Girls' Angle Bulletin

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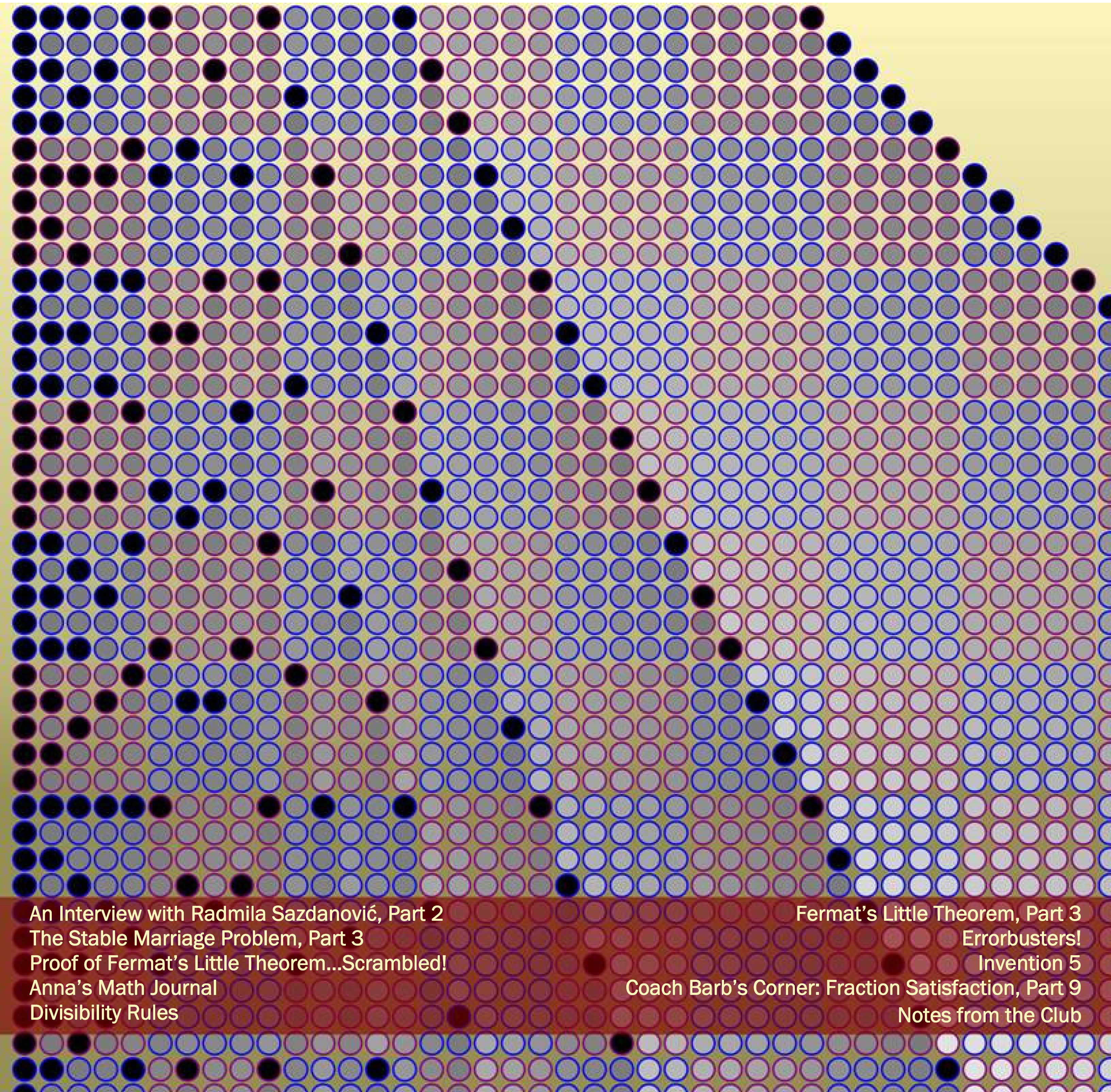
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

To improve at thinking, you have to try to understand things well. Math provides a wealth of things to understand at all levels of complexity. Start at whatever level you're capable of then challenge yourself to grasp ever more complex things. Check your understanding by solving problems. Don't fear failure, and above all, don't give up!

- Ken Fan, President and Founder

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Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Divisibility* by Ken Fan. Lately, exploration of the concept of divisibility has been a major theme at the club.

An Interview with Radmila Sazdanović, Part 2

Ken: How tantalizing! Do you have any advice for how to best approach learning mathematics?

Radmila: There is no one best way, but it is definitely useful to have an active approach. Try to relate the math you are learning to the math you already know. When you learn something new try to understand why is it true, why is it important, how can it be generalized? Looking for generalizations of mathematical concepts or new relations between object that are well known often lead to new discoveries!

Ken: When you get stuck on a problem, what kinds of things do you do to try to get unstuck?

Radmila: Most practical solution is to partition the problem into things that are difficult for me and things that I know I can do more quickly. Then I do the easy things first and hope that by the time I have to deal with the not-so-easy parts I will have new ideas how to approach them. Doing math is a bit like putting the pieces of some bigger puzzle together except that you do not know exactly how it should look like in the end.

Ken: …Did math always come easily to you?

Radmila: *Math demystified: you have to work hard.*

Ken: You are also a visual artist. For you, what is the relationship between the two worlds of art and math? Has one world informed or inspired work in the other?

Radmila: In the Renaissance – and earlier – the concepts "science" and "art" were not distinct. The idea of visual perspective originated in art studios and preceded projective and descriptive geometry by roughly 2 centuries. Leonardo Da Vinci and Albrecht Dürer were studying polyhedra.

Since Romanticism the artist has been viewed as someone who does not think analytically, maybe has messy hair, etc. But it is not true that artists are not analytical. Cubism, for example, is an attempt to render in a single painting the information about an object obtained from several different viewpoints. This is not so different from the insight underlying Einstein's theory of relativity, namely that the way we perceive something depends upon our own position (and momentum). It is maybe just a coincidence, or maybe not, that Cubism and relativity were roughly contemporaneous, and that Einstein had messy hair, too. (Hilbert just wore a silly hat.)

Ken: I really like "Sea Pearls" and "Seven Towers." Can you elaborate on how these works relate to the hyperbolic plane?

Radmila: Both "Sea Pearls" and "Seven Towers" are tessellations of the hyperbolic plane. If you are not familiar with the hyperbolic plane, you can think about it as a special disk in which dots that are really far away from us (hence should look small) are those lying close to the boundary circle. Tessellation is an arrangement of shapes closely fitted together, especially of polygons in a repeated pattern without gaps or overlapping.

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For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Prof. Sazdanović, most of Part 3 of the Stable Marriage Algorithm, and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
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The Stable Marriage Problem[1]

Part 3. Provable Truths

by Emily Riehl, illustrated by Julia Zimmerman, edited by Grace Lyo

In the last two installments, we introduced a problem and proposed a means for finding its solution. The problem was to arrange marriages with the goal of stability: no unmatched couple should simultaneously prefer each other to their current partners. The matches are determined by the **girl-proposing algorithm**: the girls propose to their top choices and the boys reject all but their best suitor. The rejected girls then make a second proposal, and then the boys have a chance to reject previous suitors in favor of a new one. This process repeats until everyone is engaged.

Some proofs. It remains to prove that this idea actually works. We separated this question up into two parts:

Theorem 1. *The Algorithm always terminates.*

Theorem 2. *The arranged marriages are stable.*

Let's prove them.

Proof of Theorem 1. The algorithm stops when every boy is engaged — this also means that every girl is engaged, since the numbers of boys and girls are the same. But note that once a boy becomes engaged he never becomes unengaged; he might change partners but he is not allowed to break an engagement in favor of becoming single again. So the first moment that no boy is single is exactly when the last boy is proposed to for the first time. And this must happen eventually because the number of boys who have been proposed to will only ever increase, and any perpetually single girl will eventually propose to all the boys because she has ranked each boy in the village on her list. □

A more precise argument can be used to determine the maximum number of days it will take before the algorithm terminates. Complexity theory, a branch of computer science, investigates how long it takes to complete a given task.

Now let us prove that our algorithm solves the original problem: arranging **stable marriages**.

Proof of Theorem 2.

[1] This content supported in part by a grant from MathWorks.

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(Unscramble this) Proof of Fermat's Little Theorem

Every word in S_w can be obtained by starting with w and, one by one, moving the first letter to the end of the word. Since w is a word with p letters, if you repeat this process p times, you arrive back at w. Therefore, S_w has at most p words in it.
Theorem. Let p be a prime number. Then p divides $n^p - n$ for any positive integer n.
Notice that for any word w' in S_w, we have $S_{w'} = S_w$. This implies that S_w and S_v are either equal to each other or disjoint for all words w, v in W. Therefore, we can form a partition of W by taking the sets S_w for w in W and removing duplicates.
Let w be a word in W. Let S_w be the set of words that can be obtained from w by cyclically rotating the letters of w. That is, if w' can be obtained from w by taking a block of letters from the front of w and shifting them to the end of w, then w' is in S_w. Since cyclically rotating letters of a word preserves its length, S_w is a subset of W.
If S_w has less than p words, it must mean that as you rotate by one letter at a time, you get some word duplication. That is, there must be some word v in S_w where, if you remove some k (less than p and greater than 0) letters from the front of v and move them to the end of v, you get back the word v. But this means that the first letter of v must be the same as the $(k + 1)$st letter of v, which in turn must be the same as the $(2k + 1)$st letter of v, etc. More precisely, the first letter of v must be the same as the $(mk + 1)$st letter of v for any integer m, where by "$(mk + 1)$st letter of v" we mean that you have to consider the remainder of the quantity $mk + 1$ divided by p (and if the remainder is 0, that stands for the last letter of v).
Since there are exactly n different words in W where every letter is the same (one for each letter in the alphabet), our partition of W contains exactly n subsets with a single word, with the rest containing p words. We conclude that W has $n + pM$ words, where M is some integer (equal to the number of subsets in our partition that have p words).
We claim that the remainders of $mk + 1$ for $m = 1, 2, 3, \ldots, p$ are all distinct, because if any of these had the same remainder, say $ak + 1$ and $bk + 1$, with $a < b$, then p would divide their difference $(bk + 1) - (ak + 1) = (b - a)k$, but both $b - a$ and k are positive integers less than p, so that is impossible. Thus, every remainder is obtained as m changes from 1 to p. But that means that the first letter of v must be the same as every other letter of v. and S_w consists of this single word.
To summarize, S_w either contains p words or 1 word, and if it contains 1 word, it is a word whose letters are all the same.
Proof. Imagine an alphabet with n letters. Let W be the set of all p-letter words made using this alphabet. (We're not concerned with whether the words make sense or are pronounceable.)
Therefore, $n^p = n + pM$, and this tells us that p divides $n^p - n$.
There are n choices for each letter, so the number of words in W is n^p.
For example, if $p = 5$ and w is the word "GOOSE," then S_w = {GOOSE, OOSEG, OSEGO, SEGOO, EGOOS}.

Reorder the rectangles so that the resulting text forms a coherent proof of Fermat's little theorem. (Note: The above proof is a combinatorial proof and does not address the case when $n \leq 0$. As an additional problem, extend the proof to cover all integers n.)

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna verifies 3/7's claim about the reciprocal of rates in Coach Barb's Corner on page **Error! Bookmark not defined.**.

First, I'll write down a clear statement of the problem.

Problem. Show that the arithmetic mean of the reciprocals of two positive numbers is always greater than or equal to the reciprocal of their arithmetic mean, with equality if and only if the numbers are equal.

For more context, see this issue's installment of Coach Barb's Corner.

I'll rewrite the problem as an inequality. Here's what I have to show.

Suppose $0 < x \leq y$.

Show $\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right) \geq \frac{1}{\frac{x+y}{2}}$

So I want to show that this expression is nonnegative, and zero only when $x = y$.

$$\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right) - \frac{1}{\frac{x+y}{2}}$$

I'll start by simplifying.

$$= \frac{1}{2}\left(\frac{x+y}{xy}\right) - \frac{2}{x+y}$$

$$= \frac{(x+y)^2 - 4xy}{2xy(x+y)}$$

$$= \frac{x^2+2xy+y^2-4xy}{2xy(x+y)}$$

$$= \frac{x^2-2xy+y^2}{2xy(x+y)}$$

$$= \frac{(x-y)^2}{2xy(x+y)} \geq 0 \quad \text{with equality iff } x=y \quad \square$$

Wow! Everything works out so nicely. The denominator is positive, and the numerator is a square. So the numerator will be greater than 0, unless $x = y$. Perfect!

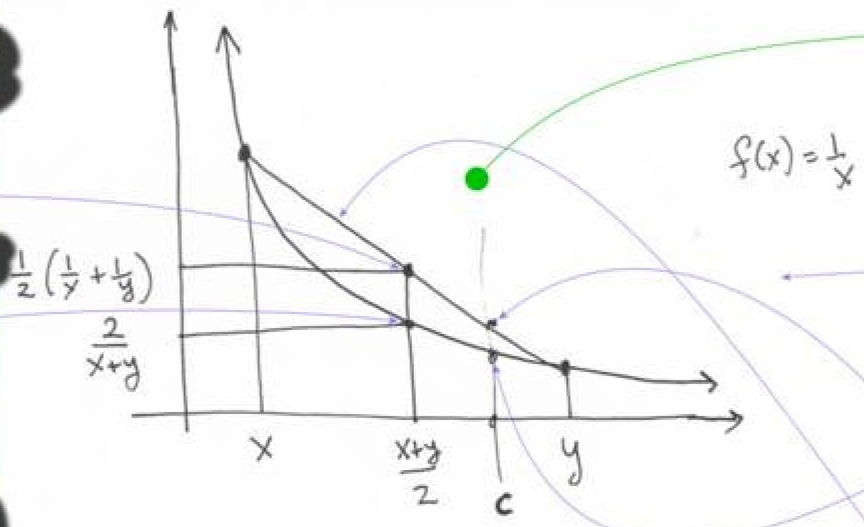
Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

I wonder what this all means graphically. I'll try to graph the situation.



Graphically, the inequality means that this point...

...is above this point.

This is the graph of $f(x) = 1/x$.

This point is $(c, 1/c)$.

The fact that the graph always lies below the "chord" means that the region above the graph is a convex set.

*A region is **convex** iff the line segments that connect any two points in the region are also contained in the region.*

Actually, between x and y, it looks like the graph of $f(x) = 1/x$ lies entirely beneath the straight line segment that connects $(x, 1/x)$ and $(y, 1/y)$. I'll try to prove that.

$$x < c < y$$

$$\text{Claim: } \frac{1}{c} < \frac{1}{x} + (c-x)\left(\frac{\frac{1}{y}-\frac{1}{x}}{y-x}\right),$$

And this point also has horizontal coordinate c. And I can get the vertical coordinate by moving along this line segment. From the point $(x, 1/x)$, I go over $c - x$, so I can get the vertical change by multiplying $c - x$ by the slope.

I'll proceed the same way as before. I'll try to show that this is always greater than 0 (for $x < c < y$).

$$\frac{1}{x} - \frac{1}{c} + (c-x)\left(\frac{\frac{1}{y}-\frac{1}{x}}{y-x}\right)$$

Now I'll start simplifying...

$$= \frac{1}{x} - \frac{1}{c} + (c-x)\left(\frac{\frac{x-y}{xy}}{y-x}\right) = \frac{1}{x} - \frac{1}{c} - \frac{c-x}{xy}$$

$$= \frac{cy - xy - c(c-x)}{cxy} = \frac{cy - xy - c^2 + cx}{cxy}$$

Wait a sec...factoring out y from the first two terms in the numerator will give me $y(c - x)$, and then I can factor out the $(c - x)$...

$$= \frac{(c-x)y - c(c-x)}{cxy} = \frac{(c-x)(y-c)}{cxy} > 0$$

This works out very nicely too! The denominator is again positive and when $x < c < y$, the numerator is also positive, so the whole expression is indeed positive!

ABB 2.27.13

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Divisibility Rules

by Ken Fan / edited by Jennifer Silva

Are there easier ways to figure out if a number is divisible by another number than performing long division? The answer is yes. Perhaps you are already familiar with "casting out 9's" as an example of one of these "divisibility rules."

Casting out 9's says that a number is divisible by 9 if and only if the sum of its digits is also divisible by 9. For example, the sum of the digits of 1,234 is 10, which is not divisible by 9. Therefore, 1,234 is not divisible by 9. (For more on casting out 9's, see the series *Prueba del 9* in Volume 2, Numbers 1 through 5, of this Bulletin.)

The following table lists the more common divisibility rules:

A number is divisible by…	**if…**
2	its units digit is 0, 2, 4, 6, or 8.
3	the sum of its digits is divisible by 3.
4	the 2-digit number comprised of its rightmost 2 digits is divisible by 4.
5	its units digit is 0 or 5.
6	the divisibility rules for both 2 and 3 work.
8	the 3-digit number comprised of its rightmost 3 digits is divisible by 8.
9	the sum of its digits is divisible by 9.
10	the units digit is 0.
11	the alternating sum of the digits is divisible by 11.

These rules may seem like a lot to memorize. Some rules pay attention to the rightmost digit, others have you adding up all the digits, and one asks for the alternating sum of the digits. In this article, we're going to show that these rules are all reflections of a single uniform procedure. We'll then apply this procedure to obtain a divisibility rule for 7.

The Basic Principle

The basic principle we will use over and over is this: if two numbers differ by a multiple of d, then the two numbers leave the same remainder when divided by d.

If you don't understand this principle, perhaps the following way to visualize divisibility will help. When we say a number n is divisible by d, this means that when you divide n by d you get a whole number. Imagine that you work as a packager for a chocolate factory. You've got a vat of chocolates that you have to put into boxes. You must pack exactly d chocolates in each box. If n is divisible by d, then you're in luck. You'll be able to put each and every chocolate into a box with none left over. But if n is not divisible by d, then you'll end up with a few chocolates left over, not enough to fill another box. The number left over is the remainder when you divide n by d.

Suppose you've just spent hours packing chocolate boxes and there are a few chocolates left over. After all that packing, you're about to treat yourself to the remaining chocolates as a small gift for all of your hard work. But just as you're about to pick up one of those succulent morsels, your boss drops another load of chocolates onto your desk. If the number of additional chocolates is divisible by d, then you can imagine that they'll all perfectly fill out some number of boxes, and you'll still have the exact same set of chocolates left over on your desk. This illustrates that adding a multiple of d doesn't affect the remainder.

Applying the Basic Principle

If we want to figure out the remainder that a number n leaves when divided by d, the basic principle allows us to add or subtract multiples of d to our heart's content. If we knew the largest multiple of d less than or equal to n, we could simply subtract that multiple to find the remainder. But the problem is that this multiple is not readily apparent without dividing n by d. To avoid having to divide, we'll instead exploit the information that the number gives us by virtue of being represented in the usual decimal notation. When written in this format, the powers of 10 play a prominent role. Each digit tells how many groups of some power of 10 that the number contains. Thus, the number 352 has 3 groups of $10^2 = 100$, 5 groups of $10^1 = 10$, and 2 groups of $10^0 = 1$. So, *if we find a multiple of* d *that we can subtract from each power of 10 to yield a small number, we can remove these multiples from each group of* 10^p, thereby obtaining a much smaller number that will have the same remainder as the original number.

For example, consider the number 5000. Let's use the idea presented in the preceding paragraph to figure out the remainder of this number when we divide by 9. The numeral 5000 represents 5 groups of 1000. The number 999 is the closest multiple of 9 to 1000. When 999 is subtracted from 1000, the answer is 1. So we subtract 999 from each group of 1000, reducing it to a group consisting of just 1 object. Since there were 5 groups of 1000, we end up with 5 groups of 1, or 5 objects. Therefore, 5000 and 5 leave the same remainder when you divide by 9. If we divide 5 by 9, we get 0 with a remainder of 5. So 5000 is not divisible by 9 and, in fact, leaves a remainder of 5 when divided by 9.

For divisibility by 9, we can see that the numbers 9, 99, 999, 9999, etc., are each 1 away from 10, 100, 1000, 10000, etc., respectively. So we can reduce each group of 10^p objects to a single group consisting of 1 object *without affecting the remainder*. The figure below illustrates this process for the number 542.

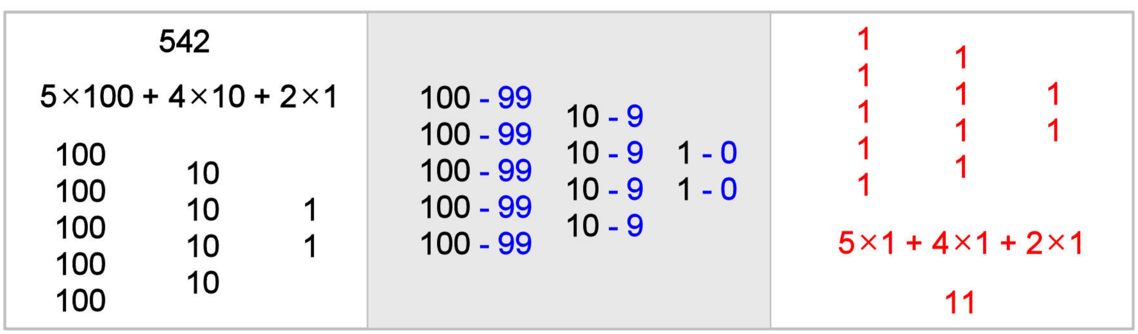


Figure 1. Suppose we want to know what the remainder will be if we divide 542 by 9. The left panel breaks the number 542 into groups of 100's, 10's, and 1's, according to the number's customary decimal representation. In the second panel, the multiples of 9 nearest to each power of 10 are subtracted from it. The basic principle explained in the text tells us that subtracting multiples of 9 will not affect the remainder. After subtracting the multiples of 9, the 100, 10, and 1 in the decimal expansion are effectively replaced with 1, 1, and 1. We conclude that 542 and $5(1) + 4(1) + 2(1) = 11$ leave the same remainder when divided by 9. Since 11 leaves a remainder of 2 when divided by 9, so will 542. Of course, when you apply this procedure, you can skip all of the intervening steps and simply add up the digits. The intervening steps are shown here to explain why adding up the digits works.

Let's apply this method to obtain divisibility rules for the numbers 2 through 11. For each divisor d, we need to figure out what multiple of d is closest to each power of 10. We can then subtract this closest multiple from the power of 10 to obtain a magical sequence of numbers that will produce our divisibility rule.

	10^6	10^5	10^4	10^3	10^2	10^1	10^0
2	10^6	10^5	10^4	10^3	10^2	10^1	0
	0	0	0	0	0	0	1
3	999999	99999	9999	999	99	9	0
	1	1	1	1	1	1	1
4	10^6	10^5	10^4	10^3	10^2	8	0
	0	0	0	0	0	2	1
5	10^6	10^5	10^4	10^3	10^2	10^1	0
	0	0	0	0	0	0	1
6	1000002	100002	10002	1002	102	12	0
	-2	-2	-2	-2	-2	-2	1
7	999999	100002	10003	1001	98	7	0
	1	-2	-3	-1	2	3	1
8	10^6	10^5	10^4	10^3	96	8	0
	0	0	0	0	4	2	1
9	999999	99999	9999	999	99	9	0
	1	1	1	1	1	1	1
10	10^6	10^5	10^4	10^3	10^2	10^1	0
	0	0	0	0	0	0	1
11	999999	100001	9999	1001	99	11	0
	1	-1	1	-1	1	-1	1

In the table above, each divisor's row is split into two rows of numbers, one blue and the other red. The blue numbers are the nearest multiples of the divisor to the power of 10 heading that column. (Sometimes, there are two nearest multiples to a power of 10. However, the basic principle informs us that it doesn't matter which multiple you use.) The red numbers are the result of subtracting the nearest multiple from the corresponding power of 10. This color-coding corresponds to the color-coding in Figure 1.

If you know modular arithmetic, then you can also interpret the red numbers as numbers close to zero that are congruent, modulo d, to each power of 10. Robert Donley made similar computations in *Fermat's Little Theorem, Part 2*, in the previous issue of this Bulletin. Using the technique he used, the magic sequence can be constructed by multiplying each magic number by 10 to get the next (modulo d). The red numbers are eventually periodic. Can you see why?

Let's examine the row corresponding to the divisor 4. The magic numbers, read from right to left, are 1, 2, 0, 0, 0, …. This means that for a given number n, we compute the generally much smaller number by taking 1 times its units digit plus 2 times its tens digit; this smaller number will leave the same remainder as n when divided by 4. For example, to find the remainder of 4,267 when divided by 4, we compute $2(6) + 7 = 19$. Since 19 leaves a remainder of 3 when divided by 4, so does 4,267. (If you don't see what remainder you get when you divide 19 by 4, you can apply the rule again to 19: $2(1) + 9 = 11$, and again to 11: $2(1) + 1 = 3$.)

Perhaps you are thinking, "Wait a minute! The divisibility rule for 4 says that you should take the rightmost 2 digits and consider them as a 2-digit number, not twice the tens digit plus the units digit!" That works too, and aligns with our analysis. It corresponds to leaving the first power of 10 (which is 10 itself) alone, and not subtracting a multiple of 4 from it to get the magic number 2. Doing so makes it easier to find a reduced number to work with because the 2-digit number comprised of the rightmost 2 digits of the original number is sitting before you in plain sight. Replacing the 10 with a 2 requires more computation, but the reduced number will be smaller than the number you get by chopping off the rightmost 2 digits. In practice, use whatever method you find easier, but do understand how the two are related and that both are applications of the same general method.

Divisibility by 7

Let's examine the row corresponding to the divisor 7. The magic numbers read from right to left are: 1, 3, 2, -1, -3, -2, and then they start repeating. We ask, what is the remainder when 104,202 is divided by 7?

We write the digits of the number we are testing with the corresponding magic red numbers below:

1	0	4	2	0	2
-2	-3	-1	2	3	1

We then compute the number obtained by adding up the products of each digit and its corresponding magic red number: $1(-2) + 4(-1) + 2(2) + 2(1) = -2 - 4 + 4 + 2 = 0$. We conclude that 104,202 will have the same remainder as 0 does when you divide it by 7; that is, 104,202 is a multiple of 7! Verify this by using long division. Do you find long division faster or slower than using the divisibility rule?

Notice that the magic number in the 10^p place is the negation of the magic number in the 10^{p+3} place. We can use this fact to conclude that any number whose digits are of the form ABC,ABC will be divisible by 7. Using the divisibility rule for 7, can you see right away that the number 123,124,234,235,345,346 will leave a remainder of 3 when divided by 7?

The Method in a Nutshell

Fix an integer $d > 1$. Let's group the integers into d different sets according to the remainder that the integers leave when divided by d. (Integers are in the same set if and only if they differ by a multiple of d.)

Example: if $d = 2$, then the integers will be grouped into evens and odds.

Modular arithmetic is based on the observation that addition and multiplication respect these sets. That is, if a and a' belong to a set and b and b' belong to a set (possibly, but not necessarily, the same as the one containing a and a'), then $a + b$ and $a' + b'$ will differ by a multiple of d and ab and $a'b'$ will differ by a multiple of d.

Therefore, if we expand a number n as a decimal and replace each power of 10 with a smaller number in the same set as that power, the resulting expression will evaluate to a smaller number in the same set that contains n. *That* is the method in a nutshell.

Other Bases

The magic numbers associated with each divisor are specific to decimal numbers. If you represent numbers in other bases, you have to compute new magic numbers to suit those bases. But some patterns persist in all bases. For instance, a number n leaves the same remainder when divided by b as the sum of its base $b + 1$ digits. The magic sequence for the divisor b with respect to powers of $b + 1$ is a sequence consisting of just 1's. Similarly, for $b > 2$, a number n leaves the same remainder when divided by b as the alternating sum of its base $b - 1$ digits (starting with the units digit and working left).

To test your understanding of these ideas, try to use them to produce a solution to the following problem: If $2^n + 1$ is prime, show that n is a power of 2.[1] For a hint, turn this page upside down:

Hint: Write $n = mr$, where m is odd and r is a power of 2. Develop a divisibility rule for the divisor $2^r + 1$ with respect to the base 2^r.

[1] This is problem 1.12 from Harold Davenport's *The Higher Arithmetic*, 8th edition. There, he further asks if the converse is true. That is, if n is a power of two, does $2^n + 1$ have to be a prime number?

Divisibility Rules Problems

Here are some problems that will help you master the content of Divisibility Rules on page 13. Subscribers, feel free to send in your questions and solutions to girlsangle@gmail.com.

1. (Created by Stuart Sidney) Let N be a positive integer. Let $S(N)$ be the sum of the decimal digits of N. What is $S(S(S(S(2012^{2012}))))$? What is $S(S(S(S(2013^{2013}))))$?

Divisibility Rule for 27. Problems 2-4 pertain to divisibility by 27.

2. Use the method from *Divisibility Rules* (p. 13) to deduce the following facts.

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Thank you and best wishes,
Ken Fan
President and Founder
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Fermat's Little Theorem,[1] Part 3

by Robert Donley / edited by Jennifer Silva

Robert Donley runs the YouTube channel MathDoctorBob, which has over 650 videos on close to 20 math subjects.

In the previous parts of this series, we investigated **Fermat's little theorem**: If we choose any prime p and any integer m, then p divides $m^p - m$. With $p = 7$ and $m = 4$, the theorem states that 7 divides evenly into $4^7 - 4 = 16380 = 7 \cdot 2340$. In the language of modular arithmetic, we write this as $m^p = m \pmod{p}$. (Recall that we write $a = b \pmod{n}$ if and only if n divides evenly into $a - b$. Alternatively, a and b have the same remainder upon division by n.)

In Part 1, we gave a proof using the binomial theorem, which relied on properties of prime-numbered rows in Pascal's triangle. In part 2, we recast the theorem as a statement about periodicity of remainders modulo p for powers of a fixed integer. In this part, we will exploit another property in the set of remainders for a given divisor n to give yet another proof of Fermat's little theorem. This time, however, our approach will actually yield a generalization of the theorem, freeing it from the condition that p be a prime number.

Before presenting this new proof, let's do some exercises to get a sense for what happens when p is not a prime number. So let's begin by following the development in Part 2 and consider the remainders of geometric sequences with respect to a fixed number. However, unlike last time, we'll allow this fixed number to be any positive integer, not just prime numbers. So let d be any positive integer. For example, let $d = 8$. Let's examine the remainders of powers of 3 when divided by d. As noted in Part 2, we don't actually have to calculate the powers of 3 to do this since we're only interested in the remainders these powers leave after dividing by 8. We readily compute that the remainders of the powers of 3 when divided by 8 are:

$$3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, \ldots$$

Please verify and complete the following table:

Table of Powers Modulo 8

n	1	2	3	4	5	6	7	8	9	10	11
0^n	0	0	0	0	0	0	0	0	0	0	0
1^n	1	1	1	1	1	1	1	1	1	1	1
2^n	2	4	0	0	0	0	0	0	0	0	0
3^n	3	1	3	1	3	1	3	1	3	1	3
4^n											
5^n											
6^n											
7^n											

What patterns do we observe now? First, if m is even, say $m = 2k$ where k is an integer, then $m^n = (2k)^n = 2^n k^n$ will be divisible by 8 when $n \geq 3$ (and possibly for smaller values of n, too). That is, powers of even numbers will eventually be divisible by 8. While these cases fall into a predictable pattern, we never return to the original remainder unless it is 0.

[1] This content supported in part by a grant from MathWorks.

On the other hand, we see a familiar pattern if m is odd. The sequence of remainders is immediately periodic, and there exists some k such that $m^k = 1 \pmod 8$. Last time, we saw that in the case when d is a prime number that does not divide m, we could define the order $o_d(m)$ to be the smallest positive integer k such that $m^k = 1 \pmod d$. In a similar manner, we see that 3, 5, and 7 all have order 2 modulo 8.

Make a table similar to the table one the previous page for $d = 6, 9, 10$, and 12. For what values of m will the powers of m eventually leave a remainder of 1 when divided by d?

Did you see that the answer is exactly when m and d share no common factor other than 1? That is, there exists a positive integer k such that $m^k = 1 \pmod d$ if and only if m and d are relatively prime. See if you can prove this. A proof can be constructed along the lines of those used in Part 2. This observation enables us to extend the definition of order to all positive integers d and m relatively prime to d. In this case, we define $o_d(m)$ to be the smallest positive integer k such that $m^k = 1 \pmod d$.

(By reworking the development in Part 2 with a non-prime modulus d, we developed a sense for some of the similarities and differences between prime and non-prime moduli. Recall the Challenge from Part 2 which asked you to show that when d is a prime number, there exists m such that $o_d(m) = d - 1$. Give an example to show that this is no longer true when d is not restricted to being prime.)

Finding a positive integer k where $m^k = 1 \pmod d$ was key to our proof of Fermat's little theorem last time. Our observations to this point show that this is only possible when m and d are relatively prime. Therefore, the concept of being relatively prime must be very important. For this reason, let us define U_d to be the set of all integers m between 1 and d that are relatively prime to d. Note that we can multiply two elements of U_d to get another element of U_d, modulo d. That is, if m and m' are relatively prime to d, then so is mm'. For example, when $d = 8$, we find that $U_8 = \{1, 3, 5, 7\}$ and we have the following multiplication table working modulo 8:

Multiplication in U_8 modulo 8				
b \ a	**1**	**3**	**5**	**7**
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

What do you notice about this table? What do you think would be true not just of this modular multiplication table, but for the modular multiplication table for U_n in general?

Let's define $\varphi(d)$ to be the number of elements in U_n. For example, $\varphi(8) = 4$. This function is known as **Euler's totient function**. We may now state the generalization of Fermat's little theorem, due to Euler:

Euler's theorem. Let d be a positive integer. If m and d are relatively prime, then d divides $m^{\varphi(d)} - 1$. That is, $m^{\varphi(d)} = 1 \pmod d$.

Proof.[2] Choose m relatively prime to d, and consider modular multiplication by m as a map from U_d to U_d. If $mm_1 = mm_2$ (mod d), then d divides $mm_1 - mm_2 = m(m_1 - m_2)$. Since m and d are relatively prime, d must divide $m_1 - m_2$, that is, $m_1 = m_2$ (mod d). Thus, modular multiplication by m is a one-to-one mapping: distinct elements of U_d map to distinct elements. Since U_d is a finite set, modular multiplication by m merely permutes the elements of U_d. That is, if $U_d = \{m_1, \ldots, m_{\varphi(d)}\}$, then we also have $U_d = \{mm_1, \ldots, mm_{\varphi(d)}\}$, where the elements in each list are the same but may be listed in different orders. This fact can be seen in the modular multiplication table for U_8 on the previous page: note that all 4 elements of U_8 appear in each row (not including the row heading, of course).

Now consider the product of all elements of U_d modulo d. We have

$$m_1 \cdots m_{\varphi(d)} = mm_1 \cdots mm_{\varphi(d)} = m^{\varphi(d)}(m_1 \cdots m_{\varphi(d)}) \pmod d.$$

From the first part of the proof, we can cancel each m_i from both sides of the equation, leaving us with the statement of Euler's theorem.

Notice that Fermat's little theorem follows from Euler's theorem once we observe that $\varphi(p) = p - 1$ when p is a prime number.

When $d = 8$, we know $\varphi(8) = 4$ and Euler's theorem tells us that the fourth power of any odd integer will be 1 more than a multiple of 8. In fact, if a fourth power is equal to 1 modulo 8, then the 8th, 12th, 16th, etc. powers will also be equal to 1 modulo 8. So without having to do any further computation, we can say definitively that $2013^{4000} = 1$ (mod 8).

An important feature of the above proof deserves its own mention:

Cancellation Law for U_d. Suppose m is relatively prime to d. If $mm_1 = mm_2$ (mod d), then $m_1 = m_2$ (mod d).

Notice that we can effectively realize the cancellation by m by multiplying both sides by $m^{\varphi(d)-1}$, as Euler's theorem implies. For this reason, it is sensible to define m^{-1} as $m^{\varphi(d)-1}$ in modular d arithmetic when m is relatively prime to d. If we wish to solve the equation $mx = a$ (mod d), we simply have to multiply both sides by m^{-1}.

As a final note, we indicate how to compute $\varphi(d)$. If p is a prime number and $k \geq 1$, then U_{p^k} is obtained by removing the multiples of p from $\{1, 2, 3, \ldots, p^k\}$. Since there are exactly $p^k/p = p^{k-1}$ multiples of p removed, we see that $\varphi(p^k) = p^k - p^{k-1}$. To compute $\varphi(d)$ in general, we can use the fact that $\varphi(ab) = \varphi(a)\varphi(b)$ when a and b are relatively prime. (Can you prove this?) Thus, if the prime factorization of d is $p_1^{k_1} \cdots p_j^{k_j}$, where the p_i are distinct prime numbers, then

$$\varphi(d) = d\,\frac{(p_1 - 1)\cdots(p_j - 1)}{p_1 \cdots p_j}.$$

For instance, if $d = 72$, then $p_1 = 2$ and $p_2 = 3$ and we find $\varphi(72) = 72 \cdot 1/2 \cdot 2/3 = 24$.

---

[2] The proof given follows a line of reasoning due to James Ivory. For further reading, see "Euler's Theorem" by Keith Conrad at www.math.uconn.edu/~kconrad/blurbs/ugradnumthy/eulerthm.pdf.

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

In the review chapter that I teach for my precalculus class, we study one section on simplifying rational expressions. By "rational expressions," I mean fractions that have polynomials in both the numerator and the denominator. One pet peeve I have regarding the simplification of rational expressions is that students often forget that the expression

$$\frac{(x-1)(x-2)}{(x-1)(x+2)}$$

is *not* exactly equivalent to the expression $\frac{x-2}{x+2}$. In the April 2011 *Errorbusters!*, we discussed how to simplify rational expressions correctly, but we did not identify the difference between the two rational expressions I have just shown you. It is true that the first expression simplifies to the second by cancellation of the common factor of $x - 1$ in both the numerator and denominator. However, notice that one can substitute $x = 1$ into the simplified expression (to obtain -1/3), whereas $x = 1$ cannot be substituted into the first expression because you cannot divide by 0. So the original expression cannot have $x = 1$ in its domain, whereas the simplified expression can.

Therefore, to make it clear that we don't wish to inadvertently widen the domain when we simplify a rational expression, we need to stipulate that we are not allowing the simplified expression to be evaluated at any points missing from the original expression's domain. In other words, for our current example, we need to explicitly state that $x \neq 1$ for the simplified expression $\frac{x-2}{x+2}$; without such a statement, there would be no way for someone to know of this condition from the expression alone.

Let's try to simplify a more complicated rational expression: $\frac{x^4-5x^2+4}{x^3-2x^2-5x+6}$, where we will take the domain to be the set of all real numbers where the denominator is not equal to 0. We begin by factoring both the numerator and the denominator as much as possible. A trick that you can use to factor this particular numerator is to set $y = x^2$ and note that

$$x^4 - 5x^2 + 4 = y^2 - 5y + 4 = (y - 1)(y - 4) = (x^2 - 1)(x^2 - 4).$$

Using differences of squares, we can factor this further:

$$(x^2 - 1)(x^2 - 4) = (x + 1)(x - 1)(x + 2)(x - 2).$$

For the denominator, we observe that there must be at least one real root since it is a cubic polynomial (with real coefficients). By the rational root theorem, we know that if the root is a rational number, then it must be an integer that divides 6, i.e., it must be ±1, ±2, ±3, or ±6. We'll test each candidate, starting with 1. We compute $1^3 - 2(1^2) - 5(1) + 6 = 0$. This shows that 1 is a root! So $x - 1$ is a factor. Dividing, we find that $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$. We then factor the quadratic factor to get $x^3 - 2x^2 - 5x + 6 = (x - 1)(x + 2)(x - 3)$.

Now we can simplify the original rational expression:

$$\frac{x^4-5x^2+4}{x^3-2x^2-5x+6} = \frac{(x+1)(x-1)(x+2)(x-2)}{(x-1)(x+2)(x-3)} = \frac{(x+1)(x-2)}{x-3}.$$

We *must* specify that $x \neq 1$ or -2 in our new expression. To be thorough, we should also state that $x \neq 3$ since the denominator is 0 when $x = 3$. This condition will not likely be neglected, however, because the simplified expression manifestly cannot be evaluated when $x = 3$.

Let's try another example:

$$\text{Simplify } \frac{x^3 - 19x + 30}{x^3 + 7x^2 + 2x - 40},$$

where the domain is all real numbers where the denominator is not 0. Both the numerator and denominator are cubic polynomials (with real coefficients), so each has at least one real root. Hopefully they'll be integers! For the numerator, potential rational roots are all divisors of 30, which are: ±1, ±2, ±3, ±5, ±6, ±10, ±15, and ±30. By systematically trying these one after the other, we happen to find that $x = 2$ is a root. You could continue through the list of divisors of 30 to try to find other integer roots, but we'll factor out $x - 2$ to get a quadratic, and then factor the quadratic: $x^3 - 19x + 30 = (x - 2)(x^2 + 2x - 15)$. By factoring the quadratic, we arrive at the complete factorization $x^3 - 19x + 30 = (x - 2)(x - 3)(x + 5)$.

The same approach enables us to factor the denominator. Eventually, we find that

$$x^3 + 7x^2 + 2x - 40 = (x - 2)(x + 4)(x + 5).$$

We can now simplify our original expression:

$$\frac{x^3 - 19x + 30}{x^3 + 7x^2 + 2x - 40} = \frac{(x-2)(x-3)(x+5)}{(x-2)(x+4)(x+5)} = \frac{x-3}{x+4}.$$

We *must* specify that $x \neq 2$, -4, or -5.

Here's another example for you to try. Simplify

$$\frac{x^3 + 3x^2 - 10x - 24}{x^3 + 4x^2 - 9x - 36},$$

where x can be any real number where the denominator is not 0. Can you show that this rational expression is equivalent to

$$\frac{x+2}{x+3}$$

for real numbers $x \neq -4$, -3, or 3?

For practice, simplify each expression and specify the domain of the simplified expression. In each case, the domain of the given expression is the set of all real numbers where the denominator is not equal to 0. Note that the numerator of the fourth expression is the same as the denominator of the fifth expression. The answers can be found on page 29.

1. $\dfrac{x^2 - 25}{x^2 - 3x - 10}$ 2. $\dfrac{x^2 - 7x + 12}{x^2 - 5x + 4}$ 3. $\dfrac{x^3 + 5x^2 - 4x - 20}{x^3 + 10x^2 + 31x + 30}$ 4. $\dfrac{x^3 + 6x^2 + 11x + 6}{x^3 + 3x^2 - 9x - 27}$

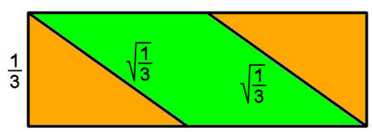
5. $\dfrac{x^3 + 2x^2 - x - 2}{x^3 + 6x^2 + 11x + 6}$ 6. $\dfrac{x^3 + 5x^2 + 2x - 8}{x^2 + 6x + 8}$ 7. $\dfrac{x^4 - 10x^2 + 9}{x^3 + 2x^2 - 11x - 12}$ 8. $\dfrac{x^4 - 34x^2 + 225}{x^4 - 13x^2 + 36}$

Invention 5

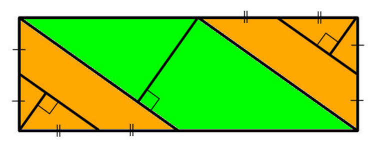
In the previous issue, we described 8 math problems where we claimed that different people would likely come up with different solutions. Here, I'd like to describe two different, partial, solutions to Invention 5, which was to find a way to cut a square into pieces that can be rearranged to form N squares all the same size.

We'll address the case $N = 3$, where we have to cut a square into pieces that can be reassembled to form 3 squares all the same size.

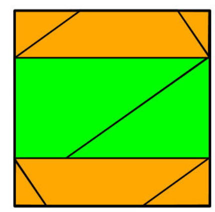
Method 1. For definiteness, assume that the given square is a unit square. A natural way to start is to cut the square into 3 identical 1 by 1/3 rectangles. Then, if we can figure out how to convert one such rectangle into a square, we just repeat the process on the other two.



Since we know that the desired squares will be $\sqrt{1/3}$ on a side, we try to make shapes that have sides with this length. This can be done with two parallel cuts forming two right triangles and a parallelogram.

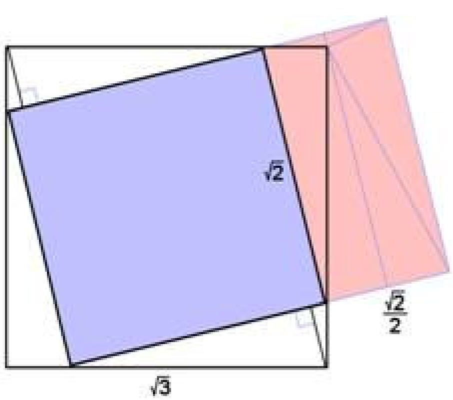


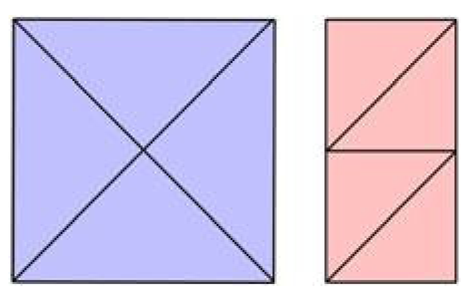
We then make cuts so that the triangles and parallelograms can be formed into rectangles that all have one side of length $\sqrt{1/3}$.

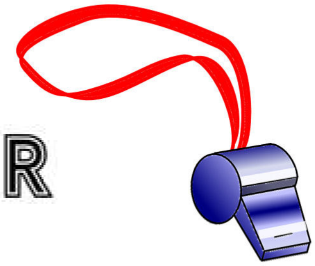


These rectangles can then be stacked to form a square, and we're done.

Method 2. This method was invented by Luyi Zhang, a Girls' Angle mentor. She chose to start with a square $\sqrt{3}$ units on a side so that its total area is 3 square units. The goal then becomes to build 3 unit squares. She began by inscribing a square of area 2 (shown in blue) inside the square of area 3. By snipping 2 of the 4 right triangular fragments along the altitudes to their hypotenuses, we get 6 right triangles that can be rearranged to form the pink rectangle in the figure. Because the pink rectangle consists of pieces left after removing 2 square units from a square of area 3 square units, it must have area 1 square unit. Therefore, the pink rectangle has dimensions $\sqrt{2}$ by $\sqrt{2}/2$. The blue square and pink rectangle can then be dissected into 4 isosceles right triangles which can be fit together to form the sought after 3 unit squares (see bottom right). We'll leave further details to the reader. The motivation behind inscribing a square with area 1 square unit less than the area of the given square is that it might lead to an inductive method for solving the general case (of dissecting a square into pieces that can be rearranged to form N squares). That is, to dissect into N + 1 squares, we inscribe a square of area N into a square of area N + 1 and use induction to treat the square of area N and find a way to fashion the 4 left over right triangles into a unit square. Can you extend Luyi's method to the general case?







COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 9

Oh look! Here comes 3/7, and she's carrying a bucket and wearing prize ribbons on her dress.

You: Hi, 3/7. Those are nice ribbons. Have you been to a horse show?

$\mathbf{\frac{3}{7}}$: Why, no. I've been to something far more exciting: snail races.

You: Snail races? I didn't know that was a sport.

$\mathbf{\frac{3}{7}}$: Oh yes, dear, it's all terribly exciting. I'm campaigning to have them added to the Olympics.

You: Really?

$\mathbf{\frac{3}{7}}$: Oh, yes. Let me show you some of my prizewinners. Here is Bertha. She blasts along at 1/2 meter per hour. This one, Fiona, races at 1 meter per hour. This is Cordelia. She was clocked at 3/2 meters per hour.

You: Oh … well. So Fiona is right in the middle, between Bertha and Cordelia, speed-wise.

$\mathbf{\frac{3}{7}}$: She is indeed, darling. I love them so! My heart just swells with pride as I watch them compete in the 100-meter sprint.

You: What? At Bertha's pace of 2 hours per meter, the race takes over a week!

$\mathbf{\frac{3}{7}}$: Yes, dear, but the action at the front of the race happens much earlier. Fiona only takes one hour per meter so she finishes in a mere 100 hours. Cordelia's pace of 2/3 of an hour per meter has her finish in 66 hours and 40 minutes. That's less than 3 days.

You: Oh. I see. Hey, I just noticed something!

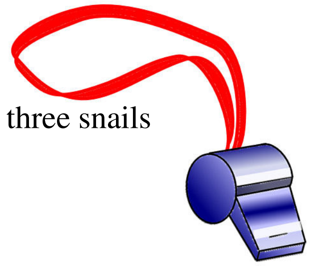
$\mathbf{\frac{3}{7}}$: Do share it darling. But please remember, hay is for horses, one of many animals far inferior to my lovely snails.

You: Sorry. What I noticed is that although the snails' speeds are equally spaced on the number line – 1/2, 1, and 3/2 meters per hour – their paces, by which I mean the time it takes them to travel a meter – 2, 1, and 2/3 hours per meter – are not.

$\mathbf{\frac{3}{7}}$: That's correct, they are not. Nor would I expect them to be. Does that bother you?

You: A little.

$\mathbf{\frac{3}{7}}$: Well, then, let's talk about it.



You: Umm. Okay. Here's what bugs me. In both cases we're describing the same three snails moving along.

$\mathbf{\frac{3}{7}}$: Racing, my dear.

You: As they are racing along, not only are their speeds equally spaced, but Fiona will be Dear Reader,

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 12 - Meet 1 Mentors: Rediet Abebe, Ruthi Hortsch, Lauren McGough, Isabel Vogt, Jessica Wang
January 31, 2013

One thread at the club concerns the "divisibility rules." None of our members knew a divisibility rule for 7, so we developed one at the club. See page 13. You know you've mastered the divisibility rule for 7 when you can determine, relatively quickly and entirely in your head, the remainder that results when you divide the following number by 7:

111,222,333,444,555,666,777,888,999,999,888,777,666,555,444,333,222,111.

What is it? Can you quickly figure out, entirely in your head, the remainder after you divide this next number by 7?

111,222,333,444,555,666,777,888,999,888,777,666,555,444,333,222,111.

At this meet, **#27** mentioned that her favorite number is 27. See page 17 for a divisibility rule for 27 and other problems.

Session 12 - Meet 2 Mentors: Rediet Abebe, Elenna Capote, Jordan Downey, Isabel Vogt
February 7, 2013

Some members unscrambled the proof of Fermat's little theorem on page 10. For yet another proof of Fermat's little theorem, see page 18.

Session 12 - Meet 3 Mentors: Rediet Abebe, Elenna Capote, Ruthi Hortsch, Jessica Wang
February 14 2013

Some members began thinking about permutations. For example, suppose you have n sticks, no two of which have the same length. They are arranged from shortest to longest, left to right. You want to reverse their order so that they go from longest to shortest, left to right. To do this, you are allowed to make certain moves. With each move, you can switch the positions of two adjacent sticks. What is the minimum number of moves needed to accomplish this?

Session 12 - Meet 4 Mentors: Rediet Abebe, Jordan Downey
February 28, 2013

More work on permutations, Fermat's little theorem, divisibility, and a new Community Outreach problem from Jane Kostick.

Calendar

Session 11: (all dates in 2012)

September	13	Start of the eleventh session!
	20	
	27	Charlene Morrow, Mt. Holyoke
October	4	
	11	Pardis Sabeti, Broad Institute/Harvard
	18	
	25	Anoush Najarian, MathWorks
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Session 12: (all dates in 2013)

January	31	Start of the tenth session!
February	7	
	14	
	21	No meet
	28	
March	7	
	14	Iris Ortiz, Cambridge Systematics, Inc.
	21	No meet
	28	
April	4	Crystal Fantry, Wolfram Research
	11	
	18	No meet
	25	Ashlee Ford Versypt, MIT Dept. of Chemical Eng.
May	2	Emily Riehl, Harvard University
	9	

The deciphered secret message at the end of the *Notes from the Club* of the previous issue is:
GOOD JOB, YOU WOULD HAVE BEEN ABLE TO SOLVE ONE OF THE MATH TREASURE HUNT PROBLEMS! LET US KNOW YOU SOLVED IT BY EMAILING US THE EXACT NUMBER OF POSITIVE DIVISORS OF TWO THOUSAND AND THIRTEEN. HAPPY NEW YEAR!

Here are the solutions to this issue's *Errorbusters!* problems on page 22:

1. $\frac{x+5}{x+2}$, $x \neq$ -2, 5
2. $\frac{x-3}{x-1}$, $x \neq$ 1, 4
3. $\frac{x-2}{x+3}$, $x \neq$ -5, -3, -2
4. $\frac{(x+1)(x+2)}{(x+3)(x-3)}$, $x \neq \pm 3$
5. $\frac{x-1}{x+3}$, $x \neq$ -1, -2, -3
6. $x-1$, $x \neq$ -4, -2
7. $\frac{(x-1)(x+3)}{x+4}$, $x \neq$ -4, -1, 3
8. $\frac{(x+5)(x-5)}{(x+2)(x-2)}$, $x \neq \pm 2, \pm 3$

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, custom content production including our magazine, the Girls' Angle Bulletin, and various outreach activities such as our Math Treasure Hunts and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The print version (beginning with volume 3, number 1) costs $36 for an annual subscription and brings with it access to our mentors through email. Subscribers may send us their solutions, questions, and content suggestions, and expect a response. The Bulletin targets girls roughly the age of current members. Each issue contains a variety of content at different levels of difficulty extending all the way to the very challenging indeed.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We also aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club where they work directly with our mentors exploring mathematics. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin. Remote members may email us math questions (although we won't do people's homework!), send us problem solutions for constructive comment, and suggest content for the Bulletin. To become a remote member, you can simply subscribe to the print version of the Bulletin.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership/Bulletin Subscription

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: ________________ Email: ______________________

For **membership applicants only**, please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ \$216 for a one session Membership (which includes 12 two-hour club meets)
  - ☐ \$36 for a one year Remote Membership (which includes 1-year subscription to Bulletin)
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at \$20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: _______________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: __________________________________

Girls' Angle

A Math Club for Girls