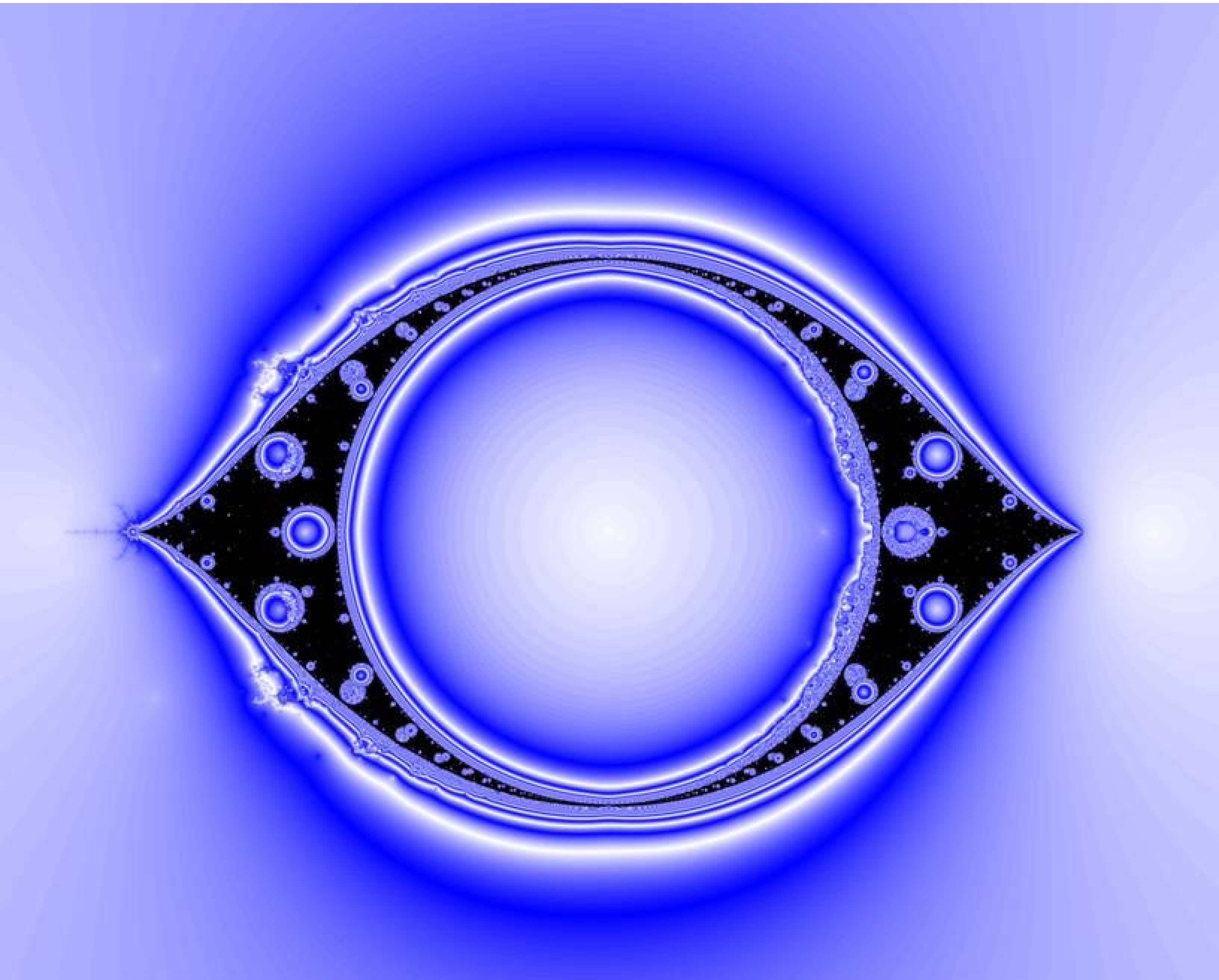
Girls' Angle Bulletin

April 2013 • Volume 6 • Number 4

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Girls' Angle tries to be a safe haven where members can explore the wondrous universe of mathematics in peace. We avoid mention of current events, unless they have compelling math educational value. But, the recent events in Boston were impossible to ignore. We are grateful to law enforcement for their bravery and professionalism and to the many citizens who displayed extraordinary acts of goodness throughout this tragic period. We extend our deepest sympathy to colleagues, friends, and family of the victims and dedicate this issue to the memory of the victims.

- Ken Fan, President and Founder

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Girls' Angle Bulletin

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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

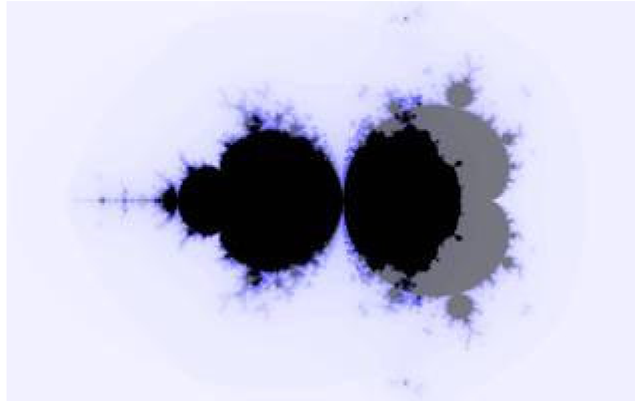
Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: A bifurcation locus created by Laura DeMarco using Dynamics Explorer Tool, a program developed by Suzanne Lynch Boyd and Brian Boyd.

An Interview with Laura DeMarco



Laura DeMarco is a professor of mathematics at the University of Illinois at Chicago. She received her doctoral degree in mathematics from Harvard University under the supervision of Prof. Curtis McMullen.

Ken: Normally, I begin interviews by asking how you became interested in mathematics. But I was looking at your website and watched an amazing video that you posted showing the "bifurcation locus" of some mathematical object. What is that? And, how do you see this movie as a mathematician?

Laura: In my field of research, there are very impressive computer tools to visualize some of the most remarkable phenomena. The pictures and movie on my webpage are illustrations of bifurcation loci for certain dynamical systems. I am most interested in studying the dynamics of rational maps – and how the dynamical concepts relate to algebraic or geometric concepts.

A dynamical system is a configuration of objects that evolves in time. For example, our solar system, consisting of the sun and 8 planets and a bunch of moons, is a dynamical system: as time passes, the planets move around the sun and the moons move around the planets. As another example, imagine a billiard ball bouncing around on a rectangular billiard table. The "configuration" of the ball at any moment in time consists of its location on the table and the direction it is rolling. As time passes, the configuration changes. Or, as an abstract mathematical example, imagine that the objects are the whole numbers (the integers – negative, zero, or positive) and the time evolution is that with each passing second, every number gets squared. So 0 never moves, because $0^2 = 0$. But $2^2 = 4$ and $4^2 = 16$ and $16^2 = 256$. The object "2" heads off to infinity as time passes.

My goal is to understand stability: what is it and when are we confident that it exists? Stability can mean many things, but here is the idea. Suppose we start with some initial configuration (e.g., the planets as they are today) and we examine what happens after 1,000,000 years and again after 1,000,000,000 years. Then suppose we perturb the initial configuration (maybe a giant meteor hits the Earth and the Earth moves a little bit off its orbit), and we look at the new outcome after 1,000,000 years and again after 1,000,000,000 years. If the new outcome is basically the same as the original outcome, we say the system is stable.

To be successful, we need true concentration, over many days, weeks, months, years even, on a single problem.

Is the solar system stable? Is the path of a billiard ball stable? Is the squaring operation on whole numbers stable?

Think about these questions. You can read Laura's answers to them at the end of this interview.

Ken: There's also a stunning image on your webpage of mostly oranges and greens. One thing that caught my eye about it is that in the outer regions, and some other places, there seems to be a deformed grid work very similar to images that appear in a recent Girls' Angle Bulletin (see Volume 5, Number 6) which showed images of conformal maps in the complex plane. Does

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For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Prof. DeMarco, Part 4 of the Stable Marriage Algorithm, and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
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The Mandelbrot Set

How is the Mandelbrot set made? Here, we explain.

The Mandelbrot set (see image a the top of page 5 and on the next page) is named after Benoit Mandelbrot. Technically, the Mandelbrot set is a subset of the complex plane, indicated by the black region in the image on page 5. We will give two descriptions of it side-by-side. They are identical descriptions, but the one on the left uses complex numbers, while the one on the right only assumes familiarity with the Cartesian plane. Can you reconcile the two descriptions?

Let c be a complex number.	Let $P = (a, b)$ be a point in the Cartesian plane.
Consider the function f_c that maps the complex number z to the complex number $z^2 + c$.	Consider the function g_P that maps the point $p = (x, y)$ to the point $(x^2 - y^2 + a, 2xy + b)$.
Consider the infinite sequence of complex numbers $0, f_c(0), f_c(f_c(0)), f_c(f_c(f_c(0))), \ldots$	Consider the infinite sequence of points $(0, 0)$, $g_P((0, 0)), g_P(g_P((0, 0))), g_P(g_P(g_P((0, 0)))), \ldots$
The point c is contained inside the Mandelbrot set if and only if this sequence is bounded (i.e., all these complex numbers are contained inside some circle in the complex plane).	The point P is contained inside the Mandelbrot set if and only if this sequence is bounded (i.e., all the points in the sequence are contained inside some circle in the plane).

In most images of the Mandelbrot set, the complex plane (or plane) is stunningly multi-colored. The coloring in these images relates to when the associated sequence (constructed above) falls outside of the circle of radius 2 centered at the origin. Here are details:

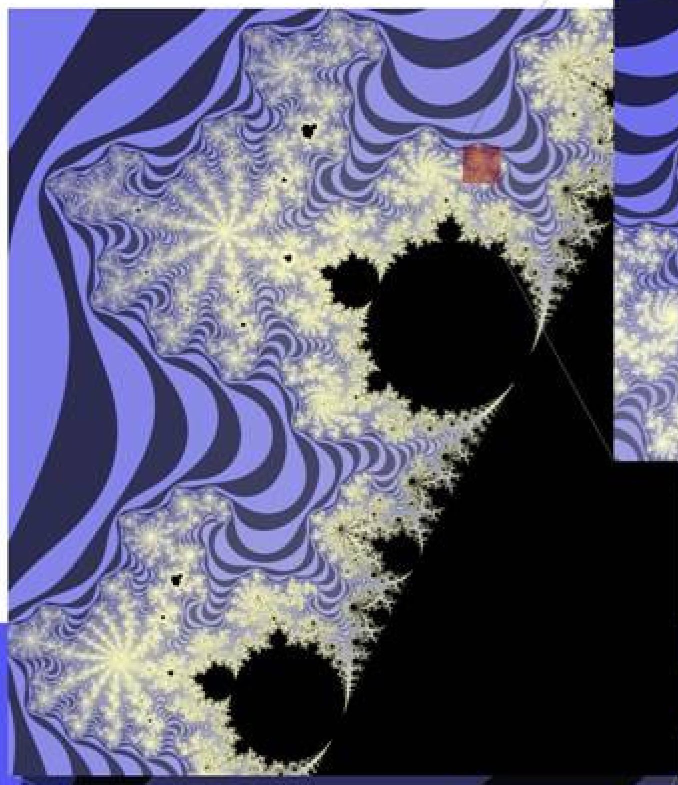
For each complex number c, let I_c be the position of the first term in the sequence $$0, f_c(0), f_c(f_c(0)), f_c(f_c(f_c(0))), \ldots$$ that falls outside the circle of radius 2, centered at 0. (Note: I_c can be infinity.)	For each point P, let I_P be the position of the first term in the sequence $$(0, 0), g_P((0, 0)), g_P(g_P((0, 0))), \ldots$$ that falls outside the circle of radius 2, centered at the origin. (Note: I_P can be infinity.)
If c is in the Mandelbrot set, i.e., if $I_c = \infty$, color it black. Otherwise, color it using a color determined by the index I_c.	If P is in the Mandelbrot set, i.e., if $I_P = \infty$, color it black. Otherwise, color it using a color determined by the index I_P.

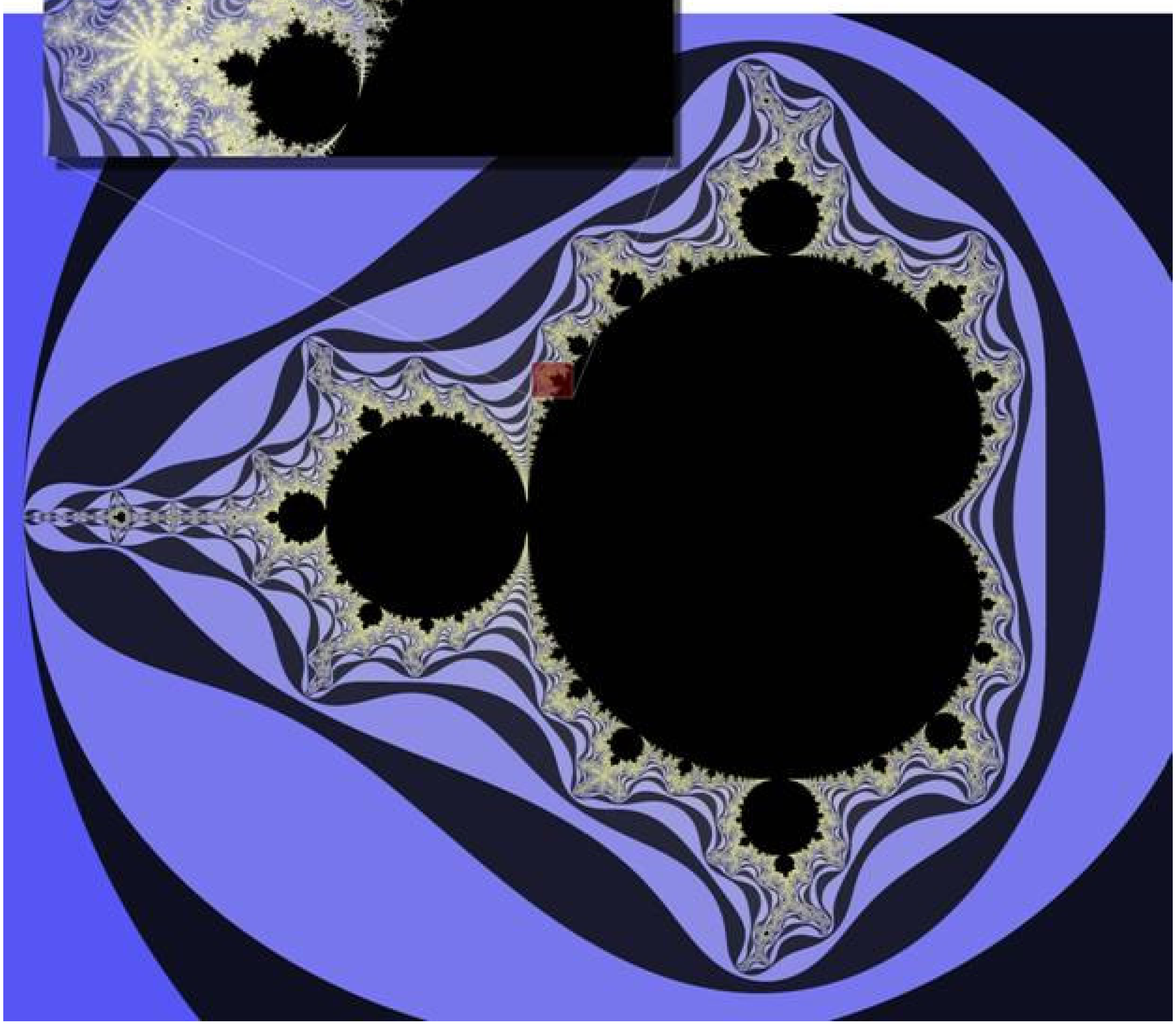
The choice of colors is entirely up to you. Traditionally, points in the Mandelbrot set are colored black, but there's no law that says that must be the way things are done. Happy coloring!

Exercise. Show that a point outside the circle of radius 2 centered at the origin will map to a point also outside this circle under f_c (or g_P, if you're thinking in terms of the plane). Thus, once the sequence leaves the circle of radius 2 centered at the origin, it never gets back inside. (Sometimes complex numbers are referred to as "points" in the complex plane.)

The Mandelbrot Set

Images created with Mathworks' MATLAB.





The Stable Marriage Problem[1]

Part 4. The National Resident Matching Program

by Emily Riehl, illustrated by Julia Zimmerman, edited by Grace Lyo

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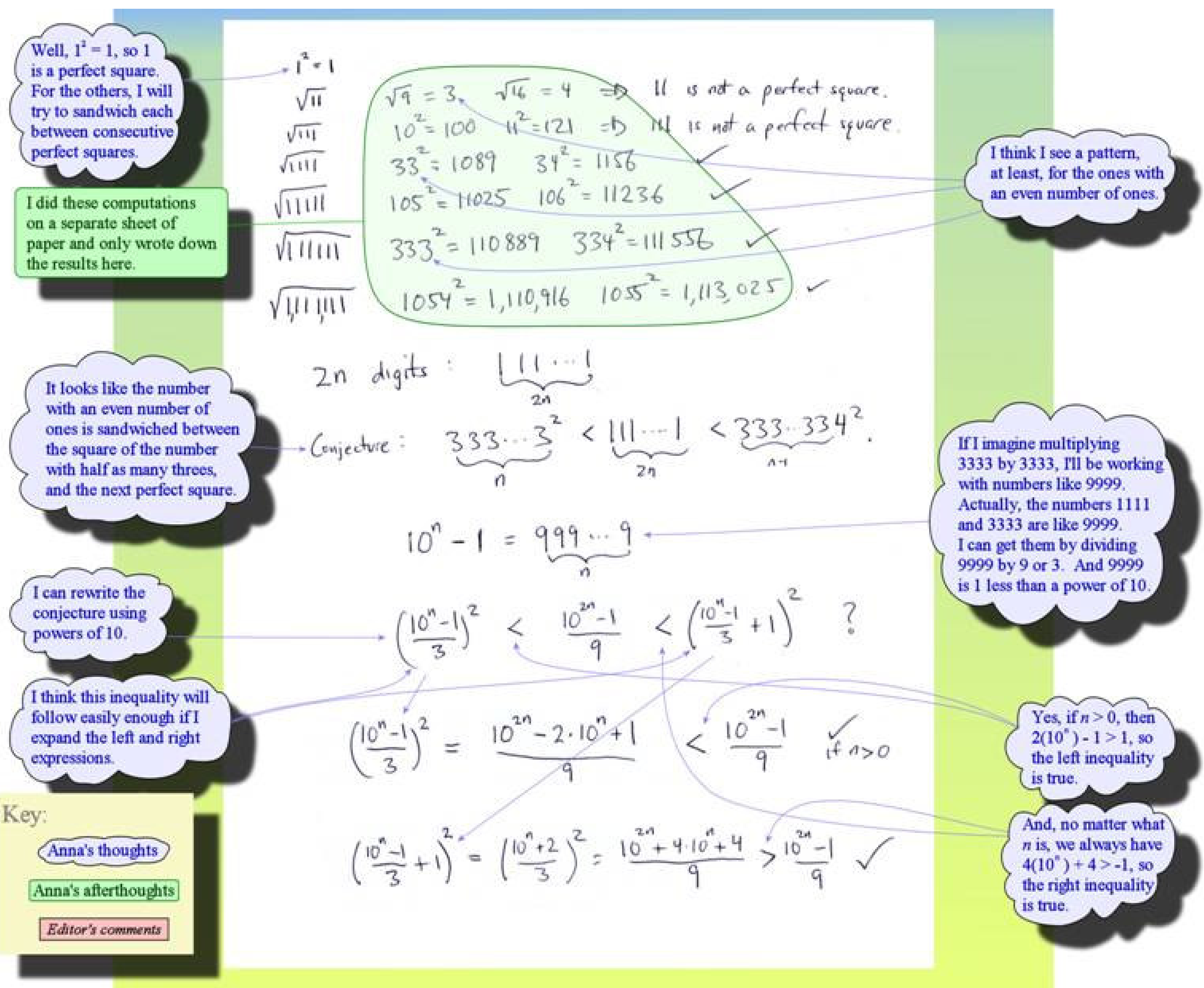
Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna is asked to show that the only number of the form 111…1 that is a perfect square is 1.



OK, now how about the case where there are an odd number of digits.

$2n+1$ digits : $\underbrace{111\cdots1}_{2n+1}$

$1^2 = 1$

$10^2 < 111 < 11^2$

$105^2 < 11111 < 106^2$

$1054^2 < 1111111 < 1055^2$

It's interesting that digits are preserved, but I don't see a pattern for each new units digit.

I don't see a pattern, so I'll try something else. I'll try to look for a pattern in perfect squares.

I'll quickly make a table of the first 10 perfect squares.

n	n^2
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

Hey, the units digit of a perfect square is not often equal to 1...

The units digit of a perfect square only depends on the units digit of its square root. Maybe the units and tens digits of a perfect square depend only on the units and tens digit of the square root. If that's true, this computation should show it...

$(10m+d)^2 = 100m^2 + 20md + d^2$

$(100m+t)^2 = 10000m^2 + 200mt + t^2$

...and it does. Here, *t* is a 2-digit number, and only t^2 contributes to the units and tens digit. So all I have to do is check to see that none of the squares of the numbers from 0 to 99 end in 11...

$(50m+t)^2 = 2500m^2 + 100mt + t^2$

...actually, I just need the coefficient of *mt* to be 100, so I could look at $50m + t$. That would reduce my work by 50%...

$(10T+D)^2 = 100T^2 + 20TD + D^2$

...I'll let $t = 10T + D$, where *T* and *D* are digits, square this, and try to figure out what values of *T* and *D* make this square end in 11.

Can $20TD + D^2$ end with 11?

$D = 1$ or 9.

From the table I made, I already know that *D* must be 1 or 9, so I'll consider these two cases.

$D = 1$: $20T + 1 \Rightarrow$ endings can be 01, 21, 41, 61, 81

$D = 9$: $180T + 81 \Rightarrow$ endings can be 81, 61, 41, 21, 01

This settles it! No square ends in 11.

Hmm. What I've done is to consider the situation modulo 100. I wonder if an argument would work with a smaller modulus.

Modulo 2 doesn't help because perfect squares can be odd. Modulo 3 also won't help because the numbers that consist of just ones can leave remainders of 0, 1, or 2 modulo 3. So, I'll check what happens modulo 4.

$\underbrace{11\cdots1}_{n} \equiv 3 \pmod 4$ when $n > 1$.

~~$(4m+K)^2$~~

n	$n^2 \pmod 4$
0	0
1	1
2	0
3	1

Modulo 4, perfect squares can only be congruent to 0 or 1...but numbers of the form 111...1, other than 1, are congruent to 3, modulo 4. So that settles it too, and much faster.

No number of the form $\underbrace{222\cdots2}_{n}$ is a perfect square (because such numbers are 2 modulo 4).

Thinking modulo 4 also makes it clear that none of the numbers of the form 222...2 are perfect squares because these numbers are all congruent to 2 modulo 4. I wonder about other numbers whose decimal expansion consists of only one type of digit.

Which numbers whose decimal representations use only a single digit are perfect squares?

And there's still the question of whether there's a pattern to the square roots of the largest perfect squares less than 111...1 in the case where there are an odd number of ones.

Does 1, 10, 105, 1054, ... belong to a sequence where a new digit is added each time?

ABB 4.27.13

Uncomfortable with divisibility issues or modular arithmetic? Check out Divisibility Rules *and Bob Donley's articles in the previous issue.*

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Fermat's Little Theorem,[1] Part 4: Finding Inverses

by Robert Donley / edited by Jennifer Silva

Robert Donley runs the YouTube channel MathDoctorBob, which has over 650 videos on close to 20 math subjects.

In our several passes at Fermat's little theorem, we have progressed from a statement about integer division and remainders to a picture with modular arithmetic, repeating patterns, and methods for solving equations. Our main result from Part 3 was

Euler's Theorem. Fix an integer $n > 1$, and choose any integer m relatively prime to n. Then $m^{\varphi(n)} = 1 \pmod{n}$.

Here $a = b \pmod{n}$ means that n divides $b - a$, in other words, that a and b leave the same remainder upon division by n. The Euler totient function $\varphi(n)$ gives the number of integers between 1 and n, inclusive, that are relatively prime to n. When n is a prime, then $\varphi(n) = n - 1$, and Euler's theorem becomes Fermat's little theorem.

We used this result to solve equations, modulo n, using the Cancellation Law:

Cancellation Law. Suppose m and $n > 1$ are relatively prime. If $mb = mc \pmod{n}$, then $b = c \pmod{n}$.

In our setting, we may obtain the Cancellation Law by application of the multiplicative inverse m^{-1} of m to both sides of the equation. A **multiplicative inverse** of m modulo n is an integer x such that $mx = 1 \pmod{n}$. If such an x exists, it is unique modulo n. To see this, if x and y are inverses to m, then $mx = my = 1 \pmod{n}$. If we multiply by x on both sides of the equation $1 = my \pmod{n}$, we obtain

$$x = x(my) = (xm)y = (1)y = y \pmod{n}.$$

In this part, we describe three methods for finding multiplicative inverses.

Method 1. Suppose $xm = 1$ (modulo n), that is, $xm - 1$ is divisible by n. In other words, there exists an integer k such that $xm - 1 = nk$, which means that $xm = nk + 1$. So if x is a multiplicative inverse to m, then x divides $nk + 1$. Conversely, if x divides $nk + 1$ for some integer k, then x is a multiplicative inverse to $(nk + 1)/x$. One approach to finding multiplicative inverses is to list the ways of writing the numbers $nk + 1$, for $k = 0, 1, 2, 3, \ldots$ as products. For instance, working modulo 7, we examine the numbers $7k + 1$, starting with $k = 0$. When $k = 0$, $7k + 1 = 1$, and $1 = 1 \cdot 1$ and $1 = -1 \cdot -1$, therefore, 1 and 1 are multiplicative inverses to each other, as are -1 and -1 (or 6 and 6, if you prefer positive integers). When $k = 1$, $7k + 1 = 8$, and $8 = \pm 1 \cdot \pm 8$ and $8 = \pm 2 \cdot \pm 4$, where the signs must be taken to be the same within each equation. We already found that 1 and 8 are multiplicative inverses (remember that $8 = 1 \pmod{7}$), as are -1 and -8. But 2 and 4 are a new pair of multiplicative inverses, as is -2 and -4 (or 5 and 3, if you prefer positive integers). This exhausts all of the remainders modulo 7 except for 0, which doesn't have a multiplicative inverse.

[1] This content supported in part by a grant from MathWorks.

Note that this method allows us to quickly see some useful facts, such as that when n is odd, then $2^{-1} = (n + 1)/2 \pmod{n}$. Also, if $n = 1 \pmod 3$, then $3^{-1} = (2n + 1)/3 \pmod{n}$, and if $n = 2 \pmod 3$, then $3^{-1} = (n + 1)/3 \pmod{n}$. For example, $3^{-1} = (2(100) + 1)/3 = 67 \pmod{100}$.

Method 2. With Euler's Theorem, we can find inverses when m and n are relatively prime. In this case, $m^{\varphi(n)} = 1 \pmod{n}$, so $m \cdot m^{\varphi(n)-1} = 1 \pmod{n}$ and $m^{-1} = m^{\varphi(n)-1} \pmod{n}$.

For example, let's find 4^{-1} modulo 9. We compute that $\varphi(9) = 6$. Therefore, the multiplicative inverse of 4, modulo 9, is $4^{\varphi(9)-1} = 4^5 = 7 \pmod 9$. In Part 3 of this series, we noted the general formula

$$\varphi(n) = n\frac{p_1 - 1}{p_1}\frac{p_2 - 1}{p_2}\frac{p_3 - 1}{p_3}\cdots\frac{p_k - 1}{p_k},$$

where the p_i are the distinct prime divisors of n. For large n, computing powers of m modulo n might become a cumbersome exercise. We recommend that the reader try to determine the inverse of 59 modulo 137 with this method now; the answer appears below.

Method 3. A third way to find an inverse utilizes the usual long division of integers. That is, if we have two positive integers m and n with $m < n$, then there exist integers q and r with $0 \le r < m$ such that $n = qm + r$. We reapply this fact to $r < m$ (in place of $m < n$) and continue. Because the remainders obtained are non-negative and strictly decrease in size, this process eventually terminates with remainder 0. We have just described

The Euclidean Algorithm. Fix positive integers $m < n$ and solve for each q_i and r_i in

$$\begin{aligned} n &= q_1 m + r_1 \\ m &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\;\;\vdots \\ r_{k-3} &= q_{k-1} r_{k-2} + r_{k-1} \\ r_{k-2} &= q_k r_{k-1} + 0. \end{aligned}$$

Then r_{k-1} is the greatest common divisor of m and n.

To ascertain why this is true, we can work up the list of equations to see that r_{k-1} divides each r_i, and thus divides both m and n. So r_{k-1} is a common divisor of m and n. Conversely, if d' is any divisor of both m and n, then we can work down the list to see that d' divides each r_i. In particular, d' divides r_{k-1}, so r_{k-1} must be the greatest common divisor of m and n.

One consequence of the Euclidean algorithm that is of particular interest to us is

Bézout's Identity for the Integers. Suppose m and n are nonzero integers. Then there exist integers x and y such that $mx + ny = \gcd(m, n)$. (Note: the integers x and y are not unique. Many pairs x and y satisfy $mx + ny = \gcd(m, n)$.)

To see this, notice that each equation we get when performing the Euclidean algorithm shows (after rearranging terms) how to express an r_i in terms of r_{i-1} and r_{i-2} (or r_2 in terms of r_1 and m in the case of the second equation, or r_1 in terms of m and n in the case of the first equation). This means we can work up the list of equations, starting with the penultimate equation $r_{k-3} = q_{k-1} r_{k-2} + r_{k-1}$, and eliminate r_i for $0 < i < k - 1$ until we're left with Bézout's identity. In more detail, we rearrange the penultimate equation to get $r_{k-1} = r_{k-3} - q_{k-1} r_{k-2}$. We then use the third to last equation, $r_{k-4} = q_{k-2} r_{k-3} + r_{k-2}$, to see that we can substitute $r_{k-4} - q_{k-2} r_{k-3}$ for r_{k-2} and obtain an expression for r_{k-1} in terms of r_{k-3} and r_{k-4}. We continue eliminating the r_i in this manner for $i < k - 1$ until we're left with the desired expression that relates m and n with r_{k-1}.

Example 1. Find the inverse of 59 modulo 137.

Since 137 is a prime number, gcd(59, 137) = 1. Using the Euclidean algorithm, we find

$$\begin{aligned} 137 &= 2(59) + 19 \\ 59 &= 3(19) + 2 \\ 19 &= 9(2) + 1 \\ 2 &= 2(1) + 0 \end{aligned}$$

Guided by Bézout's identity, we now seek x and y so that $137x + 59y = 1$. We start with the penultimate equation $19 = 9(2) + 1$, and rearrange it to get $1 = 19 - 9(2)$. We then use the second equation, $59 = 3(19) + 2$, to express 2 as $59 - 3(19)$; we substitute this expression for 2 in the equation $1 = 19 - 9(2)$, getting $1 = 19 - 9(59 - 3(19)) = 28(19) - 9(59)$. Using the first equation, we see that $19 = 137 - 2(59)$. We use this to replace the 19 in $1 = 28(19) - 9(59)$, thereby obtaining the equation $1 = 28(137 - 2(59)) - 9(59) = 28(137) - 65(59)$. This is Bézout's identity.

If we reduce this equation modulo 137, we get $-65(59) = 1$, so the inverse of 59 modulo 137 is -65 (or any number that differs from -65 by a multiple of 137, such as 72).

Example 2. Find the inverse of 21 modulo 100.

Again, we apply the Euclidean algorithm:

$$\begin{aligned} 100 &= 4(21) + 16 \\ 21 &= 1(16) + 5 \\ 16 &= 3(5) + 1 \\ 5 &= 5(1) + 0 \end{aligned}$$

Using these equations, we find Bézout's identity: $4(100) - 19(21) = 1$. Reducing this equation modulo 100, we get that the inverse of 21 modulo 100 is -19. You might want to check that Method 2 would produce the same result by computing $21^{\varphi(100) - 1}$, modulo 100.

What happens when m and n are not relatively prime? Suppose m and n share the divisor d. If m^{-1} exists modulo n, then $m^{-1}m = 1$ (mod n), i.e., there exists k such that $m^{-1}m - 1 = nk$, or $m^{-1}m - nk = 1$. But if d divides both m and n, this equation shows that d must also divide 1. Therefore, the only integers with inverses modulo n are those m relatively prime to n.

Exercise 1. Find x and y that satisfy $512x + 201y = 1$.
Exercise 2. List all integers m between 0 and 50 that are relatively prime to 50, and pair them up with their multiplicative inverses.

(Answers on page 29.)

Who’s Better?

by Lightning Factorial
edited by Jennifer Silva

Logo Design by Hana Kitasei

In the plaza in front of the Au Bon Pain in Harvard Square, the chess players concentrated on finding some way to undo their opponents’ positions. Deep in thought, they were oblivious to a young woman who sat nearby before a Scrabble board all set up, offering an open invitation to play. It wasn’t long before a man happened by and gestured interest, to which the woman answered with a motion to have a seat. I didn’t take much notice either, until I saw the first word placed on the board: ABETTER, for 74 points! My attention shifted to the unfolding Scrabble game: SIR, JAW, ZIT, 5 letters exchanged, BOINK, FEU … “feu?” I’d never heard of “feu”[1] before, but the woman didn’t challenge it. Instead, she took the lead with “RECANTS,” for 76 points! I was impressed. From there, the players kept exchanging leads, the words landing like punches in a heavyweight title bout. By the tenth round, the score was 228 to 227, and the two Scrabble players looked more focused than the chess players. Three rounds later, the man delivered a *coup de grâce* with “DOOLIES” for 69 points. The final score: 398 to 342. The woman smiled back at the man and graciously pronounced, “You’re *a better* Scrabble player than I!”

That got me thinking. Could she really make this conclusion after losing just one game to this stranger? Perhaps the man simply got lucky with the tiles. In tennis, there are plenty of “head-to-head” records that show players exchanging wins and losses. Maybe if these two played again, she would be the victor. Perhaps, I thought, the question could be analyzed mathematically. But the man laughed heartily, and I finally got the pun: “*a better*” player, ha ha!

Still, was it sensible to ask and could one answer the following question: what is the probability that the winner of a single game is the better player?

What does it mean to be “better”? To analyze the question, I decided I needed a precise definition of what it means for player A to be better than player B. It did not seem reasonable to define “better” as winning *every* game; even from my own limited experience playing Scrabble, I have won and lost games to the same person many times.

In Scrabble, the rules imply that there are finitely many possible games. So one could imagine examining the result of every possible game between the two players and recording the fraction of games, $w(A, B)$, that player A won. If this “winning fraction” were greater than ½, then I would say that player A is better than player B. There are some complications with this definition though. People and Scrabble are complex enough that a player could conceivably make different moves in the same exact situation, thereby complicating the computation of the winning fraction.

Those complications might be worth studying at some point. However, I decided to work under the assumption that given two players A and B, player A would win a definite fraction

[1] From Webster, a “feu” is a right to use land in perpetuity for a fixed annual payment.

$w(A, B)$ of the games in the long run. Another way of putting this is that player A would have a probability of $w(A, B)$ of winning any given game they played. And if $w(A, B) > \frac{1}{2}$, then I would say that, by definition, player A is better at Scrabble than player B.

Who's Better? I wanted to know the probability that player A is better than player B once player A has already won a game. But before thinking about that, I decided that it would be a good idea to answer the same question in the scenario that no game had yet been played. So, I imagined two strangers sitting down for a game of Scrabble. What's the probability that $w(A, B) > \frac{1}{2}$? In the absence of any information, it seems impossible to make any conclusion at all about who is better. After all, it's just as likely for John to be player A and Jane to be player B as it is for Jane to be player A and John to be player B, and if Jane's winning fraction over John is w, then John's winning fraction over Jane is $1 - w$. Thus, by symmetry, it is just as likely that $w < \frac{1}{2}$ as it is that $w > \frac{1}{2}$. Furthermore, it is equally likely for w to be between a and b as it is for w to be between $1 - a$ and $1 - b$.

Of course, if I knew something about the players, that could change my estimation of the various probabilities that w lies in some range. For example, if I knew that player A had won the National Scrabble Championship and player B was a foreigner currently enrolled in English as a Second Language class, I would think it very likely that $w(A, B) > \frac{1}{2}$. I might even believe there was a very good chance that $w(A, B) = 1$.

But knowing nothing at all about either player, symmetry leads me to assume that the various probabilities for w are symmetric about $\frac{1}{2}$.

Back to the original question: given that player A has won a single game against player B, what is the probability that player A is the better player?

How can this question be answered? Surely, the answer is not that player A is better than player B with 100% certainty just by virtue of player A having won a single game! After all, a player can lose to someone less skilled. For example, it's possible that by pure chance, the lesser player gets both blank tiles, all 4 "S" tiles, and is able to find good places to put the Z, Q, J, X, and K tiles. Such luck would be hard to beat!

To answer the question, I thought about probability basics. If all possible outcomes are laid out before us and each outcome is equally likely, then the probability of a desired outcome can be computed by counting. We count the number of outcomes that are desired and divide by the total number of outcomes.

What are the outcomes relevant to my question?

I imagined setting up a table with a Scrabble board and placing two seats at the table for the two players in such a way that no seat conferred a special advantage to the player in it. One seat would be marked "Player A" and the other, "Player B." I then imagined inviting huge numbers of players to play at this table. Indeed, I envisioned including every single pair of Scrabble players and having them play every single possible Scrabble game (for them). Sometimes player A wins, sometimes player A loses. Since I am interested in knowing the winning fraction for player A against his opponent, given that player A won the game, my outcomes will be all of the games won by player A. Of these, the desired outcomes are the ones in which $w(A, B)$ exceeds $\frac{1}{2}$. If I set this all up properly, I would be able to compute the answer to my question by taking the ratio of the number of desired outcomes to the total number of outcomes.

Because the winning fractions are symmetrically distributed about $\frac{1}{2}$, player A would win half of the games and would lose half of them. The outcomes that are relevant to my question are the ones in which player A wins. Therefore, the total number of outcomes that I must study is equal to half of the games played in this massive, all-encompassing tournament.

All that remains is to compute the number of these outcomes which are also desired, that is, the outcomes where w(A, B) exceeds ½. Unfortunately, we're stuck, because it isn't really practical to hold this massive, all-encompassing tournament. To make progress, we have to make a simplifying assumption about how the winning fractions are distributed. Doing so will take us further from the truth. The situation is much like when people say that the population is split 50/50 between males and females. The truth is likely to be something other than a perfect 50/50 split, though 50/50 often turns out to be a useful assumption as an approximation. But this leaves the question of what assumption we should make about the distribution of winning fractions. Our answer depends heavily upon this critical question; we'll see this as we consider two extreme cases below.

Considering extremes. Imagine a situation where, instead of Scrabble, the game being played is that of flipping a fair coin. Player A wins if heads comes up, otherwise player A loses. In this scenario, all of the analysis done so far is valid. However, since the only winning fraction possible is ½, there are no desirable outcomes: the probability that player A is a better player than player B is 0, irrespective of whether player A happens to have won a game or not.

On the other hand, suppose that the players played a silly game where the winner of the game was determined by whomever was born earlier. In this case, the only winning fractions possible are 0 and 1, because the past can't be changed. If player A wins one game against player B, then player A will always win since player A will forever be the person born earlier. So the probability that player A is a better player than player B, given that player A has won a game, is 1.

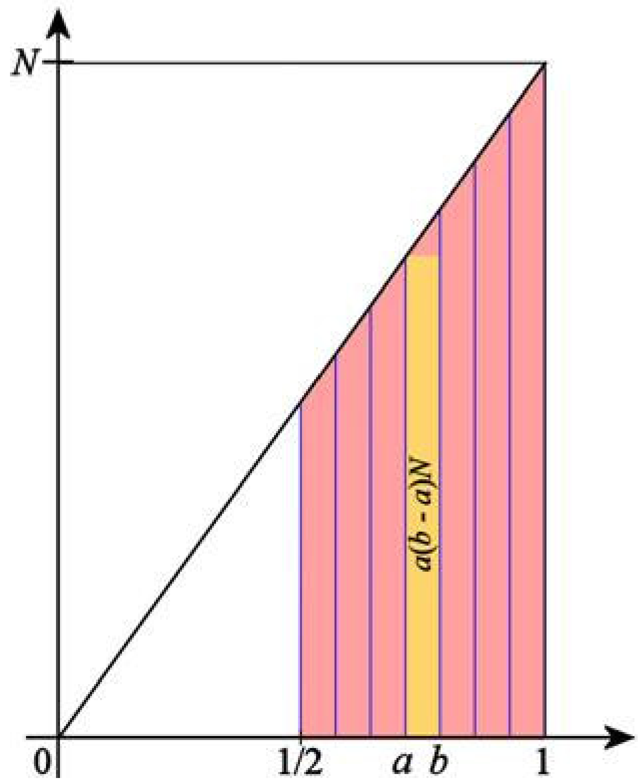
These two extreme scenarios show that the number of desired outcomes does depend on the exact distribution of winning fractions. One way to get an idea of the distribution of winning fractions is to conduct a survey. In a survey, we look at a subset of all the games in the massive, all-encompassing tournament. We try to pick this subset, which is called a "sample," so that the distribution of winning fractions in the subset accurately reflects the true distribution of winning fractions. The problem of how to pick a good sample is of extreme importance in statistics. While various methods have been invented, it is important to bear in mind that the only way to know for sure that the sample reflects the true distribution is to hold that massive, all-encompassing tournament.

Rather than conducting a survey, however, I decided to proceed by making a theoretical assumption. Doing so means that the answer I get will be divorced from the truth. But what I hope to gain by making a theoretical assumption is some insight into how the probability I'm interested in behaves, and a better sense of how these computations go. If I really needed to get a good sense of the truth, I'd have to conduct a good survey.

I decided to assume that if we pick player A from the population so that every person is equally likely to be picked, and we pick player B from the remaining population (again, with every person equally likely to be picked), player A's winning fraction over player B will be uniformly distributed between 0 and 1. What this means is that the probability that w(A, B) falls between a and b, where $0 \le a < b \le 1$, is equal to $b - a$. This assumption is definitely an oversimplification. For example, one implication of it is that the probability that player A will win every game played against player B is 0. However, in actual Scrabble play, it seems more believable that there are people who could always win against certain other people. We need to remember this limitation in our model so that we aren't misled.

Under the assumption that the winning fractions are uniformly distributed across the population of pairs of Scrabble players, we return to our plan for computing the answer. Let's suppose that in the massive, all-encompassing Scrabble tournament, N Scrabble games are played. The uniformity assumption enables us to state that the number of these games where $w(\mathrm{A}, \mathrm{B})$ is between a and b, where $0 \le a < b \le 1$, is equal to $(b - a)N$. If a is very close to b, then the fraction of these $(b - a)N$ games that end in a win for player A is approximately a.

Now we can estimate the total number of games in which player A wins and has a winning fraction greater than ½. We split up the interval from ½ to 1 into many short segments. As we just saw, for the segment with endpoints a and b, there are approximately $a(b - a)N$ games that end in a win for player A. We add up these quantities for each interval to approximate the total number of desirable outcomes. The picture at right illustrates the situation and suggests that as we shrink our intervals to get ever more accurate estimates, we end up computing the area of the trapezoidal shaded region, which is $3N/8$.



We conclude that under the simplified assumption about the distribution of winning fractions, the probability that player A is better than player B, given that player A has won a single game, is $(3N/8) / (N/2) = 3/4$.

Take it to your world. Is this the final word? Absolutely not! Remember that I made the major simplifying assumption that the distribution of winning fractions is uniform, an assumption which is most certainly false.

Consider this alternative distribution of winning fractions: assume the chances that $w(A, \mathrm{B}) = 0$, 1/3, 2/3, or 1 are all equal to ¼. Show that the probability that player A is better than player B, given that player A has won a single game against player B is equal to 5/6.

The urge to make more realistic models often leads to more sophisticated mathematics. Further study of this subject quickly steers us to calculus. If you know calculus, try this: instead of a uniform distribution, assume that the probability that $a < w(\mathrm{A}, \mathrm{B}) < b$, with $0 \le a < b \le ½$, is given by $2(b^2 - a^2)$ and that it is symmetric about ½. Show that the probability that player A is better than player B, given that player A has won a game between them, is now 2/3.

If you know the concept of the expected value, answer this: what is the expected winning fraction for player A over player B, given that player A has won a game over player B in both the uniform distribution case and the case described in the previous paragraph?

Imagine that the probability that $w(\mathrm{A}, \mathrm{B})$ is less than p is given by the function $P(p)$ and that the probability that player A is a better player than player B, given that player A has won a single game, turns out to be Q. (Also, assume that $P(0) = 0$. Note that $P(p) = 1 - P(1 - p)$.) Now suppose you decide that a better model of the distribution of winning fractions is that $(1 - y)/2$ represents both the probability that the winning fraction is 0 and the probability that the winning fraction is 1, with the probability of the winning fraction being between 0 and p given by $yP(p)$. Show that the probability that player A is better than player B, given that player A beats player B in a single game, changes to $1 - y(1 - Q)$.

What do you think the distribution of winning fractions is in tic-tac-toe? Rock, Paper, Scissors? Tennis? Chess? Or, in your favorite 2 person game?

COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 10

There's 3/7, who's in rare form. She is walking with her hands and feet pointing out to the sides. Her hat, instead of displaying the usual fruit, flowers, and stuffed bird on it, is topped with a football-shaped loop. Her outfit, which normally boasts a 3 above her black belt and a 7 below it, has a 7 on her top and a 3 on her bottoms. Something is definitely going on.

$\frac{\mathbf{3}}{\mathbf{7}}$: Hello, my dear.

You: Why are you walking like that?

$\frac{\mathbf{3}}{\mathbf{7}}$: Just what I've been waiting for someone to ask! Why do you think?

You: You remind me of pictures of ancient Egyptian art I saw at a museum, but I have no idea why.

$\frac{\mathbf{3}}{\mathbf{7}}$: You got it, girly! I am walking like the ancient Egyptians, or at least the way their art would walk if it could.

You: Have you changed your name to 7/3?

$\frac{\mathbf{3}}{\mathbf{7}}$: Oh no, I'd never do that! See the symbol on my hat? That's what the Egyptians used for reciprocal. They put the symbol on top of numbers, so I put it on my hat.

You: How are people supposed to know all that?

$\frac{\mathbf{3}}{\mathbf{7}}$: Do you know it?

You: Now I do.

$\frac{\mathbf{3}}{\mathbf{7}}$: There you have it. I am on a one-woman mission to spread the word about Egyptian fractions.

You: So if I go to Egypt, this is how they handle fractions?

$\frac{\mathbf{3}}{\mathbf{7}}$: Not now, they don't – they use the same notation we do. When I say Egyptian in this context I am referring to ancient Egypt; it's my favorite ancient civilization since it was the first one known to use fractions.

Today I am also an anachronism since I am using the ancient symbol for reciprocal together with our modern fraction notation of a bar with a counting number above – my belt does make a lovely bar, no? – and one below.

You: What did they use instead of the fraction bar?

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Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Prof. DeMarco, Part 4 of the Stable Marriage Algorithm, and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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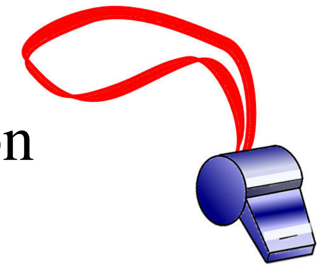

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 12 - Meet 5 Mentors: Elenna Capote, Jordan Downey, Ruthi Hortsch
March 7, 2013

We began an ongoing station about how to get unstuck when stuck trying to solve a math problem. At this particular meet, we discussed the use of related questions and explored different ways to generate them.

Suppose you are trying to answer Question A, and you get stuck. So you try to ask a new question, Question B, to get unstuck. One surefire way to establish that Question B relates to Question A is to check that the ability to answer Question A implies the ability to answer Question B.

For example, suppose you are trying to answer the question, "How many ways are there to tile an N by M rectangle with 1 by 1 squares and 1 by 2 dominos?" If you get stuck, a related question you could ask is, "How many ways are there to tile a 1 by M rectangle with 1 by 1 squares and 1 by 2 dominos?" You might find it easier to start working on the latter question because a 1 by M rectangle is a more constrained situation. And, if you could answer the first question, you would be able to answer the second.

Session 12 - Meet 6 Mentors: Jordan Downey, Isabel Vogt
March 14, 2013
Visitor: Iris Ortiz, Cambridge Systematics, Inc.

Iris Ortiz was born in Puerto Rico. In 2nd grade, she was a bit intimidated by math when she saw 3rd graders doing multiplication. But in 7th grade, her English teacher suggested that she think about a career in engineering, saying that she was good at math. Eventually, she would major in Civil Engineering and, later, receive a masters degree in Transportation from MIT.

Today, she works as a consultant at Cambridge Systematics and for the past 5 years she has been working in transportation finance. The question she studies is: How do you raise revenues for transportation projects such as the building of bridges or tunnels, or for street improvement?

In Massachusetts, a major source of revenue for transportation projects comes from commuters who use the Massachusetts Turnpike. This money is collected through a system of toll booths. The revenue generated by a single toll booth is related to the toll fares charged and the number of commuters who pass through. However, the number of commuters is affected by the weather, holidays, special events, and unusual events. For instance, Iris described a situation where road usage was reduced because a freshly-painted tunnel was giving off strong fumes so unpleasant that toll operators decided to let commuters pass through without paying so they didn't have to stop and roll down their windows. There's also feedback between the toll fare and the number of commuters: the more you charge, the fewer people will use the road. In some places, officials take advantage of this relationship to control traffic congestion by deliberately varying the toll fare.

Now, consider that the Massachusetts Turnpike doesn't have just a single toll booth. It has many, and these are divided into 6 different regions: west of 128/95, from 128/95 to the city, in the city, the Sumner tunnel, the Callahan Tunnel, and the Tobin bridge. Iris explained mathematical tools for collecting data and managing such a complex system. How much money is involved with toll collection on the Massachusetts Turnpike? In one recent year, there were approximately 194,000,000 transactions generating about $282,000,000 in revenues.

If you are comfortable with calculus, here is a toll booth inspired question: It is discovered that for every d dollar increase in a toll fare, the number of people who will use the road decreases by a factor of f, where $0 < f < 1$. In other words, if N people used the road when the toll fare cost C dollars, then fN people will use the road if the toll fare is raised to $C + d$ dollars. What fare optimizes revenues? See page 29 for the answer.

Session 12 - Meet 7 Mentors: Isabel Vogt
March 28, 2013

To learn math well, you must learn actively. It's just like tennis. You cannot learn to play tennis by reading or watching others play. You have to go out on the court and hit the ball yourself. When you read a math book, always have pencil and paper on hand and work out the solutions to problems.

To underscore this point, we involved members in an experiment. Members were split into two groups. Both groups were given written instructions that explained how to tie a fisherman's knot, which is an elegant way to tie two strings together. Everyone was told to study the instructions as long as needed until each could confidently state, "I believe I can tie a fisherman's knot without these instructions." The individuals in one group were told to do this by reading the instructions only. Those in the other group were also given rope and told to actively practice tying the knot.

After everyone said that they believed they could tie the fisherman's knot from memory, we took a 5 minute social break.

After the break, everybody was given rope and asked to make a fisherman's knot.

Within 2 minutes, everybody in the "active learning" group, i.e., the group that had rope, succeeded in making a fisherman's knot. After ten minutes, a couple of girls in the "passive learning" group managed to tie the knot, but they were the only ones from that group that were able to. Hopefully, the point was made.

Session 12 - Meet 8 Mentors: Elenna Capote, Jordan Downey
April 4, 2013
Visitor: Crystal Fantry, Wolfram Research

Crystal Fantry was born in Pennsylvania to parents neither of whom do math. But when she was in 3rd grade, her idea of a fun vacation was solving logic puzzles at the beach. Eventually, she earned a master's degree in mathematics from Boston College and began working for Wolfram Research where she is the Senior Education Content Director.

Among many other projects, Crystal runs the Mathematica Summer Camp, a program where participants learn to program in Mathematica. Open to 11th and 12th graders, each participant completes their own application. Crystal gave us a glimpse of the power of Mathematica by showing us a few of the applications that students wrote at this camp.

She then demonstrated Wolfram Alpha, a web-based question answering service. For example, **Molly** gave Wolfram Alpha the mathematical expression $x^3 + 8/x^3 + 21$. In the blink of an eye, Wolfram Alpha produced a webpage containing the graph of the equation at two different scales, a common, equivalent form of the expression, exact formulas for the real and complex roots, including a plot of these roots on the complex plain, the maximal subset of the real numbers for which the expression makes sense and the associated range, the derivative of the expression, with respect to *x*, the indefinite integral, with respect to *x*, the locations of its local minimum and maximum, and two series representations.

Crystal concluded by having the girls build a paper model of the Wolfram Alpha logo, which is a rhombic hexecontahedron. The rhombic hexecontahedron is a polyhedron with 62 vertices, 120 edges, and 60 faces all congruent to a **golden rhombus**, which is a rhombus where the ratio of the lengths of its two diagonals equals the golden ratio $(1+\sqrt{5})/2$. Can you determine the surface area and volume of a rhombic hexecontahedron with edge length *s*?

Session 12 - Meet 9 Mentors: Isabel Vogt, Luyi Zhang
April 11, 2013

To further reinforce the difference between active and passive learning, members worked on mastering an algorithm for computing square roots.

Session 12 - Meet 10 Mentors: Melody Chan, Jordan Downey, Luyi Zhang
April 25, 2013
Visitor: Ashlee Ford-Versypt, MIT Dept. of Chemical Engineering

Ashlee Ford-Versypt was born in Snyder, Oklahoma, population 1300. Her theme was "Change," for change aptly describes her own upbringing, the type of mathematics she uses in her career, and the application that she applies this math to.

She attended a high school with fewer than 50 graduates each year and no advanced math or science classes. At the time, she had no idea that a decade later, she would be a postdoctoral researcher in the MIT Department of Chemical Engineering. Interestingly, she found her first college math class difficult, but she figured out how to seek help and ask questions, and ended up earning a minor in mathematics. She attended graduate school at the University of Illinois where she got an MS and a Ph.D. in chemical engineering with a minor in computer science and applied math. She credits special opportunities throughout her upbringing for helping her become an engineer. For example, she took a special biochemistry class during one summer, she wrote a website from scratch, and, through the Girl Scouts, she attended an inspirational speech delivered by former NASA astronaut Sally Ride.

She then gave a brief introduction to the mathematics of change, i.e., calculus. She explained the derivative and the integral and had members approximate the derivative of $f(x)=\sqrt{x}$ at $x = 1/4$ and its integral from $x = 0$ to $x = 1$.

She explained that in her work, she must solve equations that involve functions and their derivatives. These **differential equations** are used to model many physical phenomena, such as diffusion and heat flow. Her specific application is in biotechnology where she models the release of medicine from special pills. Such pills are useful if a patient needs medicine released into their system in a controlled manner, rather than getting lots of medicine but only each time they swallow a pill. With controlled release, the medicine can be gradually introduced over time ensuring that the amount of medicine in the system remains between effective limits. However, to accomplish this kind of controlled release, it is necessary to use mathematics to model the release and find solutions to these mathematical models.

Calendar

Session 12: (all dates in 2013)

January	31	Start of the twelfth session!
February	7	
	14	
	21	No meet
	28	
March	7	
	14	Iris Ortiz, Cambridge Systematics, Inc.
	21	No meet
	28	
April	4	Crystal Fantry, Wolfram Research
	11	
	18	No meet
	25	Ashlee Ford Versypt, MIT Dept. of Chemical Eng.
May	2	Emily Riehl, Harvard University
	9	

Session 13: (all dates in 2014)

September	12	Start of the thirteenth session!
	19	No meet
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	
	12	

Answers to the exercises at the end of Fermat's Little Theorem, Part 4, page 17: Exercise 1. There are many solutions. One can take $x = 53 + 201k$ and $y = -135 - 512k$ for any integer k. Exercise 2: There are $\varphi(50) = 20$ numbers between 0 and 50 that are relatively prime to 50 and they pair up as follows:

a	1	3	7	9	11	13	19	21	23	33	49
a^{-1}	1	17	43	39	41	27	29	31	37	47	49

The answer to the toll fare question on page 27 is that revenue is optimized when the toll fare is set to $-\dfrac{d}{\log f}$ dollars, where log represents the natural logarithm.

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, custom content production including our magazine, the Girls' Angle Bulletin, and various outreach activities such as our Math Treasure Hunts and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The print version (beginning with volume 3, number 1) costs $36 for an annual subscription and brings with it access to our mentors through email. Subscribers may send us their solutions, questions, and content suggestions, and expect a response. The Bulletin targets girls roughly the age of current members. Each issue contains a variety of content at different levels of difficulty extending all the way to the very challenging indeed.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We also aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club where they work directly with our mentors exploring mathematics. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin. Remote members may email us math questions (although we won't do people's homework!), send us problem solutions for constructive comment, and suggest content for the Bulletin. To become a remote member, you can simply subscribe to the print version of the Bulletin.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership/Bulletin Subscription

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

For **membership applicants only**, please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership (which includes 12 two-hour club meets)
  - ☐ $36 for a one year Remote Membership (which includes 1-year subscription to Bulletin)
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: __________________________________

Girls' Angle

A Math Club for Girls