Girls' Angle Bulletin

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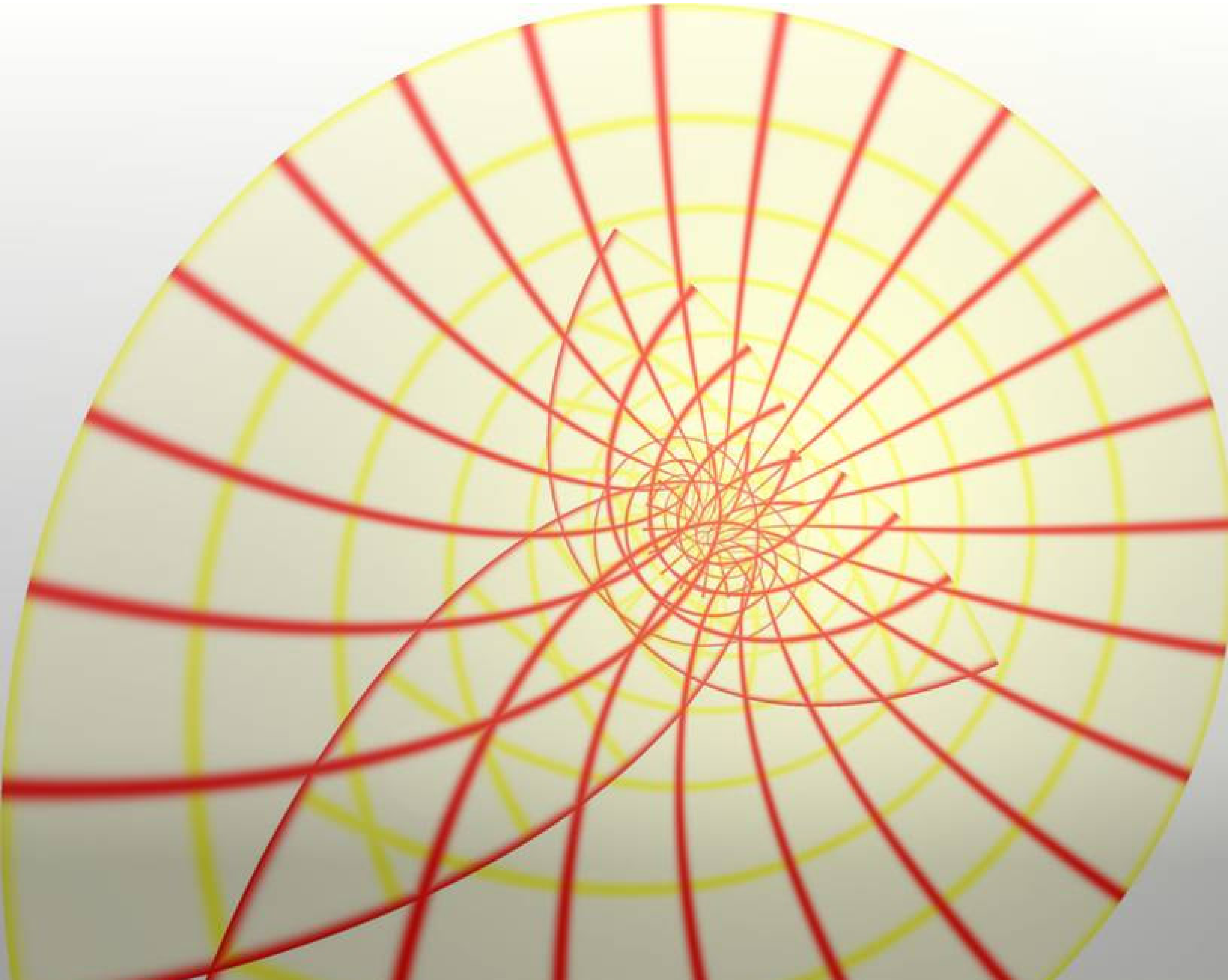
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

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- Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Angel of Geometry. Rotation of the image of a 2 by 2 square centered at the origin of the complex plane under the function $f(z) = z^5 + z^4 + z^3 + z^2 + 1$.

An Interview with Jean Pedersen, Part 3[1]

Ken: Can you explain to us a favorite result that you proved?

Jean: I think the result that I have had the most fun with recently is one that I had a role in discovering after Peter Hilton died. It came about when I was grading final exams for a very elementary mathematics course and students routinely made a certain mistake in computing the "symbol" for constructing a given regular polygon. I began to suspect that there was something important in the mathematics of these symbols and had a hunch that the mathematics connected with the students' wrong answers just might be trying to tell us something that was important. I had an idea about what question these incorrect results might be answering. After examining several cases it looked very promising. So I enlisted two of my colleagues — Robert Bekes, who was teaching combinatorics at the time, and Bin Shao, who is very good at writing computer programs — to work with me on finding out what was going on. The whole story, about Mad Tea Party

[1] Photographs in Part 3 were taken by the author and students during the dodecahedron-building contest.

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Girls' Angle is now selling advertising space in the electronic version of the Bulletin.

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Mathematical Buffet

Conformal Images by Ken Fan

The images in this Mathematical Buffet were created using MATLAB, a powerful suite of mathematical software produced by MathWorks.

To fully understand this Mathematical Buffet, you need to know about complex numbers.

Here's a crash course: Complex numbers are a set of numbers, just like the integers, rational numbers, or real numbers. Like real numbers, they can be added, subtracted, multiplied, and divided. They generalize the real numbers because the real numbers are a subset of the complex numbers, similar to the way that rational numbers are a subset of the real numbers.

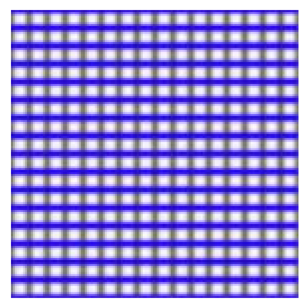
For real numbers, the number line serves as an excellent model. Each point on the number line represents a real number. For complex numbers, the plane provides a beautiful model, with each point in the plane representing a different complex number.

If we introduce Cartesian coordinates, each point is specified by a pair of real numbers (x, y). Here are the formulas for complex number addition and multiplication in terms of these Cartesian coordinates:

$$(a, b) + (x, y) = (a + x, b + y) \text{ and } (a, b) \times (x, y) = (ax - by, ay + bx).$$

Now let's think about functions that map the complex numbers to the complex numbers. We can think of such functions as maps from the plane to itself because the complex numbers are modeled by the plane. Such functions will map points of the plane to points of the plane.

Consider the 2 by 2 square with vertices at $(\pm 1, \pm 1)$. Pick a color and paint evenly spaced vertical bands in this square. Pick another color and paint the same number of evenly spaced horizontal bands to create a grid. The figure at right gives an example using blue and gray.



All pictures in this Mathematical Buffet are the images of such painted squares under complex functions of a complex variable. In other words, under a complex function, this square will be transformed, its bands twisted and turned. We're showing these images. If you know complex analysis, the functions used are all differentiable which roughly means that very tiny figures are mapped to similar versions of themselves.

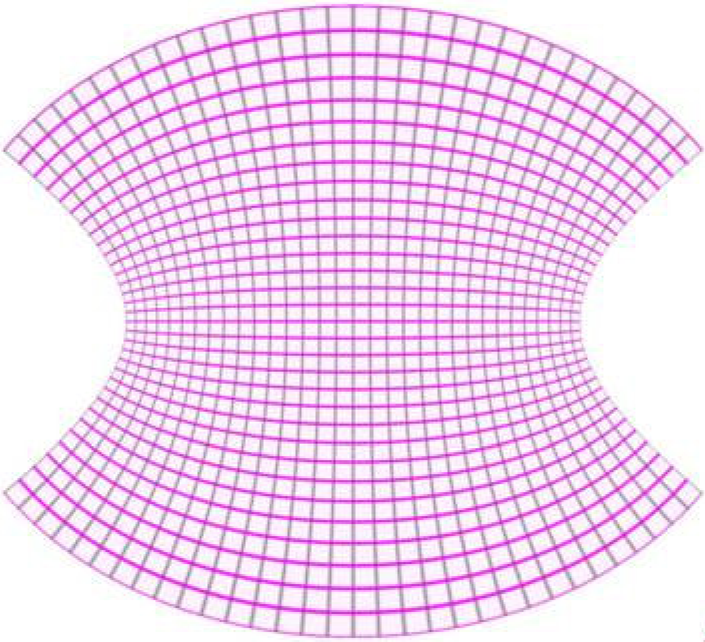
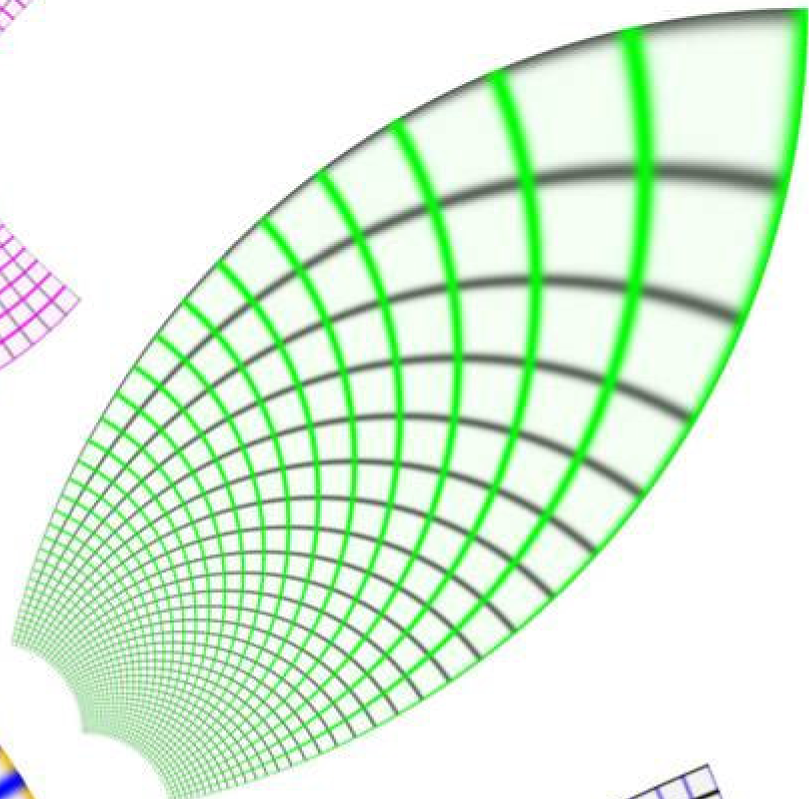
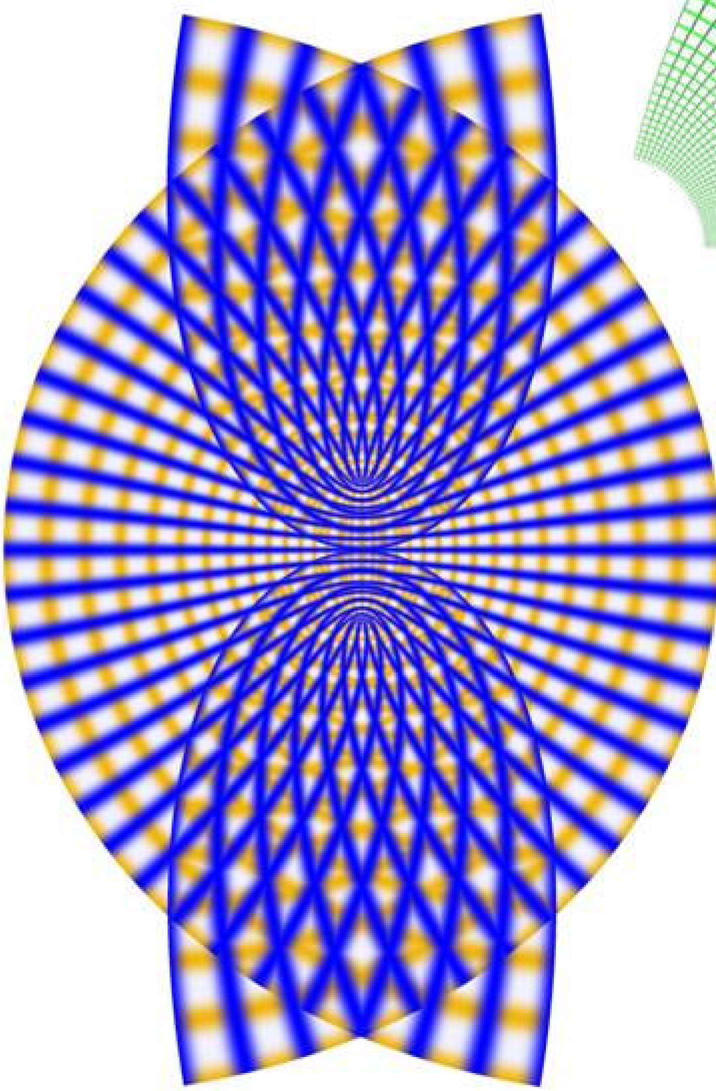
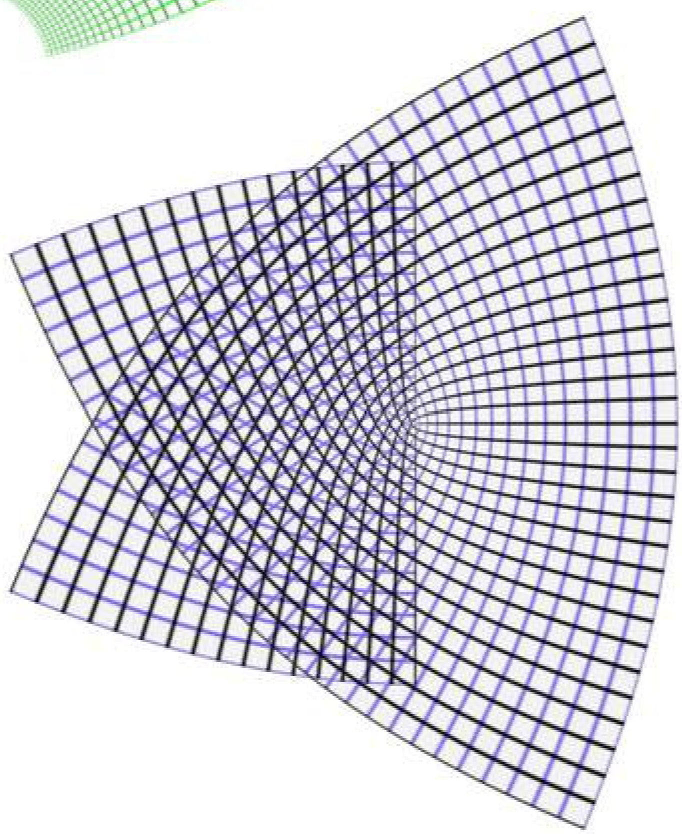
For example, consider the function $f(z) = z^2$. That is, given a complex number, this function returns the square of that number. Using the multiplication rule described above in terms of coordinates, we can compute the value of this function on the number corresponding to the point (a, b) in the Cartesian plane. We compute $(a, b) \times (a, b) = (aa - bb, ab + ba) = (a^2 - b^2, 2ab)$. This tells us that under the function f, the point (a, b) in the complex plane will be mapped to the point $(a^2 - b^2, 2ab)$. For example, $(1, 1)$ will map to $(1^2 - 1^2, 2(1)(1)) = (0, 2)$, and $(-1, 1)$ will map to $(0, -2)$. The other two corners of the square, $(-1, -1)$ and $(1, -1)$, will map to the points $(0, 2)$ and $(0, -2)$, respectively.

Here are the functions whose images are illustrated in the following pages:

$$z^2 \qquad z^3 \qquad z^4 \qquad z^5 \qquad z^3 + z \qquad e^z \qquad \sin z \qquad z^{3/2} \qquad \frac{1}{z+c}$$

where c is the complex number corresponding to the point $(5/4, -5/4)$. Can you figure out which figure correspond to which function? For the answer, turn to page 29. The cover shows a rotation of the image of the square under the function $f(z) = 1 + z + z^2 + z^3 + z^4 + z^5$. Please note that the scale of these images varies from one image to the next.

For more on complex numbers, see *Pondering Complex Numbers* on page 15.

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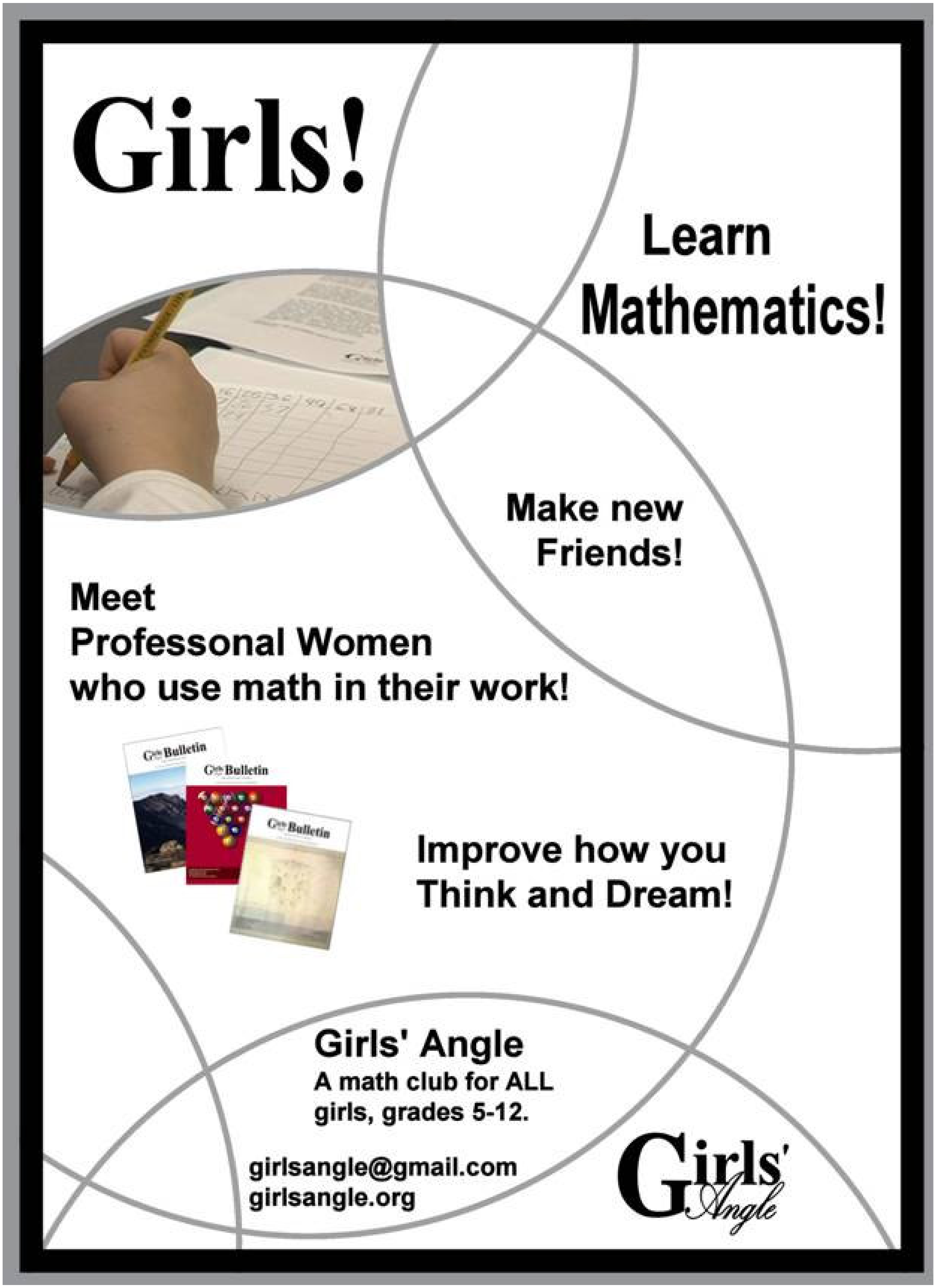
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Sprint Track Cycling

Written by Taotao Liu
Edited by Jennifer Silva

Logo Design by Hana Kitasei

I love watching the Olympics. My favorite sports to watch are diving, swimming, gymnastics, and equestrian. But one day this month, I happened to catch some footage of sprint track cycling, and I found it to be one of the most interesting sports in the Olympics.

Sprint track cycling is a 600-meter bicycle race on an indoor track (three laps of 200 meters each) that's banked like at NASCAR. Yet I immediately noticed several non-race-like attributes of the event. The first quirk was that the racers face each other one-on-one in a tournament format, rather than as eight to ten people racing simultaneously like in swimming or track and field. The next oddity is that the racers usually take the first two laps extremely slowly, and then take off as fast as they can for the final lap. It was hard for me to tell at first that there was even a race going on. The racers were going so slowly that they seemed to be just testing out their bikes to see if they worked. Also, the leader is constantly looking back over her shoulder at the other person, as if afraid she will be overtaken. I kept thinking, if she's so afraid of being overtaken, why not just go faster? To top it off, the riders don't even start at the same time. They are pushed off from the starting line by their coaches; the racer in the inside lane of the track goes first, followed by the other in the outside lane[1].

Why is this event so weird? This is a race in which the tactics are at least as important as speed. The reason tactics are required is because of **drafting**, an aerodynamic phenomenon that gives the person right behind the leading racer an advantage. Imagine walking in a really bad snowstorm, into a fierce wind. It's tough work getting where you need to go! Now imagine there's a person (of similar height and weight) walking in front of you, shielding you from that headwind. Wouldn't it be much easier to walk behind the person? When racers are going really fast, it's like braving the snowstorm: hard work. But if a racer stays close behind the person in front of her, she can have an easier time for most of the race, then pick a moment to use all of the energy she's conserved to overtake her opponent[2]. So if one racer chose to go fast at the beginning, she'd simply be setting up her opponent for a winning situation, or, to borrow a term from **game theory**, a **winning state**.

Proper game theory is the study of how to maximize your chances of winning a game, but not just any game. In game theory, games must be mathematically well-defined. The rules must be clear, and the possible moves each player can make at any point of the game must be precisely described. Winning and losing states must be spelled out. Snakes and ladders, dots and boxes, chess, go, and certain components of basketball are a few examples of the many

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[1] After some research, I found out that a coin is flipped to determine who "leads" on the inside track. The later races are all best-two-out-of-three heats, so in the second heat the other racer starts out leading. If there is a third heat, there is another coin flip to determine who leads.

[2] In some other sports, the effects of drafting are not as drastic, and the difference in sheer speed may be such that one athlete can get a big enough lead so that the trailing athlete(s) is too far back to benefit from drafting. In either case, there would be less of a need for strategic planning.

games that have been studied by mathematicians. Sprint track cycling is a complex game that depends on many hidden factors (such as the physical fitness of the individual racer) and is not the sort of game that game theorists would normally study. However, we can still apply some principles from game theory to analyze the sport and understand its seemingly bizarre features.

Count-to-Ten

But first, let's illustrate proper game theory by analyzing a simple game called "Count-to-Ten." In this game, two people count to 10 in turns. Each person may decide to count either one, two, or three more numbers, and the next person picks up where the previous person left off. Skipping numbers or turns is not allowed. Whoever says "ten" is declared the "loser," and the other player is declared the "winner." That is, saying **"ten" is a losing state**.

Here's an example of Al and Beth playing Count-to-Ten:

Al: "One, two."
Beth: "Three."
Al: "Four, five, six."
Beth: "Seven, eight, nine."
Al: "Ten."

Because Al said "ten," Beth wins. Notice that Beth intentionally went all the way up to "nine" on her last move (instead of stopping at "seven" or "eight"), forcing Al to say "ten." Since there was no way Beth could lose after she said "nine," saying **"nine" is a winning state**.

Games, Games, Games!

If you like games, here are a few that are fun to think about mathematically. For playing rules, please consult the internet.

Checkers
Chess
Domineering
Dots and Boxes
Go
Hackenbush
Hex
Mancala
Nine Men's Morris
Tic Tac Toe and variants
Toads and Frogs

For many of these games, the best strategy is unknown. Jonathan Schaeffer led a team that showed that perfect play in checkers leads to a draw. Go, an ancient Asian game, remains intractable. Curiously, in Hex, John Nash (the subject of the movie *Beautiful Mind*) proved that the first player has a winning strategy in 1952, but a complete winning strategy has yet to be devised. Maybe you can make one!

Let's see if we can go further. Working backwards, if a player says either "six," "seven," or "eight," his or her opponent can say exactly enough numbers to get up to "nine" and win. So saying **"six," "seven," or "eight" are all losing states**. On one of his turns, Al ended on "six," a losing state, giving Beth the opportunity to get up to "nine" and win. Was there a better move for Al? Yes. In order to prevent Beth from saying "nine" and winning, Al should have stopped at "five." Then Beth would have been forced to stop at either "six," "seven," or "eight," and she would have lost. So saying **"five" is a winning state**. Can you see how saying "two," "three," or "four" are losing states, and saying "one" is a winning state?

This leads to an interesting phenomenon. If both players play as well as they can, the first player will always win. By making sure to stop at "one," "five," and "nine," the second player doesn't have a chance. We say that the first player has a **winning strategy**. Many games studied in game theory have winning strategies for one of the players.

Back to Cycling

Let's try to analyze sprint track cycling from a game-theoretic point of view. By definition, being first to cross the finish line is a winning state. And, as we have already discussed, being slightly ahead when you're a significant enough distance away from the finish line (enough that your opponent has time to overtake you) is a losing state. On the other hand, being far enough ahead so that your opponent cannot draft off of you and then overtake you is a winning state. If no one wants the lead a significant distance before the finish line, both racers will go as slowly as they can for the first two laps. It is stated in the rules that the racers must start the race no slower than a walking pace. But in the middle lap, they are allowed to go slower, and even stop. Because their feet are not allowed to touch the ground, they sometimes hop up and down on their bikes to maintain a speed of zero[3].

Another maneuver that one might see makes use of the banked track. The leading racer may force the other racer to the very outside/top of the track just as they are about to make a turn. The outside racer must then pick up some speed (and likely take the undesired leading position as a result) in order to not fall off of her bike. Dutch cyclist Theo Bos used this maneuver to gain the upper hand against British cyclist Chris Hoy in their first heat for the 2008 World Championships.

Since it is unlikely that one biker can ride much faster than the other and can outstrip him for a full 600 meters, the only way to end up far enough ahead that one's opponent isn't drafting is to catch him unawares. That is why the bikers watch each other intently in the slow phase of the race, with the leader constantly looking back at his or her opponent. The extra task of having to look back all the time is another reason why the leading position in the early-to-middle part of the race is the less desirable position.

Finding the optimal time to pick up the pace is a key component of this event. Riders sometimes lose simply because they started going fast too soon, thereby ending up in the losing state of leaving too much room for their opponents to pass them. Two hundred meters away from the finish line is typically a distance through which a skilled rider who has successfully claimed the inner lane of the track can hold off his opponent.

So, all of this bizarre behavior is actually perfectly rational in light of the nature of the game.

Take it to Your World

Play Count-to-Ten with a friend! If you go first and do everything right, she won't be able to beat you. Then give your friend a chance to go first, and see if she can figure out how to win.

What if, instead of Count-to-Ten it was Count-to-Eleven? Does one player have a winning strategy? Can you analyze Count-to-*N* for any positive integer *N*?

Suppose you're in the company of more than one friend. In this case, you can all play a multi-player version of Count-to-*N*. Pick an order of play and modify the rules so that the first person to say "*N*" is eliminated. The remaining players then start again with the player who was due to go next starting the following round. Repeat until only one player, the winner, is left standing. Can we still label winning and losing states in the multi-player game?

Another fantastic game to analyze is called "Nim." In Nim, two players alternate taking objects from three heaps of objects. A player may remove any positive number of objects from any one heap. The goal is to be the last person to be able to take an object. Can you find a winning strategy for Nim? Is Nim similar to Count-to-Ten? What variants can you think of for Nim?

Next time you play a game, decide whether you can use game theory to analyze it. Can you find any winning or losing states? Is there a winning strategy? Sometimes, there's no certain winning strategy, and many games are fun because of this. But even in such cases, a game-theory-minded person might have more success at finding an advantageous strategy.

[3] According to the official rules, the maximum period for a standstill is 30 seconds, after which the leading racer must either resume or forfeit the race.

Pondering Complex Numbers

It takes time, effort, and patience to get a handle on new mathematical concepts. In this issue, complex numbers appear in both *Errorbusters!* (p. 17) and the *Mathematical Buffet* (p. 8). Complex numbers are well worth studying and mastering. Here, we state a number of truths about complex numbers for you to think about. Play with these until you are comfortably with them and see their validity. If you're having trouble and you're a subscriber, contact us! If you're really stuck, don't fret. The day will arrive when all of these statements are crystal clear.

Read the crash course on complex numbers in *Mathematical Buffet*. There, the number plane model of the complex numbers was introduced and formulas for the sum and product of complex numbers in terms of their Cartesian coordinates were given. The point (0, 1) in the complex plane is given the special label i.

Note that the multiplication symbol is often omitted in expressions. For instance, we usually denote the product $(a, b) \times (c, d)$ by writing $(a, b)(c, d)$. For more on the number plane, we also have a blog post entitled, "The Number Plane," at girlsangle.wordpress.com.

Fact 1. Consider the points on the horizontal axis of the complex plane. These have coordinates $(x, 0)$ for various real numbers x. Show that $(x, 0) + (y, 0) = (x + y, 0)$ and $(x, 0)(y, 0) = (xy, 0)$. Hint: Apply the formulas for complex multiplication explained in *Mathematical Buffet*.

This fact shows that the real numbers are embedded inside the complex numbers and the horizontal axis is a real number line. For this reason, we'll sometimes refer to the point $(x, 0)$ by just the real number x and we call the horizontal axis the **real axis**.

Fact 2. The complex number i satisfies $i^2 = -1$. (That is, $(0, 1)(0, 1) = (-1, 0)$.)

The significance of this fact is that it shows that with complex numbers, -1 has a square root. This is not true in the real numbers because the square of a real number is never negative.

Fact 3. We can express (a, b) as $a + bi$.

Writing complex numbers as $a + bi$, where a and b are real numbers is, in fact, the modern preferred notation for writing complex numbers. (Remember: the real number a is $(a, 0)$.)

Fact 4. Complex number addition and multiplication are commutative and associative. The number $0 = (0, 0)$ is the additive identity and $1 = (1, 0)$ is the multiplicative identity. Also, the distributive law holds. That is $z(w + v) = zw + zv$, for all complex numbers z, v, and w.

Fact 5. The distance of a complex number z from the origin is called the **norm** of z and is denoted $| z |$. That is, if $z = a + bi$, then $| z | = \sqrt{a^2 + b^2}$. The norm satisfies $| zw | = | z | | w |$.

Fact 6. For the complex number z, the angle formed by z, 0, and 1 (as measured counter-clockwise from the real axis in the complex plane) is called the **argument** of the complex number, and denoted arg z. The argument satisfies $\arg zw = \arg z + \arg w$ (up to a multiple of a full circle).

Fact 7. Let n be a positive integer greater than 2. The solutions to the equation $z^n = 1$ form the vertices of a regular n-gon in the complex plane. One of its vertices corresponds to 1.

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna continues her exploration of sums of the first *n* *k*th power. The mystery stymies her!

Today I'm going to try to understand why every other coefficient comes out to be zero.

Here's a summary of the relevant formulas and the conjecture.

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

$$S_k(n) = \frac{1}{k+1} n^{k+1} + C_0\binom{k}{0} n^k + C_1\binom{k}{1} n^{k-1} + C_2\binom{k}{2} n^{k-2} + \cdots + C_k\binom{k}{k}$$

$$C_p = \frac{1}{p+1}\Big(C_{p-1}\binom{p+1}{2} - C_{p-2}\binom{p+1}{3} + \cdots - (-1)^m C_{p-m}\binom{p+1}{m+1} + \cdots - (-1)^p C_0\binom{p+1}{p+1} - (-1)^{p+1}\frac{1}{p+2}\Big)$$

Conjecture: $C_{2m} = 0$ for $m \geq 1$. $\left(C_0 = \tfrac{1}{2} \right)$

If you don't understand what Anna is doing, please read the last few installments of Anna's Math Journal. This is a continuation of her investigation into sums of kth powers.

In this installment, Anna tries to unravel the mystery of the zero coefficients. But, the mystery remains.

Is she way off in left field, or is she tantalizingly close? Can you help her?

Or, maybe the conjecture isn't even true...What do you think?

What does it mean for every other coefficient to be 0? Well, this would then be a polynomial with only odd or even terms.

If true, $S_k(n) - \frac{1}{2}n^k$ would be a polynomial with only odd degree or even degree terms.

I don't want to get confused, so for definiteness, I'll start with the case where k is even.

Assume k is even, say $k = 2D$.

Then $p_k(n) = S_k(n) - \frac{1}{2}n^k$ would have only odd degree terms.

A polynomial with only odd degree terms is a so-called "odd" function, i.e. one for which $f(-x) = -f(x)$.

That would mean that $p_k(-n) = -p_k(n)$.

But what is $S_k(-n)$?

But what is the meaning of $S_k(-n)$? It's supposed to be the sum of the first n perfect kth powers...but what does this mean when n is negative?

$S_k(n)$ is the unique polynomial that satisfies

for all n → $S_k(n) - S_k(n-1) = n^k$ and $S_k(1) = 1$.

Hmmm. Actually, this consequence of my original definition of $S_k(n)$ makes sense for negative n. So I can use this to define $S_k(n)$ for all n.

I'll try to show that $p_k(n)$ is an "odd" function.

$$S_k(n) - \tfrac{1}{2}n^k - S_k(n-1) + \tfrac{1}{2}(n-1)^k = n^k - \tfrac{1}{2}n^k + \tfrac{1}{2}(n-1)^k$$

$$p_k(n) - p_k(n-1) = \tfrac{1}{2}n^k + \tfrac{1}{2}(n-1)^k$$

$$p_k(n-1) - p_k(n-2) = \tfrac{1}{2}(n-1)^k + \tfrac{1}{2}(n-2)^k$$

$$\vdots$$

$$p_k(1) - p_k(0) = \tfrac{1}{2}1^k + \tfrac{1}{2}0^k$$

Hmmm...I guess I'll try to figure out what $p_k(n)$ is. I can telescope...

$$p_k(n) - p_k(0) = \tfrac{1}{2}n^k + (n-1)^k + (n-2)^k + \cdots + 1^k$$

Backward

Oh dear! I've regressed instead of progressed! Help!

ABB 8.27.12

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

In my column in the August 2011 issue of this Bulletin, we discussed the mistake of "forgetting the negative root" when solving a quadratic equation. Another common error that I have seen is similar: I'll call it "forgetting the zero root."

Suppose we have to solve the equation $4x^3 - 36x = 0$. Perhaps the simplest way to start is to factor out the common factor of $4x$ to get $4x(x^2 - 9) = 0$. At this point, it is okay to simplify the equation by dividing both sides by 4, yielding $x(x^2 - 9) = 0$. The number 4 is simply a constant, and dividing both sides of an equation by a nonzero constant preserves equality. So why not divide through by the whole factor of $4x$ instead and simplify the equation even more? Many students do just that, but this gets them into trouble because x is a variable and, as such, could be equal to 0. As you know, dividing by 0 is never a good idea! In this case, dividing both sides by x to get $x^2 - 9 = 0$ leads one inevitably to omit the solution $x = 0$. The only roots of $x^2 - 9$ are 3 and -3 (don't forget the negative root!); the root 0 of the original polynomial $4x^3 - 36x$ has been lost.

We've been using the word **root** a lot. What exactly is a root, anyway? A root of a polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is a solution to the equation that we get by setting the polynomial equal to 0: $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$.

Oftentimes we are given equations that involve polynomials such as $5x^5 = 20x^3 - 20x$, where neither side of the equation is 0. In such cases, if we rearrange the terms so that all of them sit on one side of the equation, the result will be a polynomial equation where one side *does* equal 0: $5x^5 - 20x^3 + 20x = 0$. Thus, solving the equation $5x^5 = 20x^3 - 20x$ means finding the roots of the polynomial $5x^5 - 20x^3 + 20x$. In other words, *every polynomial equation can be converted into a problem of finding the roots of some polynomial*.

But let's get back to the main topic. The way to correct the mistake of "forgetting the zero root" is simply to pause a moment before dividing through by x and record the fact that either $x = 0$ or we can divide by x. In the case of our original example, we must say that either $x = 0$ or $x^2 - 9 = 0$. This way, we'll find all three solutions to the original equation: -3, 0, and 3.

There is an easy way to avoid this error and tell right away whether or not a polynomial has 0 as a root. In a polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, the term a_0 is called the **constant term**. Polynomials that have a constant term of 0 (or, as it is often put, "no constant term") are exactly those polynomials that have 0 as a root. Let's prove this fact!

First, suppose that we have a polynomial with no constant term. Then the polynomial must be of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x$ for some choice of $n > 0$ and coefficients a_k, where at least one of these coefficients must be nonzero. If we now substitute 0 for x, we see that every term of this polynomial evaluates to 0, so the whole polynomial does, too. That is, $x = 0$ is a solution to the equation $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x = 0$, and this exactly means that 0 is a root of $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x$. By the way, notice that when a polynomial has no constant term, it can be factored by x. We'll return to this fact later, but first let's prove the other direction: suppose that a polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ has 0 as a root. We must show that this polynomial has no constant term, or in other words, that $a_0 = 0$. But having 0 as a root means that when we substitute in $x = 0$, the polynomial evaluates to 0. If we substitute 0 for x we get, $a_n(0)^n + a_{n-1}(0)^{n-1} + \ldots + a_1(0) + a_0 = 0$, which simplifies to $0 + \ldots + 0 + a_0 = 0$, giving the desired fact that $a_0 = 0$.

Now take a look at the fifth-degree equation $5x^5 = 20x^3 - 20x$ mentioned earlier. Rearranging terms, we get $5x^5 - 20x^3 + 20x = 0$. But the polynomial on the left has no constant term! So 0 is a solution of the equation. We can record this solution, *then* proceed to find the other solutions by dividing both sides by the common factor $5x$ to get $x^4 - 4x^2 + 4 = 0$. To solve this fourth degree equation, we can factor the left side of this equation to get $(x^2 - 2)^2 = 0$, which has solutions $x = \pm\sqrt{2}$. So the solutions to the original equation are $x = -\sqrt{2}$, 0, and $\sqrt{2}$.

Earlier we noted that if 0 is a root of a polynomial, then the polynomial is divisible by x. It is actually true more generally that a polynomial has a root α if and only if it is divisible by $x - \alpha$. That is, a polynomial has a root α if and only if we can factor out $x - \alpha$. Let's prove this.

First, suppose that our polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is divisible by $x - \alpha$. This means that $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = (x - \alpha)q(x)$ for some polynomial $q(x)$. Plugging in $x = \alpha$ gives $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 = (\alpha - \alpha)q(\alpha) = 0 \cdot q(\alpha) = 0$. So α is a root of the polynomial.

Now suppose that α is a root of the polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$. We need to show that $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is divisible by $x - \alpha$. We can always apply the division algorithm and divide $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ by $x - \alpha$. The result of this division will be two polynomials $q(x)$ and $r(x)$ such that

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = (x - \alpha)q(x) + r(x),$$

where the degree of $r(x)$ is less than the degree of $x - \alpha$. (Recall that the **degree** of a polynomial is the highest exponent of x that appears in any of its terms after it has been written in simplified form. By convention, the degree of the zero polynomial is declared to be negative infinity.) But $x - \alpha$ has degree 1, so $r(x)$ can have at most degree 0. In other words, $r(x) = r$ for some constant r. Since α is a root of the original polynomial, we can substitute in α for x to get

$$0 = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 = (\alpha - \alpha)q(\alpha) + r = 0 \cdot q(\alpha) + r,$$

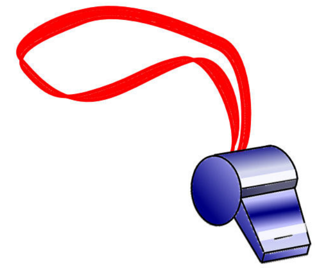
from which we see that $r = 0$. Therefore, $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = (x - \alpha)q(x)$, or, in other words, $x - \alpha$ divides evenly into $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$.

This seems like a good time to introduce the amazing **Fundamental Theorem of Algebra**, which states that *every* polynomial in one variable with complex number coefficients has at least one (complex number) root! (If you've never heard of complex numbers, you can find a crash course in the *Mathematical Buffet* on page 8, but we urge you to consult a more thorough treatment of the subject in a good textbook.) A stronger-sounding but equivalent version of the Fundamental Theorem of Algebra states that *every* polynomial in one variable with complex number coefficients can be factored into linear factors. That is, every polynomial of degree n with complex coefficients can be written as a product $c(x - \alpha_1) \cdots (x - \alpha_n)$, where c and α_k are complex number constants. Notice that it is not true that polynomials with real number coefficients always have a real number root. For example, consider the polynomial $x^2 + 1$ which is greater than 0 for all real numbers x, and hence has no real number root. This is one of the reasons why it can be advantageous to work with complex numbers instead of real numbers.

For practice, determine whether each of the following equations has 0 as a solution or not. Then find all other solutions, if possible. The answers can be found on page 29.

1. $x^2 = 0$	2. $x^3 = 2x^2$	3. $2x = x^2 + 1$	4. $x^2 - 3x = -2$
5. $x^4 + 2 = 3x^2$	6. $x^5 = x^3$	7. $3x^5 + 15x^3 = -12x$	8. $6x^3 - 18x^2 = 60x$

COACH BARB'S CORNE



by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 6

Oooh look! Here comes our old friend, 3/7. Something smells good. Coach Barb is still a little nervous around the dame, so you talk to her.

> Dear Reader,
>
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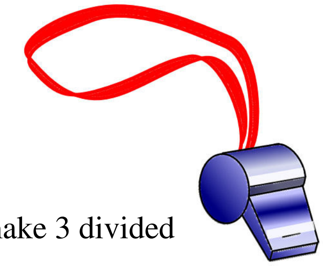
>
> Thank you and best wishes,
> Ken Fan
> President and Founder
> Girls' Angle: A Math Club for Girls

$\frac{3}{7}$: Very nice. Let's hear those comparisons.

You: Well, I have 6 cups of flour. You used 3 ½ cups so I have enough flour to make 6 divided by 3 ½ = 6 divided by 7/2 = 6 times 2/7 = 12/7 as much as you did.

$\frac{3}{7}$: Continue.

You: For sugar I have 2 cups and you used one cup, so I have enough sugar to make two times as much as you did.



$\frac{3}{7}$: Lovely.

You: For butter I have 3 cups and you used 1 ½ cups, so I have enough butter to make 3 divided by 1 ½ = 3 divided by 3/2 = 3 times 2/3 = 2 times as much butter as you used.

$\frac{3}{7}$: Yes. So?

You: So although I have enough sugar and butter to make twice as much shortbread as you did, the amount of flour is only enough to make 12/7 as much. I know flour is the ingredient that is limiting the size of my batch because 12/7 is less than 14/7, the latter of which is equal to 2.

$\frac{3}{7}$: Are you complaining? There is nothing that quite sets my teeth on edge as much as a complaining child.

Shortbread cookies

You: Oh, no. I'm grateful as can be. I'm going to be able to make 12/7 as much shortcake as you made. I can hardly wait!

$\frac{3}{7}$: Splendid. What do you call 12/7?

You: Well, it's a fraction. I used it to compare how much flour I had to how much you used. Since it was the smallest fraction of each of the ingredients' similarly composed fractions, I knew that was the fraction of your batch I could make.

$\frac{3}{7}$: Might you call it something else?

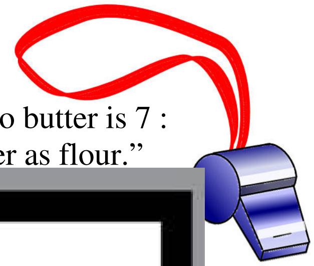
You: Huh? Well, I could say 12 : 7 is the ratio of flour I have to the flour you used. Using the ratio 12 : 7, instead of the fraction 12/7, is just like a word choice – two different ways of saying the same thing.

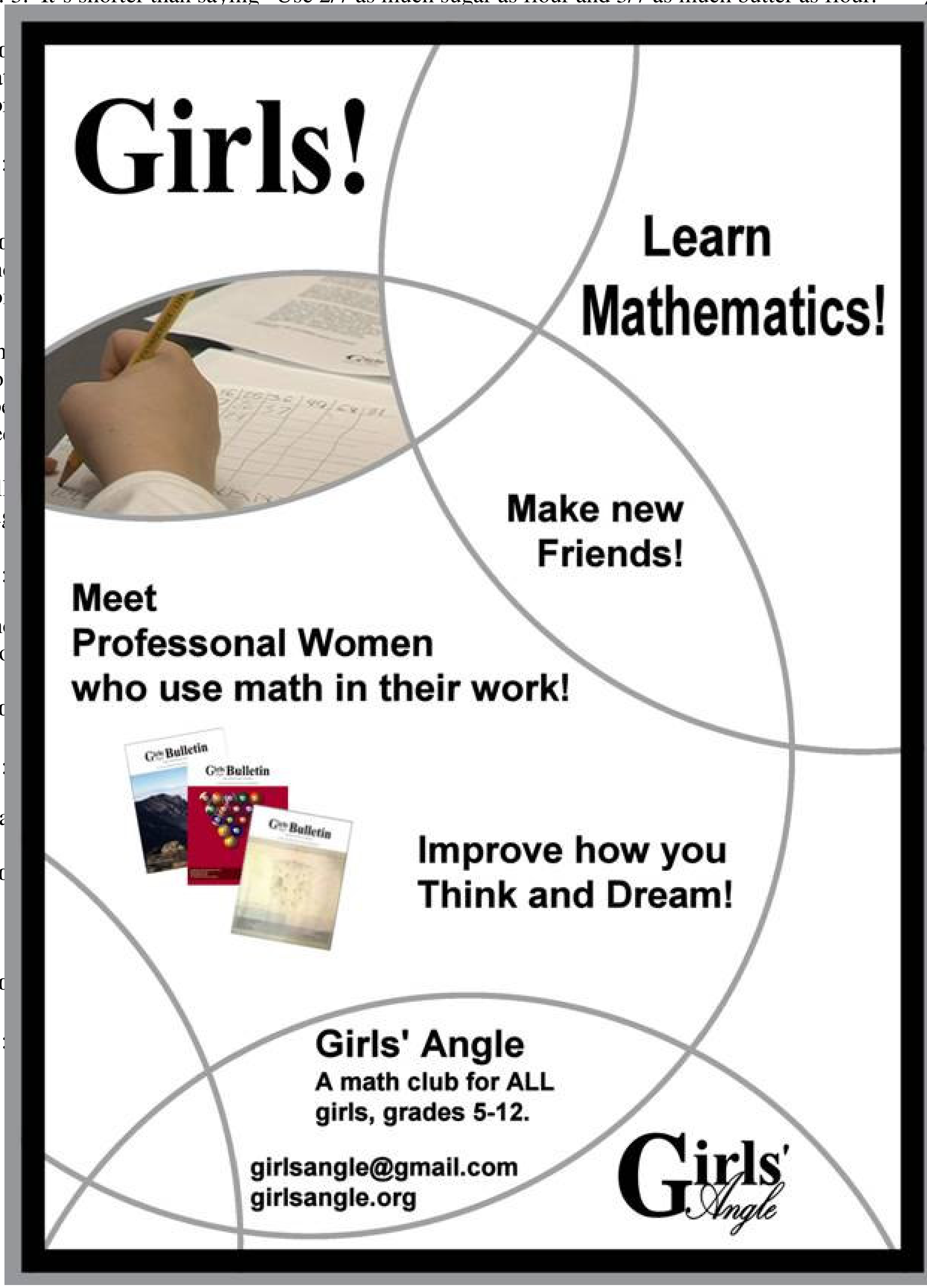
$\frac{3}{7}$: Yes, dearie. Sometimes when I need a little change-of-pace I call myself 3:7 instead of 3/7. But I can do something special with a ratio.

You: Really, what?

$\frac{3}{7}$:

I can concisely convey my shortbread recipe. Here it is: the ratio of flour to sugar to butter is 7 : 2 : 3. It's shorter than saying "Use 2/7 as much sugar as flour and 3/7 as much butter as flour."





Summer Fun!

In the last issue, we invited members to submit solutions to a batch of Summer Fun problem sets.

In this issue, we give solutions to many of the problems. These solutions will sometimes be rather terse and, in some cases, are more of a hint than a solution. We prefer not to give completely detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics really well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil and move your mind! Your might discover something new to the world.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially curt will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members *and* Subscribers*: Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Note: For solutions to the Coin Fountains problem set, we refer the reader to the very readable paper by Odlyzko and Wilf: "*n* Coins in a Fountain," *Amer. Math. Monthly* **95**, pp. 840-843, 1988.

Summer Fun!

Grand Slam Season

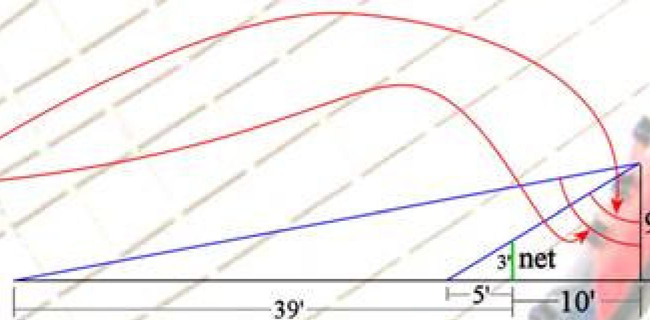
by Lightning Factorial

Just in time for the US Open… here are the solutions to some tennis-inspired questions..

1. It would take 117/220 of a second, or approximately 0.53 seconds.

2. We can use the Pythagorean theorem to determine the distance traversed. The answer is $\frac{\sqrt{27^2+39^2}}{2}$ feet per second, which is approximately 23.7 feet per second or about 16 mph. Would you be able to run that ball down?

3. Arctan $\frac{10+39}{9}$ – Arctan $\frac{10+5}{9} \approx 20.5°$.



Tiebreakers

3. The probability that the first player wins is $1 - 2p + 3p^2 - 2p^3$. Such a tiebreaker is inherently unfair. If you have a great service game and you're about to play your identical twin, make sure your twin serves first!

4. Let w be the probability that the player who serves first wins the tiebreaker. Notice that the probability that the person who serves second wins the tiebreaker is equal to $1 - w$. (In theory, the tiebreaker might never end, but the probability that this happens is zero.) There are four outcomes possible after the first two rallies. With probability $p(1 - p)$, the first player serving wins both points and the tiebreaker. With probability $(1 - p)p$, the first player serving loses both points and the tiebreaker. With probability $(1 - p)^2$, the first player serving loses the first point and wins the second. Finally, with probability p^2, the first player serving wins the first point and loses the second. In the last two cases, because players must win by 2, the situation has become as if a new tiebreaker began except that now the person who served second gets to start the serve.

Therefore

$$w = p(1 - p) + (1 - p)^2(1 - w) + p^2(1 - w).$$

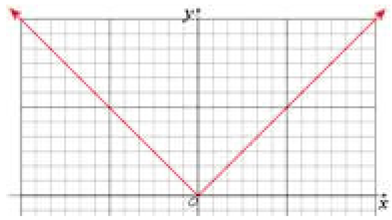
Solving for w, we find that $w = ½$, regardless of what p is. This tiebreaker is inherently fair.

5-6. Hint: Let w_n be the probability that the player who serves first wins an n-point tiebreaker (where you have to win by 2). Prove that $w_n = ½$ by induction on n. Notice that the case $n = 2$ is shown in the solution to #4. Alternatively, can you exploit the symmetry in this problem to find a more efficient solution?

Summer Fun!

Absolute Values

Solutions (Ken Fan)

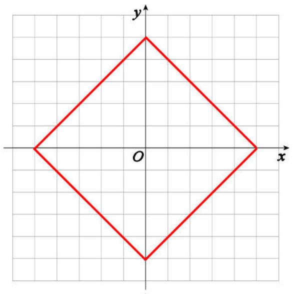


1. See figure at right.

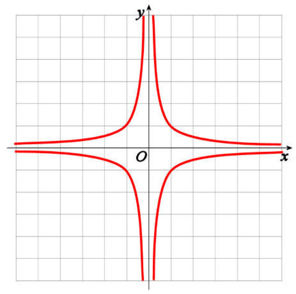
2. a. x = -5, 5; b. x = -5, 5; c. x = -4, 10; d. x = -2, 4; e. x = -7, -2; f. x = -10, 10; g. all $x \le 0$; h. $x = ½$; i. x = -3, -1, 1, 3.

3.

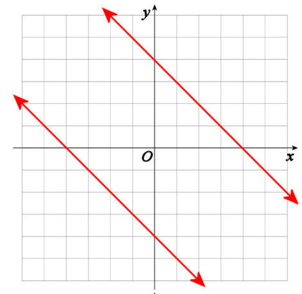
a.



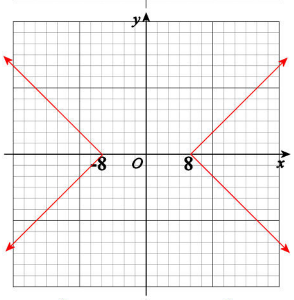
b.



c.

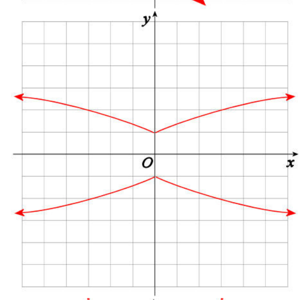


d.

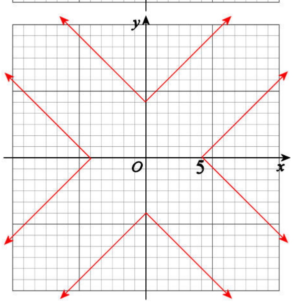


e. The point (6, -3).

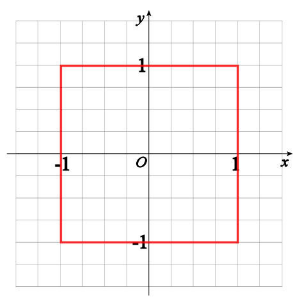
f.



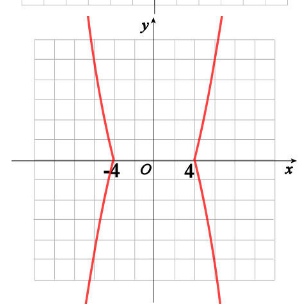
g.



h.



i.



4. $f(x) = \left|\frac{b}{2a}x\right| - \left|\frac{b}{2a}(x-a)\right| + \frac{b}{2}$

5-7. Hint: For 5 and 6, break into four cases: i. $x \ge 0$ and $y \ge 0$, ii. $x \ge 0$ and $y < 0$, iii. $x < 0$ and $y \ge 0$, and, finally, iv. $x < 0$ and $y < 0$. For 7, also use cases.

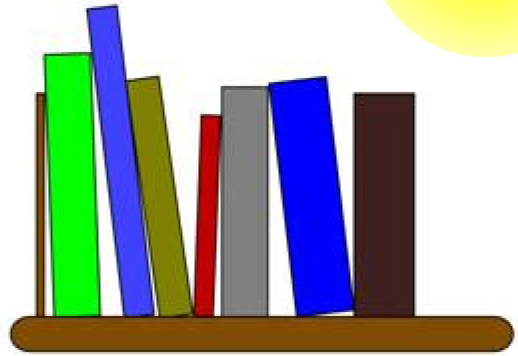
8. For the first part, suppose that there are two consecutive positive terms. Show that eventually, there must be a nonnegative number in the sequence less than both of these terms. For the second part, suppose there is a 0 in the sequence and work backward. Find a pattern to the kinds of sequences possible, and then construct a counterexample.

9. Hint: Consider the function $f(x) = (x + 1 + | x + 1 |)(1 - x + | 1 - x |)$. What are its properties? What can you do with it?

Summer Fun!

Your Choice

Solutions (Shravas Rao)



1. Suzie has 7 different objects, the books, and we want to find the number of ways to arrange them in a line. This is 7! = 5040.

2. We can consider the books in the Hunger Games trilogy as just one book. So instead of trying to arrange 7 books, we think of this as arranging 5 books, so the answer is 5! = 120.

3. 8! = 40320.

4. Because we know Jen is going to be goalie, we just need to assign 7 positions to 7 girls. The solution to this is essentially the same as that to problem 3, and the answer is 7! = 5040.

5. The girls need to choose 2 girls from 8 girls, where order does not matter. We can count this using a combination, with an answer of $\binom{8}{2} = 28$.

6. $\binom{8}{6} = 28$.

7. The answer to problems 5 and 6 are the same! If you look at the formulas we used, $\binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{(8-6)!6!} = \binom{8}{6}$. But there's also an explanation that does not use formulas. Picking a team of 6 girls out of 8 girls is the same as picking 2 girls to not be on the team. Essentially, problems 5 and 6 are counting the same thing. In fact, $\binom{n}{r} = \binom{n}{n-r}$ for all n and r, using similar reasoning.

8. $\binom{7}{5} = 21$.

9. $\binom{7}{6} = 7$.

10. The sum of the answers to problems 8 and 9 give the answer to problem 6. This is because counting the number of ways to select 6 girls from a group of 8 is the same as counting the number of ways to select a group of 6 girls that includes Nicole, and the number of ways to pick 6 girls not including Nicole. In fact, $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for all n and r using similar reasoning.

Summer Fun!

11. The solution to the problem uses a technique sometimes referred to as the "stars and bars" technique. First, we can represent the 10 wins using 10 stars, arranged in a row

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We can choose how to distribute the wins amongst the 6 girls by inserting 5 bars in between or to the right or left of the stars. If we arrange the girls in a list, the number of stars to the left of the leftmost bar is the number of wins the 1st girl got, the number of stars between the first two bars is the number of wins the 2nd girl got and so on. For example, the result of the 1st girl winning 2 games, the 2nd girl winning 4, and the rest winning 1 game each corresponds to the arrangement

☆ ☆ | ☆ ☆ ☆ ☆ | ☆ | ☆ | ☆ | ☆

We can count the number of arrangements by considering 15 symbols listed in a row, choosing 5 of them to be bars, and letting the rest be stars. This number is $\binom{15}{5} = 3003$.

12. We can count the total number of ways the girls won and lost their games by picking which of the 5 games of the first 7 they won; the results of the rest of the games follow. This number is $\binom{7}{5} = 21$.

13. 15.

14. If we let D stand for down and L for left, each path can be described as an arrangement of 4 D's and 2 L's. There are $\binom{6}{4} = 15$ such arrangements.

Generalization

15. $n!/k!$.

16. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

17. $\binom{n+k-1}{k-1}$.

18. $\frac{(2n)!}{2^n n!}$, which is the same as the product of all odd numbers between 0 and $2n$.

19. Assuming there are no byes, the answer is $\frac{128!}{2}$. Here's one way to see this: There are $\frac{128 \cdot 127}{2}$ possible pairings in the championship match. Both of these finalists emerged from the semifinals where each could have been matched with any 2 other players, giving $\frac{128 \cdot 127}{2} \cdot 126 \cdot 125$ possibilities for the final and semifinals. Each of these 4 semifinalists emerged from the quarterfinals where each could have been matched with any 4 other players, giving $\frac{128 \cdot 127}{2} \cdot 126 \cdot 125 \cdot 124 \cdot 123 \cdot 122 \cdot 121$ possibilities for the final, semifinals, and quarterfinals. Continuing this line of argument leads to the answer given above.

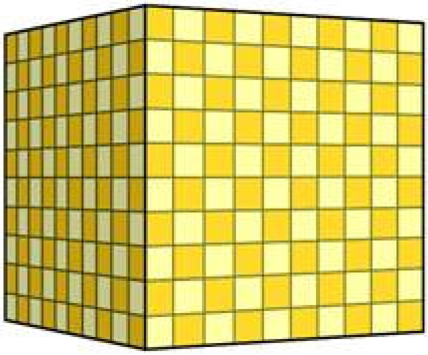
Summer Fun!

Fraction Traction

Solutions (3/7, good twin of 7/3)

1. We carefully rewrite all the fractions in terms of the common denominator of 20:

$$
\begin{aligned}
&-\frac{3}{20}+\frac{1}{2}-\frac{7}{10}+\frac{3}{4}-\frac{1}{2}+\frac{1}{10} \\
&=-\frac{3}{20}+\frac{10}{20}-\frac{14}{20}+\frac{15}{20}-\frac{10}{20}+\frac{2}{20} \\
&=\frac{-3+10-14+15-10+2}{20}=0
\end{aligned}
$$



2. 99/100.

3. The decimal form of p/q, where p/q is a fraction in lowest terms, will be a terminating decimal if (and only if) the only prime numbers that divide evenly into q are 2 and/or 5.

4. Using the arithmetic-geometric mean inequality, one can show that the answer is 2.

5. Suppose a/b and p/q are in O and in lowest terms. Then $a/b + p/q = (aq + bp)/(bq)$. Since b and q are odd, so is bq, and once a fraction has an odd denominator, its reduced form will necessarily be odd too. Similarly, the product of a/b and p/q is $ap/(bq)$ which also has the odd denominator bq. Thus, both the sum and product of numbers in O will also be in O.

6. Suppose a/b and p/q are in E and in lowest terms. Because they are in lowest terms and both b and q are even, both a and p must be odd. In the product, $ap/(bq)$, the denominator will have a factor of 2 and the numerator will be odd. When reduced, there are no factors of 2 in the numerator to cancel any of the factors of 2 in the denominator. Therefore the product will also be in E. For the sum, notice that 1/6 + 1/6 = 1/3.

7. This can be shown using similar arguments as found in the solutions to #5 and #6.

8. The sequence begins 1, 1/2, 2/3, 3/5, 5/8, 8/13, 13/12, … Notice that these are ratios of consecutive Fibonacci numbers. The limit is $\frac{\sqrt{5}-1}{2}$.

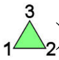
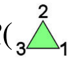
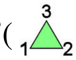
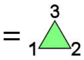
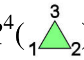
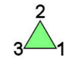
9. 1, 1, 1/2, 2/3, 6/7, 21/32, 224/339, 10848/14287.

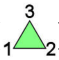
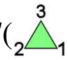
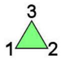
10. $\frac{\sqrt{2}}{2}$. If you assume that the limit does exist and is equal to L, then taking the limit in the recursion relation shows that $L = 1/(2L)$, from which one can get the answer. However, proving that the limit exists is challenging. Here's a hint: If one defines $e_n = a_n - L$, then to first order, $e_{n+1} = -(e_n + e_{n-1})/2$. Familiarity with calculus is helpful. If you get really stuck, write us!

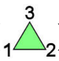
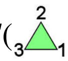
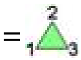
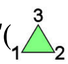
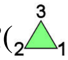
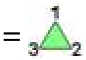
Summer Fun!

The Symmetry of Regular Polygons

Solutions (Grace Lyo)

1. $\boldsymbol{R}(\boldsymbol{R}($$)) = R($$) =$ . Similarly, $\boldsymbol{R}^3($$) =$  and $\boldsymbol{R}^4($$) =$ .

3. $\boldsymbol{F}(\boldsymbol{F}($$)) = \boldsymbol{F}($$) =$ .

4. $\boldsymbol{F}(\boldsymbol{R}($$)) = \boldsymbol{F}($$) =$ . $\boldsymbol{R}(\boldsymbol{F}($$)) = \boldsymbol{R}($$) =$ .

5a. $\boldsymbol{R}^n = \mathbf{1}$ whenever n is a multiple of 3. $\boldsymbol{R}^n = \boldsymbol{R}$ whenever dividing n by 3 gives a remainder of 1. $\boldsymbol{R}^n = \boldsymbol{R}^2$ whenever dividing by 3 gives a remainder of 2.

5b. $\boldsymbol{F}^n = \mathbf{1}$ whenever n is even. $\boldsymbol{F}^n = \boldsymbol{F}$ whenever n is odd.

6. There are 6 distinct symmetry operations for a triangle (including the symmetry operation **1**).

7. $\boldsymbol{RF} = \boldsymbol{FR}^2$

8.

Table of Symmetry Operation Compositions Y ○ X

Y \ X	1	R	R^2	F	FR	FR^2
1	1	R	R^2	F	FR	FR^2
R	R	R^2	1	FR^2	F	FR
R^2	R^2	1	R	FR	FR^2	F
F	F	FR	FR^2	1	R	R^2
FR	FR	FR^2	F	R^2	1	R
FR^2	FR^2	F	FR	R	R^2	1

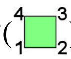
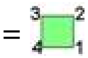
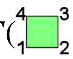
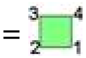
9. For a square, let $\boldsymbol{R}($$) =$  and $\boldsymbol{F}($$) =$ .

Table of Symmetry Operation Compositions Y ○ X

Y \ X	1	R	R^2	R^3	F	FR	FR^2	FR^3
1	1	R	R^2	R^3	F	FR	FR^2	FR^3
R	R	R^2	R^3	1	FR^3	F	FR	FR^2
R^2	R^2	R^3	1	R	FR^2	FR^3	F	FR
R^3	R^3	1	R	R^2	FR	FR^2	FR^3	F
F	F	FR	FR^2	FR^3	1	R	R^2	R^3
FR	FR	FR^2	FR^3	F	R^3	1	R	R^2
FR^2	FR^2	FR^3	F	FR	R^2	R^3	1	R
FR^3	FR^3	F	FR	FR^2	R	R^2	R^3	1

9 continued. Notice that $\boldsymbol{RF} = \boldsymbol{FR}^3$. There are 8 symmetry operations for the square: $\mathbf{1}$, $\boldsymbol{R}$, $\boldsymbol{R}^2$, $\boldsymbol{R}^3$, $\boldsymbol{FR}$, $\boldsymbol{FR}^2$, $\boldsymbol{FR}^3$, and $\boldsymbol{FR}^4$.

10. Substitute $n = 5$ into the solution for #11.

11. For an r-gon, label the vertices 1 through r counterclockwise around the perimeter. Let $\boldsymbol{R}$ be the transformation that rotates the numbers counterclockwise over to the next vertex. Let $\boldsymbol{F}$ be the transformation that flips the numbers of the perpendicular bisector of the side whose endpoints are labeled 1 and 2. Then, $\boldsymbol{R}^n = \mathbf{1}$ whenever n is a multiple of r. $\boldsymbol{R}^n = \boldsymbol{R}$ whenever dividing by r gives a remainder of 1. In general, if there is a remainder of k when n is divided by r, then $\boldsymbol{R}^n = \boldsymbol{R}^k$. Also $\boldsymbol{F}^2 = \mathbf{1}$. We also have $\boldsymbol{RF} = \boldsymbol{FR}^{r-1}$. There are $2r$ symmetry operations: $\mathbf{1}, \boldsymbol{R}, \boldsymbol{R}^2, \ldots, \boldsymbol{R}^{r-1}, \boldsymbol{F}, \boldsymbol{FR}, \boldsymbol{FR}^2, \ldots$, and $\boldsymbol{FR}^{r-1}$.

Summer Fun!

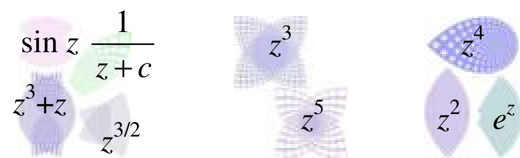
Calendar

Session 11: (all dates in 2012)

September	13	Start of the eleventh session!
	20	
	27	Charlene Morrow, Mt. Holyoke
October	4	
	11	Pardis Sabeti, Broad Institute/Harvard
	18	
	25	Anoush Najarian, MathWorks
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

If you like solving math contest problems, consider **Math Contest Prep**. The first class is September 16, 2012 at the Microsoft NERD Center.

Here's the identity of the functions in Mathematical Buffet:



Here are answers to the *Errorbusters!* problems on page 18. Here, $i = \sqrt{-1}$.

1.	0	2.	0, 2	3.	1	4.	1, 2
5.	$-\sqrt{2}$, -1, 1, $\sqrt{2}$	6.	-1, 0, 1	7.	-2*i*, -*i*, 0, *i*, 2*i*	8.	-2, 0, 5

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Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) comes with membership. We are working hard to lower the cost of the Bulletin. Until we do, however, nonmembers can receive the printed version by becoming a Bulletin Sponsor. Please contact us if interested.

The Bulletin targets girls roughly the age of current members. Each issue is likely to contain some material that feels very challenging or difficult to understand. If you are a member or Bulletin Sponsor and have any questions about the material, feel free to ask us about it!

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club and receive a printed copy of the Girls' Angle Bulletin for the duration of the membership. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin and a 25% discount for any club meet attended. Remote members may email us math questions (although we won't do people's homework!).

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership**, please fill out the information in this box. **Bulletin Sponsors** may skip this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership (which includes 12 two-hour club meets)
  - ☐ $108 for a one year Remote Membership
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

Girls' Angle

A Math Club for Girls