well as because of the way division is defined, we automatically get the division formula: $\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$.

$\frac{3}{7}$: Now how did you figure that out?

You: Well, I am still getting used to the idea of using letters for numbers, so I keep things straight by connecting the letter expressions to actual numbers in an example.



$\frac{\mathbf{3}}{\mathbf{7}}$: Perfectly clear to me, but your little sister may start to fret.

You: Well, I think to myself, what is $\frac{ac}{bd} \div \frac{c}{d}$? And as I was just explaining, it is supposed to be the number I would need to multiply with $\frac{c}{d}$ to get $\frac{ac}{bd}$. But we know that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, which precisely tells us that $\frac{a}{b}$ is the number you need to multiply with $\frac{c}{d}$ to get $\frac{ac}{bd}$. Therefore, $\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$.

$\frac{\mathbf{3}}{\mathbf{7}}$: Very nice. What did you do then?

You: I rewrote and colored the last equation because I wanted to keep it all straight.

$$\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$$

$\frac{\mathbf{3}}{\mathbf{7}}$: Nice Idea. I, myself, am rather colorful.

You: I want to start with the red thing, $\frac{ac}{bd}$, and then do something to it so I end up with the green thing, $\frac{a}{b}$. What I have to do is the blue part, which is what I'm figuring out, so I can explain it.

$\frac{\mathbf{3}}{\mathbf{7}}$: Yes.

You: Well, the result of the division of $\frac{ac}{bd}$ by $\frac{c}{d}$ has to be $\frac{a}{b}$. Looking at my starting expression, $\frac{ac}{bd}$, I see that it already has an a on the top and a b on the bottom. The only problem is that it has extra stuff – c on the top and also d on the bottom – that I want to get rid of.

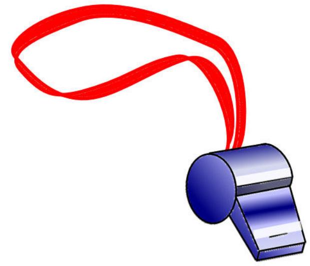
$\frac{\mathbf{3}}{\mathbf{7}}$: Mmm hmm.

You: If I could just cross out the top c and the bottom d, I'd have what I need. So then I thought about when it's okay to cross out numbers in fractions. I can cross out – cancel – numbers when the same number appears on both the top and bottom. So if I do something that puts a c on the bottom and a d on the top, then I can get rid of what I want to eliminate.

$\frac{\mathbf{3}}{\mathbf{7}}$: So what does that?

You: The way I get a c on the bottom and a d on the top is to multiply by d/c.

$\frac{\mathbf{3}}{\mathbf{7}}$: Yes.



You: So I do $\frac{ac}{bd}\times\frac{d}{c}=\frac{acd}{bdc}$. Then I cancel the c's and d's, and the result is $\frac{a}{b}$.

$\mathbf{\frac{3}{7}}$: So what does that tell you?

You: That tells me that I want the blue part – dividing by c/d – to be the same as multiplying by d/c. It shows that the "flip and multiply" song does tell you the right thing to do.

$\mathbf{\frac{3}{7}}$: Except for the "Ours not to reason why" part.

You: Yes ma'am. I also thought of another way to explain it.

$\mathbf{\frac{3}{7}}$: Oh, what riches! Do share.

You: Well, division by m/n should undo multiplication by m/n.

$\mathbf{\frac{3}{7}}$: Yes. I'd wager you can say it with fewer words!

You: $(z \times m/n) \div m/n = z$, for all z.

$\mathbf{\frac{3}{7}}$: Delightful!

You: But I was just thinking that $(z \times m/n) \times n/m$ is also z for all z.

$\mathbf{\frac{3}{7}}$: Now what wonderful property of multiplication tells you that?

You: Associativity! You see $(z \times m/n) \times n/m = z \times (m/n \times n/m) = z \times 1 = z$.

$\mathbf{\frac{3}{7}}$: Splendid! But your little sister may say, "What of it?"

Sis: Yeah, what of it?

You: Well, don't you see? Multiplication by n/m always does just what we would want division by m/n to do. It turns the number $z \times m/n$ back into z. And if two things always do the same thing, then they must be the same thing. Therefore, division by m/n *is* the same thing as multiplication by n/m … just flip and multiply!

$\mathbf{\frac{3}{7}}$: Lovely! So which explanation did you tell your dad?

You: The first one, eventually.

$\mathbf{\frac{3}{7}}$: What do you mean, "Eventually?"

You: Well, we were joking. So first I told him it was reasonable to flip and multiply because everyone else does.

Summer Fun!

The best way to learn math is to do math!

Here are the 2012 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems before seeing solutions.

By the way, some of these problems will feel quite different from the math problems you may be used to from school. Usually, problems that you get at school are readily solvable. However, some of these problems were designed to require more time and effort.

If you are used to solving problems quickly, it can feel frustrating at first to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain or rock climbing. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated at times when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Grand Slam Season

by Lightning Factorial

The French Open last month, Wimbledon now, and soon the US Open… It's grand slam season in tennis world. Not only is there a lot of tennis going on, there's a whole lot of math too, at least, if you take a shot at these grand slam inspired problems.

1. The official tennis court for singles play is 78' by 27'. Serena Williams can whip up service speeds that average over 100 miles per hour. How long does a tennis ball take to travel the length of the court if its average speed is exactly 100 miles per hour from baseline to baseline?

2. You're at the left corner of the court when you go for a down-the-line passing shot as your opponent rushes the net. Amazingly, your opponent makes a stunning leaping sharp angle volley. To get it, you must run clear across your half of the court diagonally. Fortunately, you anticipated this possibility, so you have 2 seconds to get there. How fast will you have to run?

3. You're on the center line, ten feet behind the net poised for on overhead smash. You suspect your opponent is going to make a move to one side, so you decide to go right down the center, where the net is 3 feet high. If your racquet strikes the ball 9 feet above the ground, what is the size, in degrees, of the vertical window angle you have to shoot through? Assume that the ball goes so fast its parabolic trajectory to the ground is well-approximated by a straight line.

Tiebreakers

Some sets are decided by a tiebreaker. In a tiebreaker, the first player to score at least 7 points by a margin of 2 is the winner. The players serve in turns; the first turn is one serve, and the rest are two serves. In tennis, psychological factors are critical, so it would be quite important for those who set the rules to create a system that does not have a built-in bias for one or the other player. Does the first to serve have an advantage in the tiebreaker? To examine this, let's assume that the two competitors are exactly equal in ability. When each serves, each wins the point with probability p and loses the point with probability $1 - p$. If fair, the probability that the first player wins should be one half.

3. Let's start with a simpler situation: a tiebreaker where the first player to reach 2 points wins. What is the probability that the first to serve wins? Curiously, note that if both players win every point on their serve, then the first player to serve will *always* lose this tiebreaker because even though she'll win her first point, she'll lose her next two (since the other player serves twice in a row). Is such a tie breaker inherently unfair?

4. Now suppose the tiebreaker is to 2 points, but one must win by a margin of 2 points. (This is sort of like a deuce situation except for the fact that the players alternate serving according to the ABBAABBAA… pattern.) Now what is the probability that the first to serve wins? Is such a tie breaker inherently unfair?

5. Redo problems 3 and 4 for a 3-point tiebreaker.

6. Are 7-point tiebreakers inherently fair?

Summer Fun!

Absolute Values

by Ken Fan

The absolute value of a number x, denoted $|x|$, is its distance from zero. Thus, for example, $|4| = 4$, $|0| = 0$, and $|-3| = 3$.

1. Make a graph of $y = |x|$ in the xy-coordinate plane.

Another way of describing $|x|$ is to say that $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Because of this "piecewise" nature of the absolute value function, it often helps to use cases when solving problems that involve the absolute value.

2. Find all solutions to each of the following equations:

a. $|x| = 5$ b. $|-x| = 5$ c. $|x - 3| = 7$

d. $|3x| = x + 8$ e. $5|x + 4| = |x - 8|$ f. $|x| - 3 = 7$

g. $x + |x| = 0$ h. $|x + 2| - |x - 2| = 1$ i. $|x^2 - 5| = 4$

Things get even more interesting when there is more than one variable.

3. Graph the solutions to each of the following equations in the xy-coordinate plane:

a. $|x| + |y| = 5$ b. $|xy| = 1$ c. $|x + y| = 4$

d. $|x| - |y| = 8$ e. $|x - 6| + |y + 3| = 0$ f. $y^2 = |x| + 1$

g. $x^2 - 2|xy| + y^2 = 25$ h. $|x + y| + |x - y| = 2$ i. $|y| = x^2 - 16$

4. Let (a, b) be a point in the first quadrant of the xy-coordinate plane. Let $f(x)$ be a function which is equal to 0 for $x < 0$, equal to b for $x > a$, and for values of x between 0 and a (inclusive), let $f(x) = bx/a$. Use the absolute value function to make a single expression for $f(x)$ for all real x.

5. Show that $|xy| = |x||y|$ for all x and y.

6. Show that $|x + y| \leq |x| + |y|$ for all x and y. When does equality hold?

7. Show that $||x| - |y|| \leq |x - y|$ for all x and y. When does equality hold?

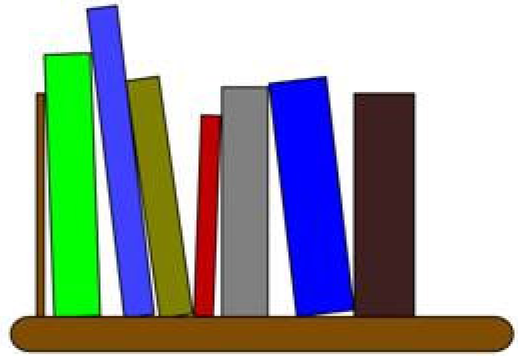
8. Let a_1 and a_2 be integers. For $n > 2$, define $a_n = |a_{n-1} - a_{n-2}|$. Show that $a_m = 0$ for some m. Would this still be true if a_1 and a_2 can be any real numbers?

9. Fix a positive integer n. Let f be a function whose domain is the set $\{1, 2, 3, \ldots, n\}$ and whose range is $\mathbf{R}$, the set of real numbers. Show that the function rule for f can be written as a single expression that involves only the basic arithmetic operations and absolute values.

Summer Fun!

Your Choice

by Shravas Rao



1. Suzie has 7 books she wants to arrange in a row on her bookshelf. How many different ways can she do this?

2. Suzie noticed that 3 of her books are part of the Hunger Games trilogy. She wants to make sure that when she arranges her books, the 3 books are always right next to each other, and are in the correct order from left to right. How many different ways can she arrange her books now?

3. A group of 8 girls are getting ready to go play soccer as a team for a competition. Their team will have a goalie, a left and right defender, a left, center, and right midfielder, and a left and right forward. How many different ways can all of the girls be assigned a position on the team?

4. Jen has been practicing to be goalie for a while, so the girls decide she can take that position. How many different ways can all of the girls be assigned a position on the team if Jen is going to be the goalie?

5. When the girls get to the competition, they find out that the soccer competition was actually a tennis competition! For the tennis competition, each team can only choose 2 members to participate. How many different ways can the girls choose who to play for their team? (Note that the girls are not assigned positions in tennis).

6. Because so many people show up to the competition, the tournament director decided to let 6 members from each team participate. Now how many different ways can the girls choose who to play?

7. What is the relationship between the answers you got for problems 5 and 6. Why is that so? Can you explain this relationship without using formulas?

8. Nicole is really good at tennis, so the girls decide to pick her to play tennis. However, they still have 5 people left to choose to play. How many different ways can they do this?

9. While warming up, Nicole sprained her ankle, and could not play tennis in the actual competition. Now how many ways can the girls choose who to play tennis?

10. How does the sum of the answers to problems 8 and 9 compare to the answer you got to problem 6? Why is that so? Can you come up with an explanation that does not use formulas?

Summer Fun!

11. The 6 girls all together won 10 games. Assuming that none of these girls played against each other, and without distinguishing between the wins, how many ways can we distribute the wins among the girls? For example, one possibility is that the first girl won 2 games, the second girl won 4, and the rest won 1 game each.

12. The girls played a total of 15 games, winning 10 games and losing 5. Also, no two games were played at the same time, so the girls could list the games in order of when they were played. When the girls looked at their results for each game, they found an interesting property. If they won the mth game, they also won the $(16 - m)$th game, and if they lost the mth game, they also lost the $(16 - m)$th game. How many different ways could they have won and lost their games? One example of a possible list of results is **WWLWWLWLWLWWLWW**, where if the mth letter is a **W**, the girls won the mth game, and if the mth letter is an **L**, the girls lost the mth game.

13. At the end of the matches, the girls learned that they had to come back the following day to participate in the doubles tournament. The same 6 girls had to split themselves up into 3 doubles teams. How many different ways could they form doubles teams?

14. After the competition is done, Angela decides to walk home from the tournament. The tournament is located at the point (2, 4) and her house is located at the point (0, 0). Assume that after every five minutes, Angela either walks 1 step down, or walks 1 step to the left. If it takes her 30 minutes to get home, how many different paths could Angela have taken to get home?

Generalization

15. Suppose you have n books and want to place them on a bookshelf. Of these n books, there are k books that you insist must appear in a specific order on the shelf. You don't care about the exact positions of these k special books. They don't even have to go next to each other. You just care that the k special books appear in a specific order from left to right. How many ways can you place the n books on the bookshelf?

16. If a team of k people must be chosen from n players, how many ways are there to create such a team?

17. Suppose k girls all together win n games. Assuming that none of these girls played against each other, and without distinguishing between the wins, how many ways can we distribute the wins among the girls?

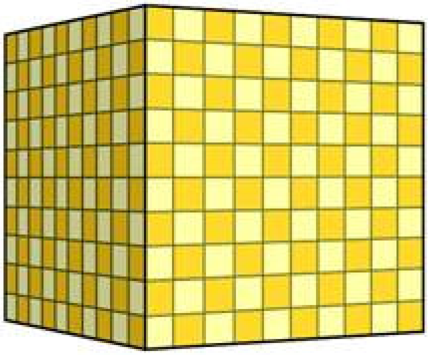
18. How many ways can $2n$ girls split themselves up into n doubles teams?

19. At the Wimbledon Championships, 128 women enter a single elimination tournament. There are a total of 7 rounds with 2^{7-R} matches played in round R. What is the *total* number of possible different match-ups in one complete Wimbledon Championship?

Summer Fun!

Fraction Traction

by 7/3 (evil twin of 3/7)



Gain traction with your fractions by solving these ten challenging problems.

1. In this issues' *Anna's Math Journal*, Anna computes that

$$-\frac{3}{20}+\frac{1}{2}-\frac{7}{10}+\frac{3}{4}-\frac{1}{2}+\frac{1}{10}=0.$$

Verify this.

2. A solid 100 by 100 by 100 cube is made out of unit cubes. What fraction of all the unit cubes' faces are hidden from view?

3. Write down a bunch (say a dozen or so) of different positive fractions in lowest terms. Convert each one of them into decimal form and take note of which ones yield terminating decimal expansions. Let p/q be a positive fraction written in lowest terms. For what q will the decimal form of p/q be terminating?

4. What is the smallest value you can obtain by adding a positive fraction to its reciprocal?

5. Let O be the set of positive fractions which have odd denominators when written in lowest terms. If x and y are in O, show that both $x + y$ and xy are also in O.

6. Let E be the set of positive fractions which have even denominators when written in lowest terms. If x and y are in E, show that xy is also in E. Find x and y in E where $x + y$ is *not* in E.

7. Let T be the set of fractions which have denominators that are powers of 2 when written in lowest terms. (Note that $2^0 = 1$, so $0 = 0/1$ and $1 = 1/1$ are both in T.) If x and y are in T, show that both $x + y$ and xy are also in T.

8. Let $f_1 = 1$. For $n > 0$, define $f_{n+1}=\frac{1}{1+f_n}$. Compute the first several terms of f_n. What is the limit, as n tends to infinity, of f_n?

I can't resist the urge to turn everything upside down… like the Fibonacci sequence.

Let $a_1 = 1$ and $a_2 = 1$. For $n > 1$, instead of defining a_{n+1} to be $a_n + a_{n-1}$, define $a_{n+1}=\frac{1}{a_n+a_{n-1}}$.

9. Compute the first 8 terms of the sequence a_n.

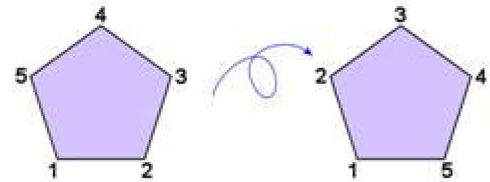
10. What do you think the limit, as n tends to infinity, of a_n is? Can you prove it?

Summer Fun!

The Symmetry of Regular Polygons

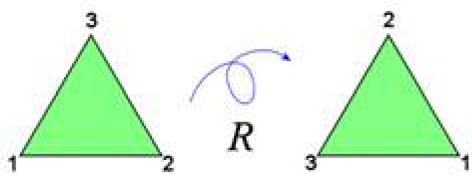
by Grace Lyo

Say you have some regular polygon with numbered vertices. You can pick up the polygon and put it back down so that it looks exactly the same as it did before, except that the numbers are indeed in different places.



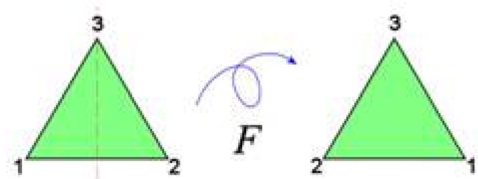
Call any such move a symmetry operation. In this problem set, we are only concerned with what symmetry operations do to the numbers on the vertices of a polygon. Since we don't care about anything else, we will say that two symmetry operations are the same if they have the same effect on these numbers.

Equilateral Triangles



One symmetry operation of an equilateral triangle is rotation counterclockwise by 120 degrees. Let's call this operation ***R***, and write $\boldsymbol{R}({}_1\overset{3}{\triangle}{}_2) = {}_3\overset{2}{\triangle}{}_1$ to indicate that ***R*** transforms the triangle ${}_1\overset{3}{\triangle}{}_2$ into the triangle ${}_3\overset{2}{\triangle}{}_1$.

You can also flip the triangle over the vertical dotted line in the picture at right. Let's call this operation ***F***, and write $\boldsymbol{F}({}_1\overset{3}{\triangle}{}_2) = {}_2\overset{3}{\triangle}{}_1$.



Problems

1. What happens if you start with the original triangle and perform ***R*** twice? What if you perform ***R*** three times? Four times?

3. How about if you perform ***F*** twice?

4. What happens if you perform ***R*** first, and then ***F***? How about ***F*** first and then ***R***?

5. Don't read this problem until you have completed problems 1-4.

Let's set up some notation. Let $\boldsymbol{F} \circ \boldsymbol{R}$ denote the result of applying ***R*** followed by ***F***.

When we apply the same symmetry operation several times in a row, we use notation that looks like exponents: we write $\boldsymbol{R}^2$ instead of $\boldsymbol{R} \circ \boldsymbol{R}$. In the previous problems, you may have noticed that $\boldsymbol{R}^2({}_1\overset{3}{\triangle}{}_2) = {}_2\overset{1}{\triangle}{}_3$ and $\boldsymbol{R}^3({}_1\overset{3}{\triangle}{}_2) = {}_1\overset{3}{\triangle}{}_2$.

In other words, performing ***R*** three times has the same effect as doing nothing at all.

Summer Fun!

Use "**1**" to represent the symmetry operation that leaves the vertices where they began. Thus, $\boldsymbol{R}^3 = \mathbf{1}$ because both $\boldsymbol{R}^3$ and **1** have the same effect on the number labels of a triangle.

a. For what numbers n is $\boldsymbol{R}^n = \mathbf{1}$? For what numbers n is $\boldsymbol{R}^n = \boldsymbol{R}$?
For what numbers n is $\boldsymbol{R}^n = \boldsymbol{R}^2$?

b. For what numbers n is $\boldsymbol{F}^n = \mathbf{1}$? For what numbers n is $\boldsymbol{F}^n = \boldsymbol{F}$?

6. How many different symmetry operations are there for a triangle?

7. Don't read this problem until you've done problem 6.

In problem 6, you found that there are 6 symmetry operations: **1**, ***R***, $\boldsymbol{R}^2$, ***F***, ***FR***, and $\boldsymbol{FR}^2$. The symmetry operations ***FR*** and $\boldsymbol{FR}^2$ are flips, but across a different line than the vertical.

Is there any way to write ***RF*** in the form $\boldsymbol{FR}^k$, for some k?

8. Complete the table below. Each square should contain a single symmetry operation (**1**, ***R***, $\boldsymbol{R}^2$, ***F***, ***FR***, and $\boldsymbol{FR}^2$).

Table of Symmetry Operation Compositions Y ○ X

Y \ X	**1**	***R***	$\boldsymbol{R}^2$	***F***	***FR***	$\boldsymbol{FR}^2$
1						
R						
$\boldsymbol{R}^2$						
F				**1**		
FR						
$\boldsymbol{FR}^2$		***F***				

9. Adapt and do all of the above problems for squares instead of triangles.

10. Adapt and do all of the above problems for pentagons instead of squares.

11. What patterns do you notice? How many symmetry operations do you think there are for a regular n-gon?

Summer Fun!

Coin Fountains

by Ken Fan

This problem set owes itself entirely to the work of Andrew M. Odlyzko and Herbert S. Wilf[1], who studied a question of Jim Propp, who asked, how main coin fountains are there?

A **coin fountain** is a connected arrangement of coins into rows in such a way that every coin not in the bottom row sits upon exactly two coins in the next lower row. Let $F(n)$ be the number of coin fountains with n coins.

1. Fill in this table. Some values are included so you can check your work.

n	1	2	3	4	5	6	7	8	9	10
$F(n)$	1	1	2			9			45	

An **(n, k) fountain** is a coin fountain with a total of n coins and k coins in its bottom row. For example, in the picture above, the top coin fountain is a (9, 6) fountain and the bottom one is a (20, 9) fountain. One last definition: Call an (n, k) fountain **primitive** if the row just above its bottom row contains $k - 1$ coins. (Neither coin fountain pictured above is primitive.) Let $f(n, k)$ be the number of (n, k) fountains and let $p(n, k)$ be the number of primitive (n, k) fountains.

2. Show that $f(n - k, k - 1) = p(n, k)$ for $n \geq k$ and $k \geq 1$.

We'll declare the convention that $f(0, 0) = 1$ and $p(0, 0) = 0$.

3. Show that $f(n,k) = \sum_{n',r \geq 0} p(n',r) f(n-n',k-r)$ for all $n, k \geq 1$. (Hint: Count $f(n, k)$ by breaking into cases indexed by r, where r is equal to the first empty position in the row just above the bottom row.)

Let $\overline{f}(x, y)$ and $\overline{p}(x, y)$ by the generating functions associated with $f(n, k)$ and $p(n, k)$, respectively. That is, $\overline{f}(x, y) = \sum_{n,k \geq 0} f(n,k) x^n y^k$ and $\overline{p}(x, y) = \sum_{n,k \geq 0} p(n,k) x^n y^k$.

4. Show that problems 2 and 3 are equivalent to $\overline{p}(x, y) = xy\overline{f}(x, xy)$ and $\overline{f} = 1 + \overline{f}\,\overline{p}$.

5. Deduce that
$$\overline{f}(x, y) = \cfrac{1}{1 - \cfrac{xy}{1 - \cfrac{x^2 y}{1 - \cfrac{x^3 y}{\ddots}}}}.$$
Set $y = 1$ and get a neat continued fraction expression for the generating function of $F(n)$.

[1]Odlyzko and Wilf, "*n* Coins in a Fountain," *Amer. Math. Monthly* **95**, pp. 840-843, 1988.

Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

<u>Session 10 – Meet 12 – May 3, 2012</u>

Mentors: Jennifer Balakrishnan, Samantha Hagerman, Connie Liu,
Jennifer Melot, Charmaine Sia

We hosted the traditional end-of-session math treasure hunt.

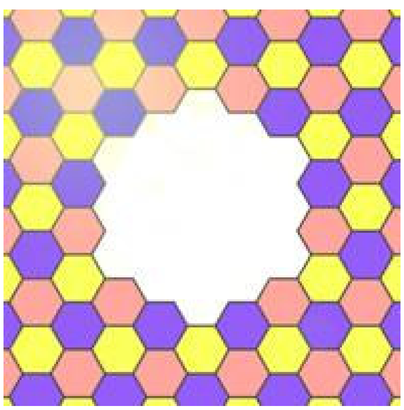
Congratulations to all participants for your excellent work and cooperation and for cracking the combination with half an hour to spare! Special thanks to Connie Liu for writing the hunt.

Here are some of the problems members had to solve:

To Infinity & Beyond!

$$\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\cfrac{6}{6+\dots}}}}$$

The fraction shown at right can be simplified to the form of $a+\sqrt{b}$, where a and b are integers. What is a?



Hexagonal Tilings

Zee and friends made a honeycomb which has the design of a tessellation made with hexagonal tiles (see figure at left). The honeycomb was missing a section. How many hexagonal tiles are missing from the missing section?

Gifts

A souvenir store sells three types of gems: aquamarines, emeralds, and rubies. All the gems of any one type are identical. Melanie wants to buy a total of 15 gems for her friends and family, but she also wants to buy at least two of each type of gem. How many ways can Melanie purchase the gems?

Sock Guarantee

Before Melanie undertook her great journey, she was looking for her socks this morning. Her sock drawer has 25 electric yellow socks, 30 blue striped socks, 33 pale purple socks, 30 royal red socks, 11 gruesome green socks, 14 midnight black socks, and 23 burnt brown socks! If she reaches into the drawer in the dark, how many socks does she need to pull out to be sure she has a matching pair?

Calendar

Session 10: (all dates in 2012)

January	26	Start of the tenth session!
February	2	
	9	Meike Akveld, ETH Zürich
	16	
	23	No meet
March	1	
	8	Julie Yoo, Kyruus
	15	
	22	
	29	No meet
April	5	
	12	Sarah Spence Adams, Olin College
	19	No meet
	26	
May	3	

Session 11: (all dates in 2012)

September	13	Start of the eleventh session!
	20	
	27	Charlene Morrow, Mt. Holyoke
October	4	
	11	Pardis Sabeti, Broad Institute/Harvard
	18	
	25	Anoush Najarian, MathWorks
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

If you like solving math contest problems, consider Math Contest Prep. The first class is September 16, 2012 at the Microsoft NERD Center.

Special Thanks to MathWorks for awarding us a grant for Bulletin Content and Production!

Here are answers to the *Errorbusters!* problems on page 14.

1. 3
2. -4
3. 2
4. 1
5. 4/3
6. -5
7. 0
8. 2
9. 4

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club and receive a printed copy of the Girls' Angle Bulletin for the duration of the membership. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin and a 25% discount for any club meet attended. Remote members may email us math questions (although we won't do people's homework!).

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership (which includes 12 two-hour club meets)
  - ☐ $108 for a one year Remote Membership
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls’ Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)’s participation in Girls’ Angle and do forever and irrevocably release Girls’ Angle and its directors, officers, employees, agents, and volunteers (collectively the “Releasees”) from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney’s fees, in any way connected with or arising out of my child(ren)’s participation in Girls’ Angle, whether or not caused by my child(ren)’s negligence or by any act or omission of Girls’ Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)’s participation in Girls’ Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls’ Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys’ fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)’s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)’s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

Girls' Angle

A Math Club for Girls