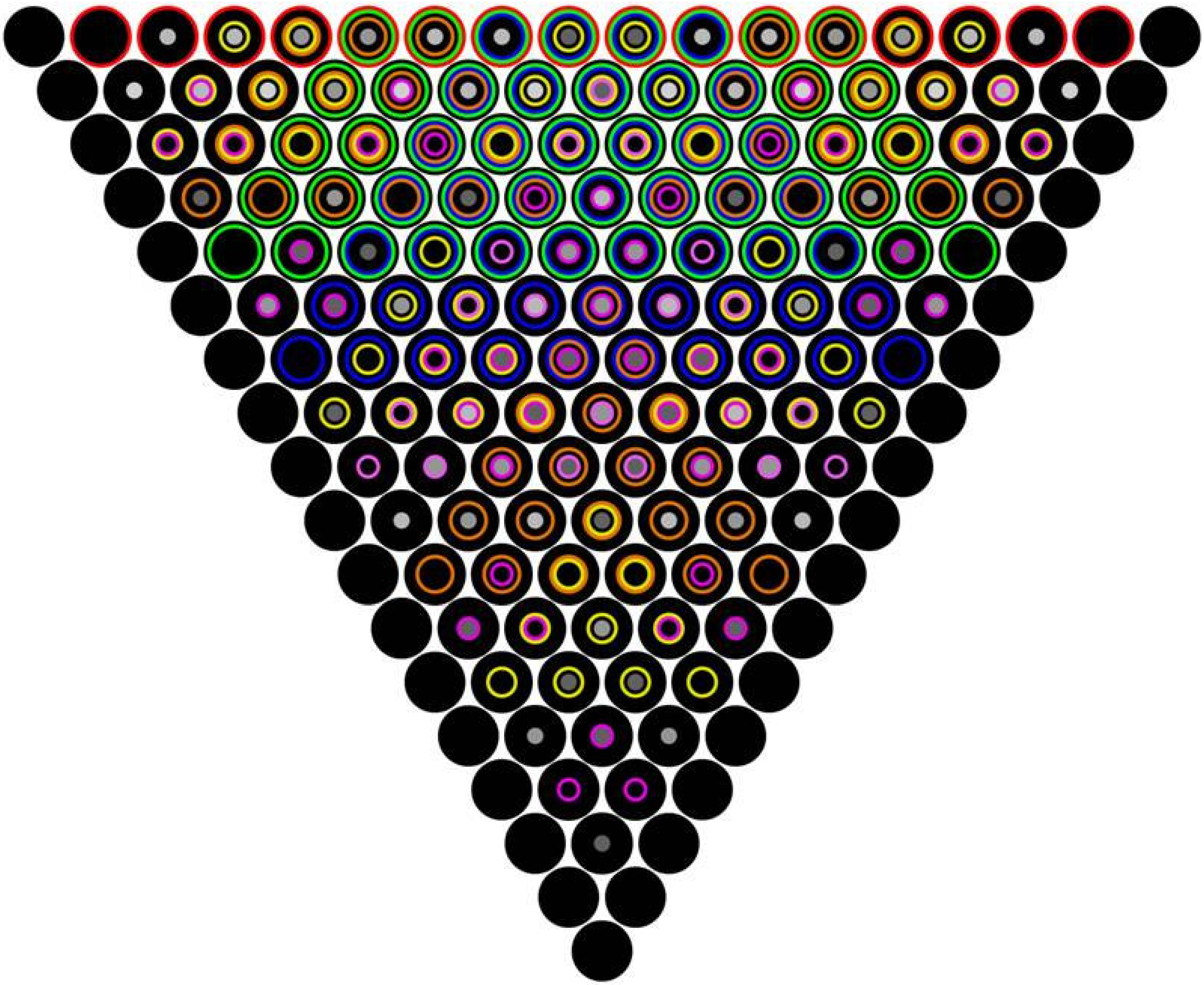
# Girls' Angle Bulletin

October 2012 • Volume 6 • Number 1

*To Foster and Nurture Girls' Interest in Mathematics*

# From the Founder

Rock climbing offers an interesting analogy to math problem solving. Advanced rock climbers scale walls that, to beginners, seem devoid of hand or footholds. Some granite puzzles require acrobatic technical maneuvers that want years to develop the skill and strength to execute. Perhaps, the most fun comes when climbing a challenging route where you nevertheless progress steadily on up. But there is one huge difference. When you err in math problem solving, you won't plunge down a precipice. So why fear math problems the size of Mt. Everest?

- Ken Fan, President and Founder

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**Girls' Angle Bulletin**
*The official magazine of Girls' Angle: A Math Club for girls*
*Electronic Version* (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

**Girls' Angle:**
**A Math Club for Girls**

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*



On the cover: Pascal's Primes. A rendition of Pascal's triangle in which prime factorization are encoded by colored concentric rings. Different radii correspond to different prime numbers.

 

# An Interview with Danijela Damjanović

Danijela Damjanović is an assistant professor of mathematics at Rice University. She is a native of Serbia and we hope that you can get a flavor of her accent in this interview.

**Ken**: Hi Danijela, Thank you for doing this interview! Congratulations on being recently awarded an NSF Career grant. In addition to supporting your research, I read that you will use the grant to create a program for teenage girls in mathematics. That sounds like a wonderful use of the funds and I want to start by asking you about this program. Will your program be open to girls all over the country? If a girl is interested in joining your program, what should she do?

**Danijela**: So far the program is opened only to high school girls from Houston. Girls who are interested should contact our secretaries at the mathematics department to get more information (wimath@rice.edu). The admission will be based on criteria such as: recommendations from teachers, previous preparation in mathematics, grades.

*Trust in yourself, no one can know better than you your own strengths and weaknesses.*

**Ken**: OK, thank you for that information. What sparked your interest in mathematics?

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

We will make the rest of this interview with Prof. Damjanović available here at some time in the future. But what we hope is that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

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Thank you and best wishes,
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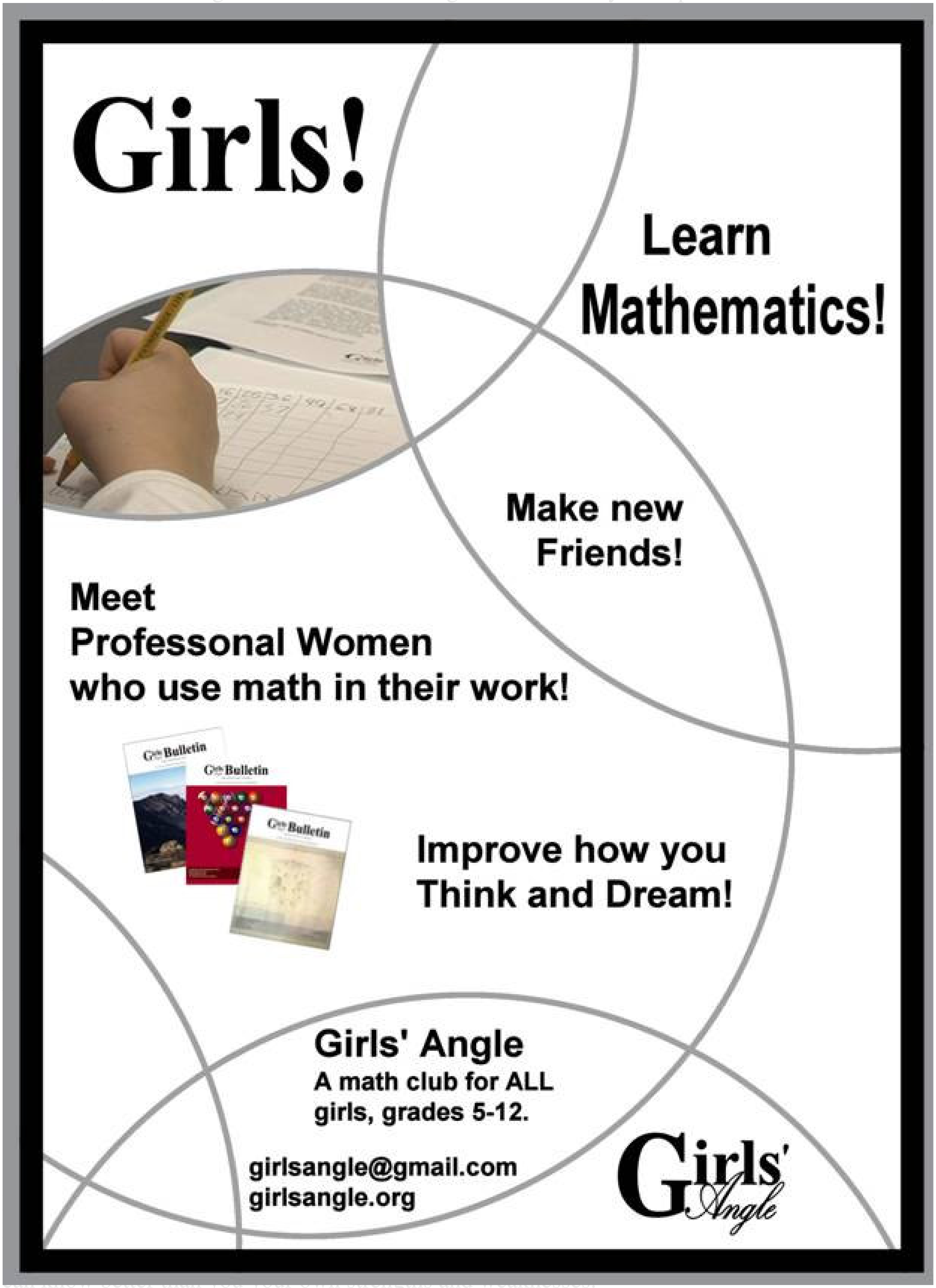
wonder if, after hitting the ball under some angle it will actually end up in a hole at the corner of



# The Cat Map

by FluffyFur | edited by Jennifer Silva

On page 4 of Prof. Danijela Damjanović's interview, she mentions two maps that are studied in dynamical systems. One of these maps is defined on a circle and the other, affectionately called the "cat map," is defined on a torus. She writes, "These are systems on the circle, or on the 2-torus, defined by simple multiplication rule ($x \to 2x \bmod 1$), or via $2 \times 2$ integer matrices acting on the 2-torus; Take for example matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the 2-torus." Here, I explain these maps in detail so that you can explore their behavior at home. Meow.

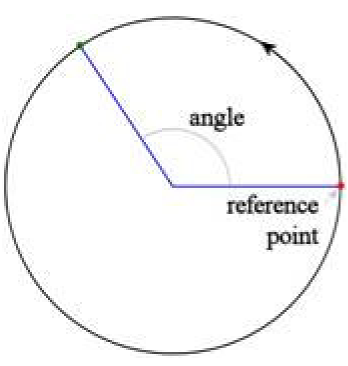
**Maps.** In this context, the word "map" means the same as the word "function." Let $X$ and $Y$ be two sets. A map from $X$ to $Y$ associates a single element of $Y$ to each element of $X$. People use the word "map" because you can think of a regular road map as a function from some region on earth to a piece of paper. Each point in the mapped region is mapped to a point on the paper. Normally, the paper is color-coded so that you can tell what point is part of a road, what is part of a lake, what is part of a forest, etc. In everyday usage, the word "map" is usually used to refer to the paper object that bears the image of the function, but the mathematical usage of the word "map" actually denotes the function.

If $m$ is a map from $X$ to $Y$ and $x$ is in $X$, we write "$m(x)$" for the point in $Y$ that is associated to the point $x$. The notation $m(x)$ is spoken, "$m$ of $x$."

You can take the sets $X$ and $Y$ to be equal to each other. In this case, the map maps from a set to itself. Both maps that Danijela mentions are of this type. One maps the circle to itself and the other maps the torus to itself.

When you have a map that maps a set to itself, an interesting possibility arises: you can iterate the map. That is, suppose $m$ maps from $X$ to itself and $x$ is in $X$. Then $m(x)$ is also an element of $X$. That means we can apply $m$ again to $m(x)$ and get the point $m(m(x))$, and again to get $m(m(m(x)))$, and keep on doing this to our heart's content. We produce a sequence of elements of $X$: $x$, $m(x)$, $m(m(x))$, $m(m(m(x)))$, etc. This sequence is called the **orbit** of $x$. People who study dynamical systems are interested in the properties of such orbits.

Let's return to the two maps that Danijela mentions, starting with the circle map.




**The Circle Map.** Danijela defines the circle map by the rule $x \to 2x \bmod 1$. If you're unfamiliar with this notation, don't worry about it because I'm going to explain it in another way.

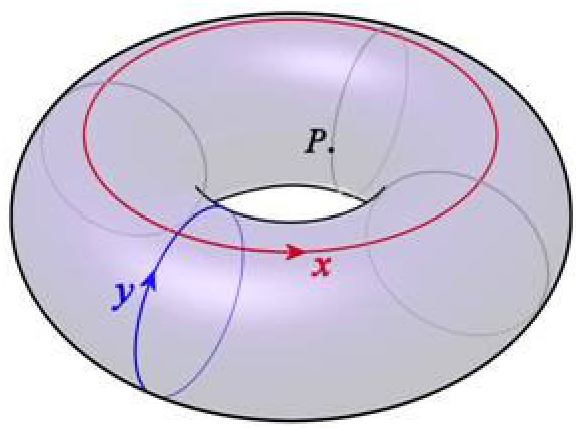
Think of a circle. Let's use angles to specify different points on the circle. To do that, we fix a reference point. We can then specify other points by giving the angle through which we must rotate from the reference point. By convention, positive angles represent counterclockwise rotations from the reference point. In this way, the reference point is at 0 degrees, and the point diametrically opposite to the reference point is at 180 degrees. Notice that the reference point is also specified by any multiple of 360 degrees, because there are 360 degrees in a full circle.

*The circle map sends the point at angle $x$ to the point with angle $2x$.* Thus the reference point, which is located at 0 degrees, is mapped back to itself since twice 0 degrees is 0 degrees as well. The point diametrically opposite to the reference point, at 180 degrees, is also mapped to the reference point, because twice 180 degrees is 360 degrees.

Even though there are many different angles that refer to the same point, the doubled angles will all consistently specify another point. This means that you don't have to worry about getting contradictory results depending on which angle you choose to refer to a point when you compute it. To be precise, if the point on the circle is referred to by both angle $x$ and angle $y$, then $x$ and $y$ differ by a multiple of 360 degrees. That means that $2x$ and $2y$ will differ by a multiple of 720 degrees, and angles that differ by a multiple of 720 degrees correspond to the same point. Thus our prescription for the circle map is **well-defined**.

Let's take a moment to relate this map back to the original notation Danijela uses to describe it: $x \to 2x \bmod 1$. First of all, Danijela measures angles in terms of full circles instead of degrees. She would specify the point at 90 degrees by 1/4 since 90 degrees is 1/4 of a full circle. When you measure angles in terms of full circles, then two angles refer to the same point if they differ by an integer (as opposed to a multiple of 360 degrees or $2\pi$ radians). So two angles $x$ and $y$ are the same if, and only if, $x - y$ is an integer. In units of full circles, the map is still defined as doubling of the angle. So when Danijela writes $x \to 2x \bmod 1$, she means that the point with full circle angle $x$ is mapped to the point with full circle angle $2x$, and the "mod 1" part says that she considers angles to be the same if they differ by a multiple of 1, i.e., if they differ by an integer. If you were to describe the circle map with her notation using degrees instead of full circles you would write $x \to 2x \bmod 360$. Let's switch to her units and measure angles in terms of full circles.

Now that we have our map, let's look at a few of its orbits. To begin, what is the orbit of the reference point, which has angle measure 0? Doubling 0 is 0 again, so this point gets mapped back to itself. It is called a **fixed** or **stationary** point. What about the point 1/5? If we apply the map over and over, here's what we get: $1/5 \to 2/5 \to 4/5 \to 3/5 \to 1/5 \to \ldots$ (The circle map sends 4/5 to 3/5 because $2 \times 4/5 = 8/5$, but 8/5 is the same angle as 3/5 when angles are measured in units of full circles.) After 4 applications of the map, we end up back where we started. Such orbits are called **periodic**. Now consider $\sqrt{2}$. If you apply the circle map to $\sqrt{2}$ over and over, will it ever come back to $\sqrt{2}$? It won't. In fact, the orbit will be densely sprinkled about the entire circle, never revisiting any point.



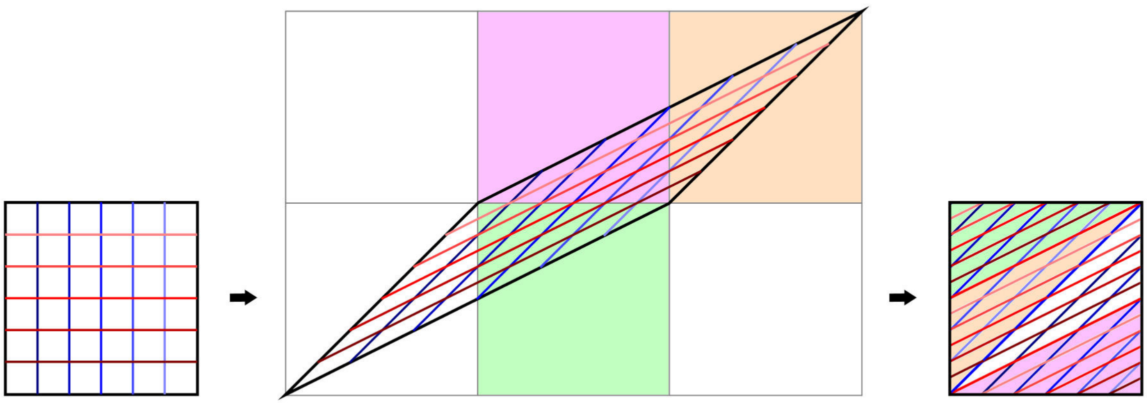

**The Cat Map.** Now to the cat map. This time, the map applies to points on a torus (the surface of a donut). A torus can be thought of as a circle of circles. Imagine the blue circle in the figure at right traveling along the red circle, sweeping out the torus. We specify points on the torus using two angles, $x$ and $y$. The angle $y$ tells how far to go around the blue circle in the direction indicated, measuring from the point of intersection between the red and blue circles. We then sweep the blue circle around the torus through the angle $x$ along the red circle. In this way, a pair of real numbers $(x, y)$ specifies a point on the torus. Let's agree to measure these angles in units of full circles. For example, the point $P$ is specified by $x = 1/2$ and $y = 1/4$.

*The cat map sends the point* $(x, y)$ *to the point* $(2x + y, x + y)$.

Let's make sure that this rule doesn't give contradictory results if we use different angles to specify the same point $(x, y)$. We have to check that the rule maps both $(x + n, y + m)$ and $(x, y)$, where $n$ and $m$ are integers, to the same point. We know that $(x, y)$ goes to $(2x + y, x + y)$ and we compute that $(x + n, y + m)$ goes to $(2x + y + 2n + m, x + y + n + m)$.

Because both $2n + m$ and $n + m$ are integers, this does indeed represent the same point on the torus as the pair $(2x + y, x + y)$.

To envision this map, we can split it into two stages. The above discussion shows how we can associate points in the plane to points on a torus, where $(x, y)$ and $(x + n, y + m)$ correspond to the same point when $n$ and $m$ are integers. That means that every point on the torus is captured (at least once, but possibly two or four times) by the unit square with lower left corner at (0, 0). So we need only to understand the cat map on this square to obtain a complete picture of it. To points $(x, y)$ in this square, we apply the cat map $(x, y) \to (2x + y, x + y)$, as the left arrow indicates in the figure below.



As we can see, the transformation $(x, y) \to (2x + y, x + y)$ turns the square into a parallelogram. If you are adept at recognizing points in the plane that correspond to the same point on the torus, the central figure above might be a satisfactory representation of the cat map. If not, we can take the additional step of translating all pieces back into the original unit square by integer amounts. The 3 by 2 rectangle in the central figure covers the torus 6 times. Three copies of the torus are colored to help you match up the parts of the central figure with parts of the square shown above right.

Here's what happens to me under this mapping:

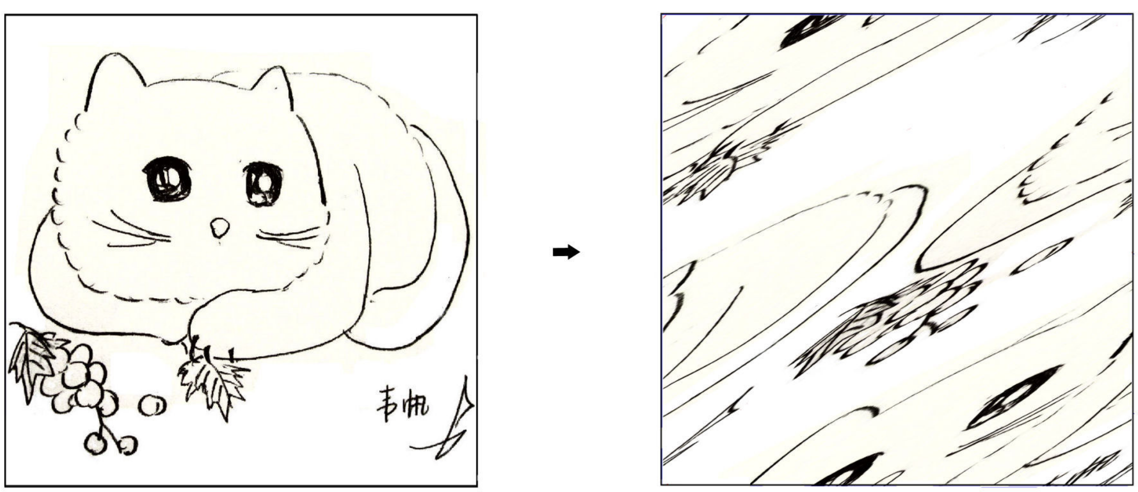


Illustration by Fan Wei

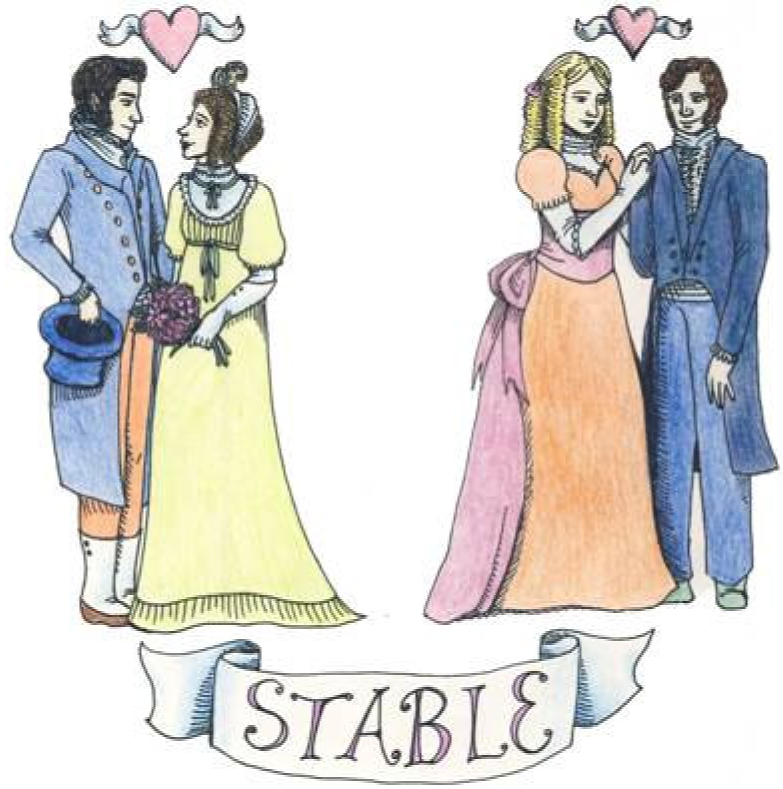
If you iterate the cat map over and over, it isn't long before the image becomes unrecognizable. Yet there is no loss of information; you can invert the cat map and apply it repeatedly to recover the original image. What are the orbits of the cat map like?

# The Stable Marriage Problem[1]

## Part 1. The Problem and a Solution

by Emily Riehl, edited by Grace Lyo,
and illustrated by Julia Zimmerman




Suppose there are an equal number of boys and girls of marriageable age, who we'll call **debutants**, living in a remote village. In the interest of societal harmony, the village elders would like to match every girl with a boy in such a way that the arranged marriages are **stable**. This means that no girl and boy who are not matched together each prefer the other to their assigned partners (the pictures show some stable and unstable matchings). The elders want to arrange stable marriages to make sure that no unmarried couple will run off and elope. Either the boy or girl in each potential couple would rather stay in their arranged marriage!


Now the village elders are confronted with quite a challenge. To start, it's not at all obvious that it is possible to arrange stable marriages for all the debutants. And even supposing this can be done in theory, it is not clear how the elder would find such a **stable matching** in practice. A guess-and-check approach, where marriages are arranged randomly and then tested for stability, is unlikely to succeed: in a village of 10 boys and 10 girls there are 3,628,800 ways to arrange (possibly unstable) marriages.[2]

Remarkably, it turns out there is a simple algorithm that assigns stable marriages for any collection of debutants, in particular proving that a stable matching always exists! Even more remarkably, the algorithm always finds a stable matching with rather amazing properties — but we'll save this part of the story for later.

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[1] This content supported in part by a grant from MathWorks.
[2] Here's an exercise: explain how this number is computed.

 

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Thank you and best wishes,
Ken Fan
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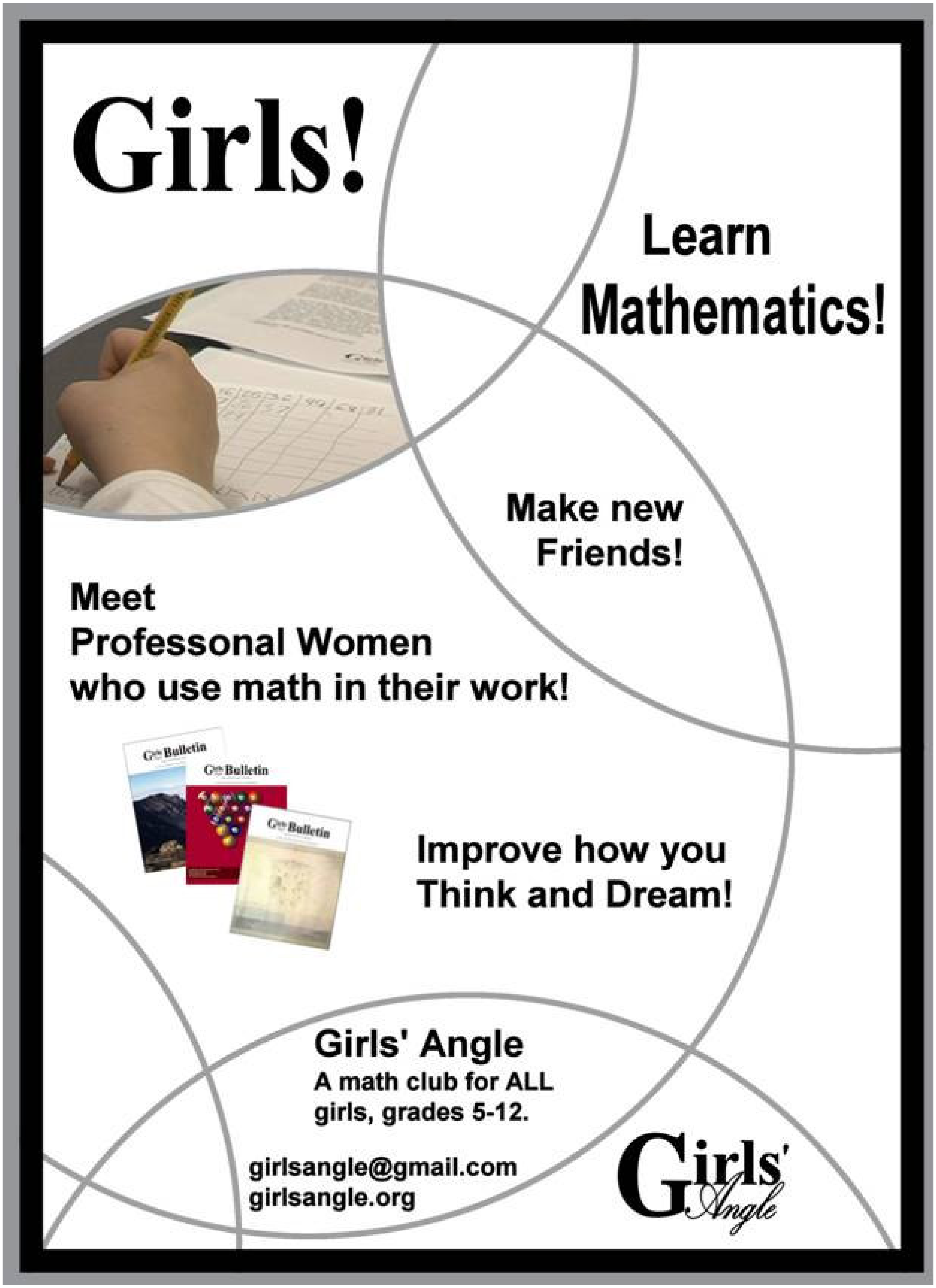
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# Another Way to Vote

by Taotao Liu | edited by Jennifer Silva

Logo Design by Hana Kitasei

It's your birthday, and you and three of your friends are going to see a movie to celebrate. But there's a problem. You don't agree on which movie to see. Rather than ruin the day by arguing about it, you all decide to put the question to a vote. However, since it is your birthday, you want to have more say than others, yet you don't want to force all of your friends to a movie that none of them want to see. Can you think of a way to modify a standard "majority wins" voting system so that both of your wishes are satisfied? Think about this before reading further.

One way to accomplish your goals is to use a **weighted voting system**. While each of your friends gets 1 vote, you are given 2, both of which you must spend on the same movie. Any movie that gets 3 votes wins. Have we accomplished what we wanted? Yes. You only need one friend to vote your way, but your three friends can still out-vote you.

In general, a weighted voting system assigns each voter a certain number of votes. All of the votes assigned to a voter must be spent on the same option. Also, a threshold number is specified which tells us how many votes an option must receive to win. Suppose that there are $n$ voters and the $k$th voter gets $a_k$ votes, and that the threshold number required for an option to win is $v$. We'll denote this system with the notation $[v : a_1, a_2, a_3, \ldots, a_n]$. Using this notation, your birthday vote would be denoted [3 : 2, 1, 1, 1]. We'll always list the weights in order from highest to lowest. This allows us to study the power structure of weighted voting systems without worrying about which individuals have the power. Also, we'll assume that none of the weights are negative.

Consider the weighted voting system [1 : 1, 1, 1]. As a voting system, this system is useless because each voter has 1 vote and the threshold is just 1 vote, so each person's choice would win. Why even have a vote? For this reason, we will require $v$ to be more than half the total number of votes. This condition guarantees that at most one option will win. On the other side of the scale, if $v$ is more than the number of votes handed out, then it will be impossible for any option to win. Therefore, we will also require that $v$ be no greater than the total number of votes.

**Equivalence of Weighted Voting Systems**

Some weighted voting systems may look different, but are, in fact, effectively the same. For instance, consider the system [15 : 5, 4, 3, 2, 1]. At first glance, it looks like the person with 5 votes has the most power. But there are a total of 15 votes to be cast, and 15 is the threshold, so this is really a **unanimous vote** in disguise. Everyone must agree in order for a choice to win. Thus, [15 : 5, 4, 3, 2, 1] is equivalent to [5 : 1, 1, 1, 1, 1].

Here's another example of equivalent systems: [10 : 11, 4, 3] and [1 : 1, 0, 0]. Can you see that these are both **one-person dictatorships**?

Now let's look at [10 : 7, 4, 3]. Which people need to agree in order to win here? In this voting system, the person with 7 votes has **veto power**, because she can veto any choice that the other two agree to by voting for something else. However, this is not a one-person dictatorship. Can you find another weighted voting system that is equivalent to [10 : 7, 4, 3]?

In general, we will say that two weighted voting systems are **equivalent** if, whenever voters cast their votes in the same way, the two weighted voting systems will yield the same outcome.

Given two weighted voting systems, one way to check that they are equivalent is to list all of their **winning coalitions**. A winning coalition is a group of voters whose combined weights are greater than or equal to the threshold. If the list of winning coalitions for the two weighted voting systems are identical, then the two weighted voting systems are equivalent.

Notice that any group of voters that contains a winning coalition is also a winning coalition. This means that listing *all* of the winning coalitions is generally more work than necessary to determine if two weighted voting systems are equivalent. Instead, we need only list the **minimal winning coalitions**. A minimal winning coalition is a winning coalition from which none of the voters can be removed without falling short of the threshold. If two weighted voting systems have exactly the same collection of minimal winning coalitions, then they would also have the same collection of winning coalitions, so they would be equivalent.

Let's use the technique of listing minimal winning coalitions to check that [51 : 49, 48, 3] is equivalent to [2 : 1, 1, 1]. Let's label the voters A, B, and C in order of the way the weights are listed. In the system [51 : 49, 48, 3], the set of all voters {A, B, C} is a winning coalition, but not a minimal one. There are 3 sets consisting of 2 voters each: {A, B}, {A, C}, and {B, C}, and one can check that all three are minimal winning coalitions. Determine all of the minimal winning coalitions for [2 : 1, 1, 1] and check that you again get all of the subsets of 2 voters.

## The Number of Types of Weighted Voting Systems

The following is a natural question to ask: for each number of voters $n$, what is the maximum number of mutually non-equivalent weighted voting systems possible? Is this number even finite? (If you know about equivalence relations and equivalence classes, how many equivalence classes are there?) Is there a nice way to explicitly list a maximal set of non-equivalent weighted voting systems?

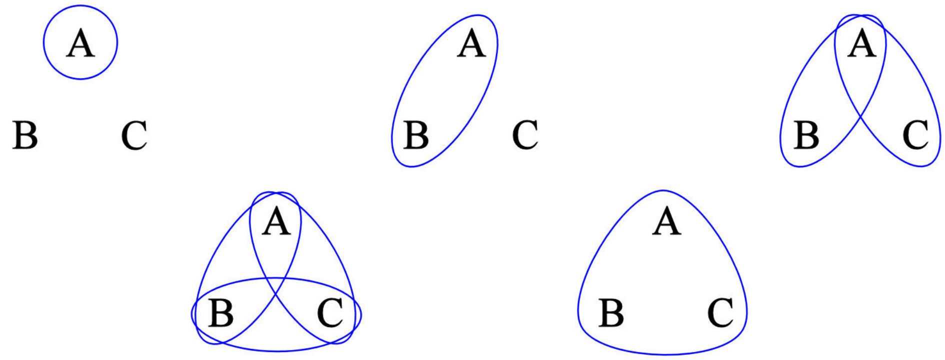
Notice that the concept of the winning coalition enables us to see that a maximal set of mutually non-equivalent weighted voting systems is indeed finite. That's because a finite set of voters only has finitely many subsets, so there are, in turn, only finitely many collections of subsets of a finite set. (A set with $N$ elements has exactly $2^N$ different subsets. So the total number of collections of subsets of a set with $N$ elements is exactly $2^{2^N}$.) The number of collections of subsets of voters serves as an extreme upper bound to the number of different collections of winning coalitions.

Let's investigate the maximal number of mutually non-equivalent weighted voting systems for $n$ voters, starting with the first case $n = 1$.

With 1 voter, there is only one possible voting situation: the 1 voter votes for an option and that option wins. Thus, all weighted voting systems for 1 voter are equivalent to [1 : 1].

With 2 voters, if the system isn't a dictatorship both voters must vote for the same option for that option to win. That is, we either have a dictatorship or a unanimous vote. Thus, all weighted systems for 2 voters are equivalent to either [1 : 1, 0] or [2 : 1, 1].

With 3 voters, can you verify that every system will be equivalent to one of the following: [1 : 1, 0, 0], [2 : 1, 1, 0], [3 : 2, 1, 1], [2 : 1, 1, 1], or [3 : 1, 1, 1]? Describe the behavior of each of these systems.




The five different collections of minimal winning coalitions for 3 voters.

Can you find the 14 different types of weighted voting systems with 4 voters?

Here are some observations that might help you to stay organized. The collections of minimal winning coalitions are rather special collections of subsets of voters. In fact, the collection of minimal winning coalitions must satisfy the following 3 properties:

1. *The collection is nonempty.*
2. *If S and T are minimal winning coalitions, then their intersection must be nonempty.*
3. *If S and T are minimal winning coalitions, then neither can be a proper subset of the other.*

Verify that these 3 properties hold for each of the five collections of minimal winning coalitions for 3 voters in the figure above. We'll leave it to you to explain why these 3 conditions must hold for the collection of minimal winning coalitions associated with any weighted voting system. To find a maximal set of non-equivalent weighted voting systems for $n$ voters, one could make a complete list of collections of subsets of these voters that satisfy the above 3 properties. Then for each collection, one could try to construct a weighted voting system that has that collection as its collection of minimal winning coalitions. Unfortunately, this technique becomes impractical even for computers and a modest number of voters.

**Take it to Your World**

Suppose you have a birthday party with more than three friends. What weighted voting systems would you want to use to pick out a movie?

Consider $[v_1 : n, n-1, n-2, \ldots, 3, 2, 1]$ and $[v_2 : n, n-1, n-2, \ldots, 3, 2, 1]$. Show that these are equivalent if and only if $v_1 = v_2$.

For the weighted voting system $[v : a_1, a_2, a_3, \ldots, a_n]$, can you give a condition on $v$ and the $a_k$ which would indicate when this system is a one-person dictatorship? A unanimous vote?

Listing all of the collections of subsets of voters that satisfy the 3 minimal winning coalition properties listed above gives an upper bound on the number of mutually non-equivalent weighted voting systems. Unfortunately, there can be such collections of subsets that cannot be realized as the collection of minimal winning coalitions for some weighted voting system. Can you construct an example?

Research the Electoral College and the method used to elect the President of the United States. What are the similarities and differences between the United States Presidential Election and weighted voting systems?

For further reading, see *Theory of Games and Economic Behavior*, by von Neumann and Morgenstern.

# Anna's Math Journal

By Anna B.

*Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.*

## Anna resolves the zero coefficient mystery!

I really want to see if the zero coefficient conjecture is true or not. Last time, I regressed, but I think I started off in a good direction. Here's where I was just before I took a wrong turn.

*If you don't understand what Anna is doing, please read the last few installments of Anna's Math Journal. This is a continuation of her investigation into sums of kth powers.*

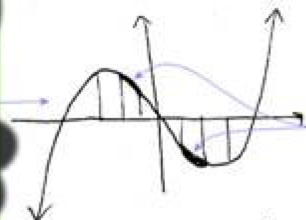
Assume $k$ is even.

Show that $p_k(-n) = -p_k(n)$ where $p_k(n) = S_k(n) - \frac{1}{2}n^k$

We know that $S_k(n) - S_k(n-1) = n^k$ for all $n$.

(Also, $S_k(1) = 1$.)

The graph of an odd function has 180 degree rotational symmetry about the origin. Hmm...



Show: $p_k(0) = 0$

and $p_k(n+1) - p_k(n) = p_k(-n) - p_k(-(n+1))$?

180 degree rotational symmetry means 0 at 0... and symmetric changes... Hmm...I should be able to relate these changes directly to the difference equation and show that they are symmetric...

Finding $p_k(0)$ should be quick because 0 is just a step away from 1, and I know $S_k(1)$ is equal to 1.

$p_k(0) = S_k(0)$ $\quad S_k(1) - S_k(0) = 1$

$1 - S_k(0) = 1$

$S_k(0) = 0$ $\quad$ So $p_k(0) = 0$ ✓

Last time, I regressed because I used the difference equation for $p_k$ to rebuild $p_k(n)$ using a telescoping sum. This time I'm trying to deduce the global symmetry by checking that the little building blocks are consistent with such symmetry.

I can get the difference equation for $p_k$ from the difference equation for $S_k$. Then I just have to check if these differences are consistent with 180 degree rotational symmetry.

From $S_k(n) - S_k(n-1) = n^k$,

$\left(p_k(n) + \frac{1}{2}n^k\right) - \left(p_k(n-1) + \frac{1}{2}(n-1)^k\right) = n^k$

$p_k(n) - p_k(n-1) = \frac{1}{2}n^k + \frac{1}{2}(n-1)^k$ for all $n$.

So $p_k(n+1) - p_k(n) = \frac{1}{2}(n+1)^k + \frac{1}{2}n^k$

Are these equal?

$p_k(-n) - p_k(-(n+1)) = p_k(-n) - p_k(-n-1)$

$= \frac{1}{2}(-n)^k + \frac{1}{2}(-n-1)^k$

$= (-1)^k\left(\frac{1}{2}n^k + \frac{1}{2}(n+1)^k\right)$

$= \frac{1}{2}n^k + \frac{1}{2}(n+1)^k$ since $k$ is even!

It works! The differences are consistent with 180 degree rotational symmetry... and, actually, when $k$ is odd, $p_k$ should be an even function and I should get the negative of this here...which I would because of the $(-1)^k$ factor!

Same argument shows when $k$ is odd, $p_k(n)$ will be an even function ... so $C_{2m} = 0$ for all $m \geq 1$!

ABB 10.27.12

The zero coefficient mystery is resolved! The secret is that $S_k$ differs from an even or odd function by $n^k/2$.

Key:

Anna's thoughts

Anna's afterthoughts

*Editor's comments*

# Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

I recently finished grading an exam for my precalculus class. I offered a free perfect quiz grade to the entire class if no one made an error that I'll call "incorrectly placing the coefficient in the denominator." With this added incentive, I hoped that I wouldn't see this mistake in the exams. Unfortunately, my bribe did not work. Several students still simplified the expression $(2y^3)^2(6y^{-1})$ incorrectly, like this:

$$(2y^3)^2(6y^{-1}) = 4y^6 \cdot \frac{1}{6y} = \frac{2}{3}y^5 .$$

The 6 does not belong in the denominator of the second factor! The exponent of -1 applies only to the $y$, not to the coefficient in front of it. If the exponent of -1 were *outside* the second pair of parentheses, then the above simplification would be valid as follows:

$$(2y^3)^2(6y)^{-1} = 4y^6 \cdot \frac{1}{6y} = \frac{2}{3}y^5 .$$

So we see that the placement of parentheses is extremely important. The correct way to simplify the original expression is

$$(2y^3)^2(6y^{-1}) = 4y^6 \cdot \frac{6}{y} = 24y^5 .$$

In a previous column, we saw that exponents distribute across products because exponentiation is repeated multiplication. That is why the factor on the left in the original expression simplifies to

$$(2y^3)^2 = 2^2(y^3)^2 = 2^2 \cdot y^{(3 \cdot 2)} = 4y^6.$$

Because the exponent of 2 is *outside* the parentheses, we must distribute it to both the coefficient 2 inside the parentheses and the factor $y^3$; the exponent applies to the entire expression in parentheses. Another common mistake I have seen with this expression is forgetting to distribute the exponent to the 2, yielding something like this:

$$(2y^3)^2 = 2(y^3)^2 = 2 \cdot y^{3 \cdot 2} = 2y^6.$$

This mistake is the opposite of the mistake that I tried to bribe my students not to make. That is, the first mistake assumed that an exponent applied to more than just the factor to which it was directly attached, whereas the second mistake failed to apply an exponent to a factor to which it was actually attached via parentheses. So what we need to watch for is the exact placement of each exponent in relation to any parentheses.

For some reason, I have yet to see the following mistake, where the exponent of 3 is misapplied to the coefficient of 2:

$$(2y^3)^2 = (2^3y^3)^2 = 2^{(3\cdot 2)} \cdot y^{(3\cdot 2)} = 64y^6.$$

Somehow, students seem to understand that the exponent of 3 belongs only to the *y* and not to the 2. But I've often seen students simplify the left-hand factor of the original expression $(2y^3)^2(6y^{-1})$ correctly, yet misread the right-hand factor in the way that I first mentioned in this column:

$$6y^{-1} = \frac{1}{6y}.$$

I don't know why there is such a temptation to apply the exponent of -1 to the coefficient of 6. If we properly apply the order of operations, the correct interpretation is

$$6y^{-1} = \frac{6}{y}.$$

Perhaps the pair of parentheses around $6y^{-1}$ is confusing. The second pair of parentheses in the original expression is there to indicate multiplication of the left-hand factor of $(2y^3)^2$ with the right-hand factor $6y^{-1}$. As I mentioned in the previous example, if the exponent of -1 were *outside* this pair of parentheses, then the exponent would apply to everything *inside* the parentheses. But the exponent is *not* outside the parentheses, therefore should not be treated as such.

I have seen similar mistakes in simplifying the expression $(3z^{-7})(4z^3)^2$. Here, I generally see the following mistake:

$$(3z^{-7})(4z^3)^2 = \left(\frac{1}{3z^7}\right)(4^2(z^3)^2) = \left(\frac{1}{3z^7}\right)(16z^6) = \frac{16}{3z}.$$

The negative sign of the -7 exponent is misapplied to the coefficient of 3, but oddly enough, the 7 is not misapplied to the 3. In fact, I have not yet seen this mistake, despite the fact that it would make much more sense to me as a mishandling of the exponent -7 with regard to the first pair of parentheses:

$$(3z^{-7})(4z^3)^2 = \left(\frac{1}{3^7z^7}\right)(4^2(z^3)^2) = \left(\frac{1}{2187z^7}\right)(16z^6) = \frac{16}{2187z}.$$

So I hypothesize that, rather than being a mere misreading of parentheses, the mistake I have called "incorrectly placing the coefficient in the denominator" happens because students are overzealous about showing that they know that the negative sign in the exponent calls for a denominator. So we must stop and ask ourselves to what exactly each exponent applies and whether we should distribute it or not. The correct simplification of the expression given earlier in this paragraph is

$$(3z^{-7})(4z^3)^2 = \left(\frac{3}{z^7}\right)(4^2(z^3)^2) = \left(\frac{3}{z^7}\right)(16z^6) = \frac{48}{z}.$$

Note that the coefficient of 3 ends up in the *numerator* of the fraction $3/z^7$ rather than in the denominator!

Here is the third expression that I put on this particular exam: $(5x^4)^{-2}$. In contrast with the other two, the negative exponent actually applies to both the 5 and the $x^4$. The most common mistakes I saw for those who chose to simplify this expression were forgetting to apply the exponent of -2 to the coefficient 5, or else completely forgetting what a negative exponent means in the first place. It is not correct that

$$(5x^4)^{-2} = 5^2(x^4)^2 = 25x^8.$$

Can you spot the error? It is also not true that

$$(5x^4)^{-2} = \frac{5}{(x^4)^2} = \frac{5}{x^8},$$

or that

$$(5x^4)^{-2} = \frac{1}{5(x^4)^2} = \frac{1}{5x^8}.$$

Can you explain why not? Likewise,

$$(5x^4)^{-2} = \frac{1}{5^2(x^4)^2} = \frac{1}{10x^8}$$

is incorrect (what happened there?), although we are getting much closer to the right answer. The correct simplification is

$$(5x^4)^{-2} = \frac{1}{5^2(x^4)^2} = \frac{1}{25x^8}.$$

### The Order of Operations

The matter that Cammie addresses pertains to the language of mathematics. Specifically, the errors committed in simplifying the expressions result from improperly applying the conventions that have been adopted for interpreting mathematical expressions. For mathematical expressions that only involve parentheses, exponentiation, multiplication, division, addition, and subtraction, the convention is to evaluate in the following order:

1. Parentheses working from the innermost out
2. Exponentiation
3. Multiplication and Division (from left to right)
4. Addition and Subtraction (from left to right)

This ordering is not something that can be deduced in the way that mathematical theorems are proven. The ordering is a convention that has been widely agreed upon to facilitate communication. There are other notational systems in use, but this is the one that is most widely assumed. Since it is a convention, you simply have to memorize it.

Thus, when we evaluate $(2x)^{-1}$, we evaluate the expression in parentheses first. That means that the exponent of -1 applies to the product $2x$ and not just to $x$. When we evaluate $3y^{-1}$, there are no parentheses, so we evaluate the exponential first, before the product. So the exponent of -1 only applies to the $y$ and not to $3y$.

Here is a tricky expression that I have never dared to put on an exam. Try simplifying the following: $(4x^{-2})^{-3}(8x^{-1})$. Note that here, the exponent of -3 is outside a pair of parentheses, so it applies to everything on the inside, whereas the exponent of -2 applies only to the first $x$ and the exponent of -1 applies only to the second $x$. So the correct simplification is as follows:

$$(4x^{-2})^{-3}(8x^{-1}) = (4^{-3}(x^{-2})^{-3})\left(\frac{8}{x}\right) = \left(\frac{1}{4^3}\right)(x^{(-2)(-3)})\left(\frac{8}{x}\right) = \left(\frac{1}{64}\right)(x^6)\left(\frac{8}{x}\right) = \frac{x^5}{8}.$$

For practice, simplify each expression, rewriting using only positive exponents. Be sure to watch out for which exponents are outside a pair of parentheses and which ones are not! The answers can be found on page 29.

| | | | |
|---|---|---|---|
| 1. $(4x^2)^{-3}$ | 2. $(4x^{-3})^2$ | 3. $(3y^{-2})(2y^2)^4$ | 4. $(3y)^{-2}(2y^4)^2$ |
| 5. $(4z^3)^{-1}(5z)^{-2}$ | 6. $(4z^{-1})^3(5z^{-2})$ | 7. $(6w^{-4})^{-1}(3w^5)$ | 8. $(6w^{-4})^{-2}(3w^5)^2$ |

# Fermat's Little Theorem[1]

by Robert Donley

Robert Donley runs the YouTube channel MathDoctorBob, which has over 650 videos on almost 20 math subjects.

Here's a fun calculator exercise known as **Fermat's little theorem**. Pick a favorite prime number $p$. Now take any integer $m$, raise it to the $p$-th power, and subtract $m$. Believe it or not, but the result will *always* be divisible by $p$.

Of course, seeing is believing, so let's check a few cases. If $p = 5$ and $m = 2$, then $2^5 - 2 = 30$ is divisible by 5. If $m = 7$, then $7^5 - 7 = 16{,}800$ is divisible by 5. If $m = 12$, then $12^5 - 12 = 248{,}820$ is divisible by 5, and so on. Checking several different primes and integers $m$ may seem convincing, but not to a mathematician! We can't check *every* prime $p$ and *every* integer $m$ by hand or even with a computer, so instead, we need a proof, which is a logical argument that establishes the truth of a mathematical statement.

As we think about how to prove Fermat's little theorem, we will come upon deeper and deeper mathematics and discover that this result is just a hint of many wondrous concepts.

Let's consider some small primes to see how a proof might go.

When $p = 2$, then $m^p - m = m^2 - m = m(m - 1)$ is the product of two consecutive integers; one of these is odd and the other even, so the product is even, i.e. divisible by 2.

When $p = 3$, then $m^p - m = m^3 - m = (m - 1)m(m + 1)$ is the product of three consecutive integers, one of which must therefore be a multiple of 3.

When $p = 5$, then $m^p - m = m^5 - m = m(m - 1)(m^3 + m^2 + m + 1)$, and the way to proceed is not so clear. By examining the first two factors, $m$ and $m - 1$, we can see that if $m$ is a multiple of 5 or one more than a multiple of 5 (that is, if $m = 5k$ or $m = 5k + 1$ for some integer $k$), then the result holds. But what about the remaining three cases where $m = 5k + 2$, $5k + 3$, or $5k + 4$?

Let's consider the case $m = 5k + 2$. Since $m$ and $m - 1$ are not divisible by 5 in this case, if the theorem is true, then $m^3 + m^2 + m + 1$ must be divisible by 5. So let's substitute $5k + 2$ into this expression for $m$ and see if we can show that it is divisible by 5. Because our result is about divisibility by 5, we can always discard multiples of 5 from our bookkeeping. That will help us simplify our computations. We'll compute $m^3$, $m^2$, $m$, and 1 and add these all up while freely ignoring multiples of 5 that appear:

$$\begin{aligned}
1 &= 1 \\
m &= 5k + 2 = \ldots + 2 \\
m^2 &= 25k^2 + 10k + 4 = \ldots + 4 \\
m^3 &= 125k^3 + 150k^2 + 60k + 8 = \ldots + 8 \\
\\
m^3 + m^2 + m + 1 &= \ldots + 1 + 2 + 4 + 8 = \ldots + 15
\end{aligned}$$

So we see that $m^3 + m^2 + m + 1 = 5K + 15$ is divisible by 5. Here, $K$ is the some integer— we could be more precise about what $K$ is, but it is not necessary because it appears in a term with a factor of 5, and we're not worrying about multiples of 5. The main point is that we need only carry the remainders to check the theorem. Thus, for the remaining cases:

| | | Divisible by 5? |
|---|---|---|
| $m = 5k + 3$: | $3^3 + 3^2 + 3 + 1 = 40$ | Y |
| $m = 5k + 4$: | $4^3 + 4^2 + 4 + 1 = 85$ | Y |

Since the contributions from the remainders are divisible by 5, the theorem is proven, for $p = 5$.

[1] This content supported in part by a grant from MathWorks.

Proving the case $p = 5$ shows us that it makes good sense to focus on remainders (after dividing by $p$). At the very least, doing so simplifies our bookkeeping. When you focus on remainders, you are using **modular arithmetic**. If we fix an integer $n$ greater than 1, the possible remainders we can get when we divide by $n$ are contained in the set $\{0, 1, 2, \ldots, n - 1\}$. We can perform operations of addition, subtraction, and multiplication as usual, except we add or subtract multiples of $n$ to get back inside this set of remainders if we fall outside of it. The number $n$ is called the **modulus** and we say we are doing arithmetic **modulo *n***.

> If you are new to modular arithmetic, convince yourself of the following fact:
>
> If $a = b \pmod{n}$ and $c = d \pmod{n}$, then
> $$a + c = b + d \pmod{n}$$
> and
> $$ac = bd \pmod{n}.$$
> For more on modular arithmetic, see *Prueba del 9* in Volume 2, Numbers 3 and 4 of this Bulletin or read any book on number theory, such as *Invitation to Number Theory* by Oystein Ore.

When $n = 12$, we are in the familiar case of a clock face with 12 replaced by 0. For instance,

$$\begin{aligned} 8 + 11 = 19 &= 7 \pmod{12}, \\ 3 - 6 = -3 &= 9 \pmod{12}, \\ 5 \times 7 = 35 &= 11 \pmod{12}. \end{aligned}$$

We write "(mod 12)" after each equation to remind us that we are doing arithmetic modulo 12. That is, $a = b \pmod{12}$ if and only if $a$ and $b$ leave the same remainder after dividing by 12. Equivalently, $a = b \pmod{12}$ if and only if $a - b$ is divisible by 12.

We restate Fermat's Little Theorem for the prime number $p$ using modular arithmetic: For all $m$, $m^p = m \pmod{p}$. To translate back to the original statement, $m^p$ and $m$ have the same remainder upon division by $p$, or $m^p - m$ has remainder 0 upon division by $p$, or $p$ divides $m^p - m$.

When we studied the case $p = 5$, it was easy to see that $m = 0$ satisfies $m^p = m \pmod{p}$. A quick computation shows that for any prime $p$, $0^p - 0 = 0$ is divisible by $p$. We then did the cases $m = 1, 2, 3$, and 4. This suggests trying to prove Fermat's little theorem by induction on $m$. We've already established the base case $m = 0$. So the question is, can we show that $(m + 1)^p = (m + 1) \pmod{p}$ assuming that $m^p = m \pmod{p}$? Let's try!

We can expand $(m + 1)^p$ using the binomial theorem:

$$(m+1)^p = \binom{p}{0} m^p + \binom{p}{1} m^{p-1} + \binom{p}{2} m^{p-2} + \cdots + \binom{p}{p-1} m^1 + \binom{p}{p} m^0 .$$

Notice that $\binom{p}{k} = \frac{p!}{k!(p-k)!}$ has a factor of $p$ in the numerator, but no factor of $p$ in the denominator when $1 < k < p$. That means that $\binom{p}{k} = 0 \pmod{p}$ whenever $1 < k < p$. (See the cover of this issue!) In other words, $(m+1)^p = \binom{p}{0} m^p + \binom{p}{p} m^0 = m^p + 1 \pmod{p}$. This is just what we need to bootstrap up from $m = 0$. That is, because $0^p = 0 \pmod{p}$,

$$\begin{aligned} 1^p = (0 + 1)^p = 0^p + 1 = 0 + 1 = 1 \pmod{p}&, \text{ so} \\ 2^p = (1 + 1)^p = 1^p + 1 = 1 + 1 = 2 \pmod{p}&, \text{ so} \\ 3^p = (2 + 1)^p = 2^p + 1 = 2 + 1 + 3 \pmod{p}&, \text{ etc.} \end{aligned}$$

The theorem is proven!

Tune in next time when we'll see Fermat's little theorem in a whole new light.

# Pondering Complex Numbers, 2

Here, we prove the 7 facts about complex numbers stated in the previous issue.

Fact 1. $(x, 0) + (y, 0) = (x + y, 0)$ and $(x, 0)(y, 0) = (xy, 0)$.

In general, $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, ad + bc)$. In the case where $b = d = 0$, these formulas reduce to $(a, 0) + (c, 0) = (a + c, 0)$ and $(a, 0)(c, 0) = (ac, 0)$. This means that if we let $R$ be the set of all complex numbers of the form $(x, 0)$, where $x$ is a real number, then $R$ is **closed** under addition and multiplication. That is, the sum and product of complex numbers in $R$ will also be in $R$. The set $R$ is a copy of the real numbers and for this reason we will sometimes write simply $a$ for $(a, 0)$, where $a$ is a real number. For this reason, the complex numbers are an extension of the real numbers.

Fact 2. The complex number $i$ satisfies $i^2 = -1$.

Recall that $i$ is defined to be the complex number $(0, 1)$. Applying the rule for multiplication of complex numbers, we find that $(0, 1)(0, 1) = (0(0) - 1(1), 0(1) + 1(0)) = (-1, 0) = -1$.

If $a$ is a real number, then $a^2 \geq 0$ and -1 does not have a real square root. However, it does have the two complex number square roots $i$ and $-i$. Indeed, every nonzero complex number has two square roots. Note that a square root of $z$ is a solution to the equation $x^2 - z = 0$. Thus, the polynomial $p(x) = x^2 - z$ has a real number solution only if $z$ is a nonnegative real number, but it will always have a complex number solution.

This last observation generalizes as the **fundamental theorem of algebra**: Any polynomial with complex number coefficients has a complex number root.

Fact 3. We can express $(a, b)$ as $a + bi$.

As explained when discussing Fact 1, the real number $a$ represents the complex number $(a, 0)$, so $a + bi = (a, 0) + (b, 0)(0, 1) = (a, 0) + (0, b) = (a, b)$. In fact, writing $a + bi$ is a more common way of expressing complex numbers than writing ordered pairs $(a, b)$.

Fact 4. Complex number addition and multiplication are commutative and associative. The number $0 = (0, 0)$ is the additive identity and $1 = (1, 0)$ is the multiplicative identity. Also, the distributive law holds.

| | | |
|---|---|---|
| Addition | Commutative law | $(a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d) + (a, b)$ |
| | Associative law | $(a, b) + ((c, d) + (e, f)) = (a, b) + (c + e, d + f) = (a + (c + e), b + (d + f))$ $= ((a + c) + e, (b + d) + f) = (a + c, b + d) + (e, f) = ((a, b) + (c, d)) + (e, f)$ |
| | Additive identity | If $(a, b) + (c, d) = (a, b)$ for all $(a, b)$, then $a + c = a$ and $b + d = b$, which implies that $c = d = 0$. |
| Multiplication | Commutative law | $(a, b)(c, d) = (ac - bd, ad + bc) = (ca - db, cb + da) = (c, d)(a, b)$ |
| | Associative law | $(a, b)((c, d)(e, f)) = (a, b)(ce - df, cf + de)$ $= (ace - adf - bcf - bde, acf + ade + bce - bdf)$ $= ((ac - bd)e - (ad + bc)f, (ac - bd)f + (ad + bc)e)$ $= (ac - bd, ad + bc)(e, f) = ((a, b)(c, d))(e, f)$ |
| | Multiplicative identity | If $(a, b)(c, d) = (a, b)$ for all $(a, b)$, then $ac - bd = a$ and $ad + bc = b$. Solving for $c$ and $d$ yields $c = 1$ and $d = 0$. |
| | Distributive law | $(a, b)((c, d) + (e, f)) = (a, b)(c + e, d + f)$ $= (a(c + e) - b(d + f), a(d + f) + b(c + e))$ $= (ac + ae - bd - bf, ad + af + bc + be)$ $= (ac - bd, ad + bc) + (ae - bf, af + be) = (a, b)(c, d) + (a, b)(e, f)$ |

Fact 5. $|zw| = |z||w|$.

Recall that $|(a, b)| = \sqrt{a^2 + b^2}$. We compute:

$$\begin{aligned} |(a, b)(c, d)|^2 &= |(ac - bd, ad + bc)|^2 \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\ &= (a^2 + b^2)(c^2 + d^2) \\ &= |(a, b)|^2 |(c, d)|^2 \end{aligned}$$

Since everything is positive, we can take principle square roots to obtain the desired fact.

Fact 6. arg $zw$ = arg $z$ + arg $w$ (up to a multiple of a full circle).

Using polar coordinates, we can write the complex number $(a, b)$ as $r(\cos \alpha, \sin \alpha)$, where $r = |(a, b)|$ and $\alpha = \arg(a, b)$. In this manner, $z = r_1(\cos \alpha_1, \sin \alpha_1)$ and $w = r_2(\cos \alpha_2, \sin \alpha_2)$. Then

$$zw = r_1r_2(\cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2, \cos \alpha_1 \sin \alpha_2 + \sin \alpha_1 \cos \alpha_2).$$

Recognizing the angle sum formulas for sine and cosine, we see that

$$zw = r_1r_2(\cos(\alpha_1 + \alpha_2), \sin(\alpha_1 + \alpha_2)).$$

Therefore, $\arg zw = \alpha_1 + \alpha_2 = \arg z + \arg w$ (up to a multiple of a full circle).

Fact 7. The solutions to the equation $z^n = 1$ form the vertices of a regular $n$-gon in the complex plane, where $n$ is an integer greater than 2.

Suppose that $z$ satisfies $z^n = 1$, where $n$ is an integer greater than 2. Fact 5 tells us that $|z| = 1$. That means we can write $z = (\cos \alpha, \sin \alpha)$ for some $\alpha$. From Fact 6, we know that $(\cos \alpha_1, \sin \alpha_1)(\cos \alpha_2, \sin \alpha_2) = (\cos(\alpha_1 + \alpha_2), \sin(\alpha_1 + \alpha_2))$. From this we deduce that $(\cos \alpha, \sin \alpha)^n = (\cos n\alpha, \sin n\alpha)$. For this to be equal to 1, we must have $\cos n\alpha = 1$ and $\sin n\alpha = 0$. The equation $\cos n\alpha = 1$ tells us that $n\alpha$ is a multiple of $2\pi$. If $n\alpha$ is a multiple of $2\pi$, then the equation $\sin n\alpha = 0$ will also hold.

So we need to find $\alpha$ so that $n\alpha = 2\pi k$ for some integer $k$. That is $\alpha = 2\pi k/n$ for some integer $k$. Notice, however, that $2\pi k/n$ and $2\pi k'/n$ correspond to the same angle exactly when $k$ and $k'$ differ by a multiple of $n$. This means that the solutions to the equation $x^n = 1$ are exactly the $n$ complex numbers $(\cos \alpha, \sin \alpha)$ where $\alpha = 2\pi k/n$ and $0 \le k < n$. These complex numbers are equally spaced around the unit circle centered at the origin of the complex number plane. When $k = 0$, this complex number corresponds to 1.

Being able to produce proofs of facts is not the same thing as understanding them. So even though we've now proven these 7 complex number facts, continue to play with them and develop your intuition about them. Gain a comfort level working with complex numbers equal to that which you enjoy when working with real numbers.

# COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

## Owning it: Fraction Satisfaction, Part 7

Here comes the grand dame, 3/7, looking like she's eager to chat.

$\frac{3}{7}$: Hello, my little fraction friend. Why the long face?

**You:** I'm confused and I hate being confused. I hate it! I hate it!

$\frac{3}{7}$: Of course, everyone hates confusion, darling. As a feeling, it's about as popular as nausea. The thing to do when one is confused is to ask questions and calmly try to answer them until one is satisfied. What is the source of your confusion?

**You**: My cousin says that if I shrink 50% and then grow 50% in height, I'll end up shorter than I started. But how can a person shrink and then grow by the same amount and end up shorter? That doesn't make any sense! Is that really true?

$\frac{3}{7}$: Yes, it is. Percentages can be tricky if you aren't used to them. I can help you if you like.

**You**: Yes, please.

$\frac{3}{7}$: Well, percentages are a particular type of fraction. Let's take 87%. It is just a number, a quantity, and, like all quantities, can be found on the number line. The percentage 87% stands for a certain fraction. The 87 is the top number of the fraction. The percent sign, %, or the word "percent," means the fraction has 100 as a bottom number. So 87% is simply an alternative way of writing 87/100.

**You**: Okay. I actually knew all that. But I still don't see how shrinking and growing 50% leaves me shorter than when I started.

$\frac{3}{7}$: There is a communication issue you need to understand. People generally use percentages when talking about the percentage *of* something. But the something is often unstated.

**You**: Okay. I know that "of," in this context, means multiplication. For instance 1/2 of 1/3 means $1/2 \times 1/3 = 1/6$.

$\frac{3}{7}$: Yes. Or we could say 50% of 1/3 is 1/6 because 50% = 50/100 = 1/2.

**You**: Okay. That's clear.

$\frac{3}{7}$: Many situations in which we use percentages, the situation is clear. Have you taken any tests at school lately?

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

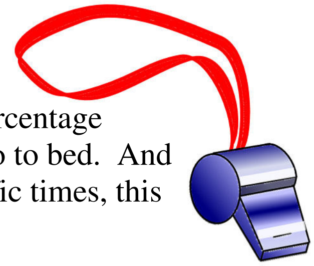
We are also committed to surviving as a nonprofit!

We will make the rest of Coach Barb's Corner available here at some time in the future. But what we hope is that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls



If I wanted to talk about the "overnight" growth as a percentage, then the percentage growth would apply to the height just before nightfall, which is the height when I go to bed. And since my magic beanstalk doesn't grow a nanometer except at those two very specific times, this bedtime height is the same as the 10:59 height.

**You**: But when you say "grows 21%," you don't say "of the initial height."

$\frac{3}{7}$: Yes, that is usually left unstated, and that is responsible for quite a bit of Percentage's tricky reputation.

**You**: Well, I am just going to state clearly what all the percentages are *of*, whenever I use percentages. For instance, if I shrink 50% of my initial height and then grow 50% of that shrunken height, then of course I'll be shorter than when I started, because the shrunken height is less than my initial height. Since I shrink as a percentage of my initial height and I grow as a percentage of the shrunken height, the amount I shrink is greater than the amount I grow. So it's clear that I'll end up shorter than when I started – 25% of my initial height shorter, to be exact.

$\frac{3}{7}$: That is a good approach to take when you're getting used to using percentages.

**You**: Actually, I have a better idea. I am going to use the confusion to my own benefit.

$\frac{3}{7}$: What a horrid idea! How, pray tell, are you going to do that?

**You**: Well, say my neighbor has 200 pieces of Halloween candy.

$\frac{3}{7}$: Yes! Ghastly holiday, unruly children everywhere …

**You**: I am going to offer him a deal. I'll say that if he'll give me 40% of his candy today, then tomorrow I'll increase his candy stock by 60%.

$\frac{3}{7}$: Oh dear.

**You**: He'll definitely accept. Then he'll give me 80 candies today – 40% of what he has now. That will leave him with 120 candies. Tomorrow I only need to give him 60% of the amount he'll have then. 60% of 120 is 72. So I'll only have to give him back 72 of the 80 candies I took. I'll have tricked him out of 8 candies!

$\frac{3}{7}$: I do declare, children today …

## Percentage Questions

Here are some questions you can use to check your understanding.

1. Convert each percentage to a fraction.

   A. 25%
   B. 80%
   C. 150%

2. Convert each fraction to a percentage.

   A. 3/4
   B. 9/10
   C. 24/25

3. As a decimal, what is 1% of 1%?

4. Ten thousand dollars is placed into a savings account. Each year, the money in the account grows by 2%. How much money is in the account after each of the 1st, 2nd, and 3rd years?

5. A company that manufactures light bulbs discovers that 3 out of every 10,000 light bulbs that the company makes is defective. What percentage of light bulbs is defective?

Answers:

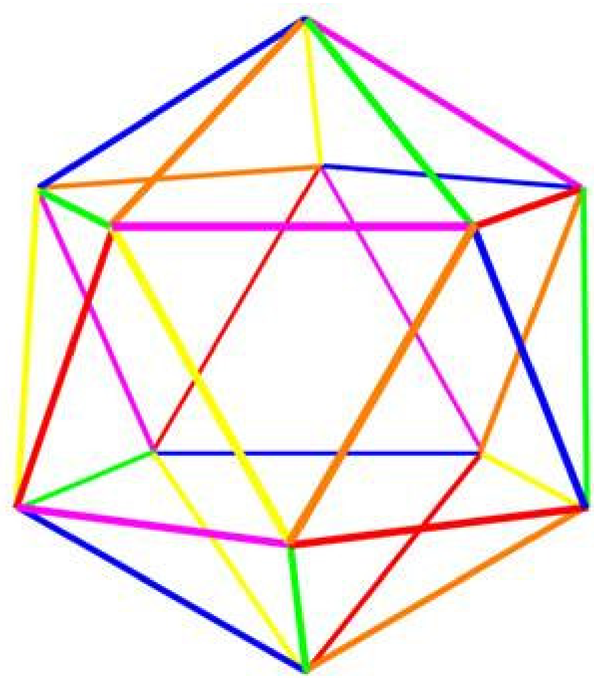
1. A. 1/4, B. 4/5, C. 3/2.
2. A. 75%, B. 90%, C. 96%.
3. 0.0001
4. $10,200, $10,404, $10,612.08.
5. 0.03%

# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

| | | |
|---|---|---|
| Session 11 - Meet 1<br>September 13, 2012 | Mentors: | Jordan Downey, Samantha Hagerman, Colleen Loynachan, Ariana Mann, Vidya Venkateswaran, Luyi Zhang |
| Session 11 - Meet 2<br>September 20, 2012 | Mentors: | Jordan Downey, Samantha Hagerman |
| Session 11 - Meet 3<br>September 27, 2012 | Mentors:<br><br>Visitor: | Vidya Venkateswaran, Luyi Zhang<br><br>Charlene Morrow, Mt. Holyoke College |

Charlene creates artworks that manifest mathematical facts. At this meet, she led us through a modular origami project, building the "Great Stellated Dodecahedron" designed by Paolo Bascetta. A stellated dodecahedron can be thought of as an icosahedron with certain triangular pyramids placed upon each on of its twenty faces. This particular origami model is constructed using 30 sheets of origami paper, one sheet for each edge of the underlying icosahedron. (For instructions, please search the web where there are many videos and written instructions for constructing this model.)



Aside from the interesting geometry, Charlene's model was carefully crafted to exhibit the combinatorial fact that 6 choose 3 is equal to 20. She did this by using 5 sheets each of 6 differently colored pieces of origami paper and carefully connecting the modular units together so that the 20 triangular pyramidal spikes exhibited each of the 20 different ways to pick 3 colors from a palette of 6.

This color scheme amounts to finding a way to color each of the 30 edges of an icosahedron 1 of 6 different colors in such a way that every one of the 20 different ways to pick 3 colors from a palette of 6 occurs in the 3 sides of some triangular face. Even more, each vertex of an icosahedron is the endpoint of 5 different edges, and these edges were each colored differently. In fact, each of the 6 different ways of choosing 5 colors from 6 was exhibited twice at such vertex hubs.

There are actually multiple ways to achieve such a coloring of the edges of an icosahedron. The figure above shows one such way. Can you find another?

| | | |
|---|---|---|
| Session 11 - Meet 4<br>October 4, 2012 | Mentors: | Jordan Downey, Vidya Venkateswaran, Luyi Zhang |

| | | |
|---|---|---|
| Session 11 - Meet 5<br>October 11, 2012 | Mentors: | Jordan Downey, Samantha Hagerman, Luyi Zhang |
| | Visitor: | Pardis Sabeti, Broad Institute/Harvard |

Born in Iran, Pardis immigrated to the United States when she was 3 years old. She holds a doctoral degree in Human Genomics from Oxford and an MD from Harvard Medical School.

Pardis described some of the work she does as the leader of a number of genomics laboratories. The human genome has 23 chromosome pairs containing approximately $3 \times 10^9$ letters. These letters determine how our cells work. In this sense, biology is information. Because of the enormous size of the human genome, Pardis uses statistical techniques to find correlations between the genetic information and traits. She can also study the genome to determine when a mutation occurred by examining how spread out among the population the mutation has become.

She has studied numerous traits, including pigmentation, sickle-cell anemia, lactose intolerance, a trait that makes people immune to Lassa virus, a trait that made Asians have more sweat pores, and a trait that makes Asians have shovel-shaped front teeth. Her work focuses on using a combination of math, statistics, and biology to study the human genome to try to improve the health of humanity.

| | | |
|---|---|---|
| Session 11 - Meet 6<br>October 18, 2012 | Mentors: | Jordan Downey, Samantha Hagerman, Lucia Mocz,<br>Luyi Zhang |

| | | |
|---|---|---|
| Session 11 - Meet 7<br>October 25, 2012 | Mentors: | Ellena Capote, Jordan Downey, Vidya Venkateswaran,<br>Luyi Zhang |
| | Visitor: | Anoush Najarian, MathWorks |

Anoush Najarian works at MathWorks, a company that makes mathematical software tools. To demonstrate the power of such mathematical tools, she had members try to determine the MathWorks FAX number from the touchtone sounds. If you try to do this using only your ear, it is quite challenging. However, using a program written in MATLAB, the tones were converted to easily distinguishable discrete information by using the mathematical technique known as the Fourier transform.

Anoush had some good advice for our members. She encouraged members not to shy away from challenges, saying, “If it is hard, that’s a good thing.” And she encouraged members to seek out a university where the teachers are experts.

In the course of her presentation, she mentioned some math puzzles that we thought our readers might like to try:

1. You’re given two ropes. Each rope takes exactly 1 hour to burn through. You are not to assume that the ropes burn through at an even rate. Using these two ropes, how you can measure out exactly 3/4 of an hour?

2. Take a chessboard and consider the points defined by the intersecting gridlines. What is the maximum number of such points that sit on a single circle?

# Calendar

Session 11: (all dates in 2012)

| | | |
|---|---|---|
| September | 13 | Start of the eleventh session! |
| | 20 | |
| | 27 | Charlene Morrow, Mt. Holyoke |
| October | 4 | |
| | 11 | Pardis Sabeti, Broad Institute/Harvard |
| | 18 | |
| | 25 | Anoush Najarian, MathWorks |
| November | 1 | |
| | 8 | |
| | 15 | |
| | 22 | Thanksgiving - No meet |
| | 29 | |
| December | 6 | |

Session 12: (all dates in 2013)

| | | |
|---|---|---|
| January | 31 | Start of the tenth session! |
| February | 7 | |
| | 14 | |
| | 21 | No meet |
| | 28 | |
| March | 7 | |
| | 14 | |
| | 21 | No meet |
| | 28 | |
| April | 4 | |
| | 11 | |
| | 18 | No meet |
| | 25 | |
| May | 2 | |
| | 9 | |

Spring **Math Contest Prep** begins January 27, 2013. For more information, please visit www.girlsangle.org/page/contest_prep_FAQ.html.

Here are answers to the *Errorbusters!* problems on page 19.

1. $\dfrac{1}{64x^6}$
2. $\dfrac{16}{x^6}$
3. $48y^6$
4. $\dfrac{4}{9}y^6$
5. $\dfrac{1}{100z^5}$
6. $\dfrac{320}{z^5}$
7. $\dfrac{1}{2}w^9$
8. $\dfrac{1}{4}w^{18}$

# Girls' Angle: A Math Club for Girls

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

**What is Girls' Angle?** Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, custom content production including our magazine, the Girls' Angle Bulletin, and various outreach activities such as our Math Treasure Hunts and Community Outreach.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

**What is the Girls' Angle Bulletin?** The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The print version (beginning with volume 3, number 1) costs $36 for an annual subscription and brings with it access to our mentors through email. Subscribers may send us their solutions, questions, and content suggestions, and expect a response. The Bulletin targets girls roughly the age of current members. Each issue contains a variety of content at different levels of difficulty extending all the way to the very challenging indeed.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We also aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

**How do I join? Membership** is granted per session. Members have access to the club where they work directly with our mentors exploring mathematics. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin. Remote members may email us math questions (although we won't do people's homework!), send us problem solutions for constructive comment, and suggest content for the Bulletin. To become a remote member, you can simply subscribe to the print version of the Bulletin.

**Where is Girls' Angle located?** Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Lauren Williams, assistant professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: A Math Club for Girls

## Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership/Bulletin Subscription

Parents/Guardians: ______________________________________________________________________

Address: ____________________________________________________________ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership applicants only**, please fill out the information in this box.

**Emergency contact name and number**: ____________________________________________________

**Pick Up Info**: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ________________________________________________________________

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____________________________________________________________________________________________

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____________________________________________________ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ______________________________________________________

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership (which includes 12 two-hour club meets)
  - ☐ $36 for a one year Remote Membership (which includes 1-year subscription to Bulletin)
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

## Girls’ Angle: A Math Club for Girls

**Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

______________________________________________________________________________,

do hereby consent to my child(ren)’s participation in Girls’ Angle and do forever and irrevocably release Girls’ Angle and its directors, officers, employees, agents, and volunteers (collectively the “Releasees”) from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney’s fees, in any way connected with or arising out of my child(ren)’s participation in Girls’ Angle, whether or not caused by my child(ren)’s negligence or by any act or omission of Girls’ Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)’s participation in Girls’ Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls’ Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys’ fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)’s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)’s participation in the Program.

Signature of applicant/parent: ________________________________________ Date: ___________________

Print name of applicant/parent: ________________________________________

Print name(s) of child(ren) in program: ________________________________

Girls' Angle

**A Math Club for Girls**