Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

From the Founder

I encourage all members and subscribers to send in questions and solutions to the Summer Fun problem sets or anything else in this Bulletin or about math in general! To learn math, you've got to do math. If you do send us stuff, here's what we'll do. We'll read it, think about it, and write back with the aim of helping you get better at mathematics. Even if you know you've got something wrong, that's okay. We'll try to help you fix it. And we'll keep things a secret…unless you want to share your insights. Take advantage of this service. Don't be shy!

- Ken Fan, President and Founder

Girls' Angle Bulletin

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Stereo pair image of Boston from Robbins Farm Park, Arlington. Cross your eyes to get a 3D illusion. For stereo pair images of the Platonic solids, see Volume 3, Number 4. Image by Ken Fan.

An Interview with Kirsten Wickelgren

> *…one good way to understand something, and know that you have understood it, is to use it.*

Kirsten Wickelgren is a fellow at Harvard University and holds an American Institute of Mathematics Five-Year Fellowship. She was a graduate student of Gunnar Carlsson at Stanford University and will be joining the faculty at Georgia Tech University in the fall.

Ken: Hi Kirsten, Thank you so much for doing this interview! I'll start by asking, what do you enjoy most about being a mathematician?

Kirsten: I have two favorites I think: the moment when you solve a problem, and hearing a compelling story about a powerful mathematical object. For example, I've always been intrigued by unique factorization. It lets you conclude that $11 \times 13 \times 19$ does not equal $7 \times 17 \times 23$ without doing any multiplying at all. Later, in college, I learned about other number systems, like $\mathbf{Z}[i]$[1], which are the complex numbers that have integer real and imaginary parts. So naturally I wondered whether there was unique factorization in these other number systems too. It turns out, often there isn't. For example, consider the set of complex numbers of the form $a+b\sqrt{-5}$, where a and b are integers. In this set of numbers, 6 can be factored in two different ways: 2×3 and $1+\sqrt{-5} \times 1-\sqrt{-5}$. Then, even later, I learned about a marvelous object called the "Picard group" which beautifully measures just how much factorization in these number systems falls short of unique factorization. Math is filled with stories like this.

Ken: What are your goals, as a mathematician?

Kirsten: I would like to understand to what extent solutions of polynomial equations are controlled by topological invariants or their analogues. For example, Gerd Faltings proved that there are only finitely many solutions in the rational numbers to polynomial equations if the solutions over $\boldsymbol{C}$ [the complex numbers] are a genus g curve with $g \geq 2$. Here, "curve" means complex dimension 1, so that's real dimension 2, and the genus is the number of holes. Such a curve looks like a donut with 2 or more holes.

Ken: What sparked your own interest in mathematics?

Kirsten: I think it was the fact that you could prove things. This isn't what I enjoy most now, but I was very impressed by this at first. I thought it meant that mathematics is true in a way other things can not be. I still find it amazing how robust math is. No matter how many ways you solve a problem, if you don't make a mistake, you get the same answer. In "The Brothers Karamazov," Dostoyevsky describes psychology as a "knife that cuts both ways" because you can prove both an assertion and its opposite with the same tools. Math is exempt from this criticism.

Ken: Can you describe one of the first mathematical concepts that got you excited about mathematics?

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[1] For more on $Z[i]$ and unique factorization, see the Summer Fun problem set on page 25.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Dr. Wickelgren, and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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Notation

by Ken Fan

Through time, ways to notate mathematical ideas have become standard. Learning the notational conventions is important for communication. The best way to learn new notation is the same way we learned the alphabet: repetition. Practice makes perfect!

Sigma Notation

Each day, a stuffed animal exchange acquires new stuffed animal dolphins. We can define a function $D(n)$ to be the number of dolphins acquired on day n. Here's a table of the values of $D(n)$ for the first ten days:

n	1	2	3	4	5	6	7	8	9	10
$D(n)$	7	9	4	5	8	4	6	6	51	2

Day 9 was a good day for dolphins at the stuffed animal exchange! How many dolphins were acquired in total over the first 20 days? Unfortunately, the table above doesn't give us data for days 11 through 20. But we can still write down a formula for this number. All we have to do is sum up the values of $D(n)$ for each value of n from 1 through 20:

$$D(1) + D(2) + D(3) + D(4) + D(5) + D(6) + D(7) + D(8) + D(9) + D(10) + D(11) + D(12) + D(13) + D(14) + D(15) + D(16) + D(17) + D(18) + D(19) + D(20).$$

That's a long formula! Would you want to write down the formula for the number of dolphins this company acquired in its first year of operation? It wouldn't be pleasant to do so. Even in words it feels long to write, though one can imagine that the formula could be of great interest.

And so a new notation is born!

Sigma notation was invented so that such sums could be written down much more efficiently. Here's how the long sum written out in full above can be written using sigma notation:

The top number gives the last value of n in the sum.

The "summation symbol" Σ is the Greek capital letter "sigma." It tells us to sum up instances of the expression that follows the symbol.

$$\sum_{n=1}^{20} D(n)$$

This is the expression that will be summed. It depends on n, and the information around the Σ symbol tells us to substitute the values 1 through 20 for n to get 20 different numbers (namely, $D(1)$, $D(2)$, $D(3)$, …, $D(19)$, and $D(20)$) and add these numbers up.

The information below the Σ tells us which variable we are summing over and its first value. The variable summed over, n in this case, is called the **index of summation**.

Sigma Notation

To gain facility with sigma notation, study the following examples and do the exercises.

Here's the sum of the first 10 (positive) perfect squares: $\sum_{k=1}^{10} k^2$. Written out in full:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2.$$

Here's the sum of the odd numbers between 101 and 125, inclusive: $\sum_{m=50}^{62} (2m+1)$, which is short for: 101 + 103 + 105 + 107 + 109 + 111 + 113 + 115 + 117 + 119 + 121 + 123 + 125.

The above sum can also be written $\sum_{m=1}^{13} (2m+99)$. Be careful not to write it like this: $\sum_{m=50}^{62} 2m+1$. Technically, the summation symbol applies only to the expression immediately following it, so $\sum_{m=50}^{62} 2m+1$ actually stands for the sum:

$$(100 + 102 + 104 + 106 + 108 + 110 + 112 + 114 + 116 + 118 + 120 + 122 + 124) + 1.$$

Exercise 1. Use sigma notation to write down the sum of the even numbers between 1 and 101.

Exercise 2. Write out $\sum_{k=1}^{10} \frac{1}{k(k+1)}$ as a sum without using sigma notation.

Exercise 3. Use sigma notation to write down the sum of an n term geometric series with first term a and common ratio r.

(See page 29 for answers to the above exercises.)

Here's the binomial theorem written using sigma notation: $(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$.

Sometimes, instead of placing indexing information above and below the Σ symbol, this information is provided by a condition, or a list of conditions, written below the Σ symbol:

$$\sum_{4<p<10} \log p, \text{ which stands for } \log 5 + \log 6 + \log 7 + \log 8 + \log 9.$$

Here's another example: $\sum_{\substack{100 \le n \le 125 \\ n \text{ odd}}} n$, which is another way to write $\sum_{m=50}^{62} (2m+1)$. For another example, take a look at how Lola Thompson uses it in her article, *An Euler φ-For-All* on page 8.

Exercise 4. Convince yourself that $a\sum_{m=1}^{n} f(m) = \sum_{m=1}^{n} af(m)$, where a is constant.

Exercise 5. Show that $\sum_{a=1}^{M}\sum_{b=1}^{N} ab = \left(\sum_{a=1}^{M} a\right)\left(\sum_{b=1}^{N} b\right)$. (This can be done by using exercise 3 twice.)

Remember: If you find yourself confused by sigma notation, you can always write out the sum in full.

An Euler φ-For-All[1]

by Lola Thompson

What is φ?

Primes are the atoms of the number universe: every whole number greater than 1 can be factored uniquely into a product of primes. When two numbers share no common prime factors, we say that they are **relatively prime**. Don't let the terminology fool you. Being "relatively prime" is not like being "relatively smart" or "relatively popular," wherein the subject is "smart" or "popular" compared with her peers. Relatively prime numbers are not necessarily any closer to being prime than other numbers. For example, 4 and 15 are relatively prime and they are both composite. As a silly example, we see that 1 is relatively prime to all other positive whole numbers, since 1 doesn't have any prime factors to share!

Rather than looking at individual pairs of numbers and asking whether they are relatively prime to one another, one could instead take a specific number like 8 and ask, "How many numbers between 1 and 8 are relatively prime to 8?" This question can be generalized by using n to represent *any* positive integer greater than 1, and asking, "How many numbers between 1 and n are relatively prime to n?" In fact, we have a special name for this counting function: we define $\varphi(n)$ to be the number of integers between 1 and n that are relatively prime to n. We call $\varphi(n)$ the **Euler totient function** (the symbol φ is the Greek letter "phi" and the word "totient" rhymes with "quotient"). To answer the question posed above, we observe that 1, 3, 5, and 7 are all relatively prime to 8. However, 2, 4, 6, and 8 all share a common factor with 8 because they're all divisible by 2. As a result, we can conclude that $\varphi(8) = 4$. On the other hand, if p is *any* prime number then $\varphi(p) = p - 1$, since 1, 2, 3, …, $p - 1$ are all relatively prime to p (remember, primes are only divisible by 1 and themselves, and 1 is relatively prime to everything).

The Euler totient function $\varphi(n)$ is defined as the number of integers between 1 and n that are relatively prime to n.

The Arithmetic of φ

The Euler totient function has a number of neat properties. For example, we can show that if n and m are any two relatively prime numbers then $\varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$. However, this property does not hold when n and m share common factors. For example, $\varphi(6) = 2$ and $\varphi(4) = 2$, so $\varphi(6) \cdot \varphi(4) = 2 \cdot 2 = 4$. Contrast this with the fact that $6 \cdot 4 = 24$ and $\varphi(24) = 8$. In short, while 6 and 4 may be lucky to have something in common, they don't play so nicely with φ.

We will illustrate a general method for proving that $\varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$ holds for any relatively prime numbers n and m by looking at the case $n = 7$ and $m = 6$. If $n = 7$ and $m = 6$, we write the numbers between 1 and 42 (= $n \cdot m$) in the chart at right.

1	7	13	19	25	31	37
2	8	14	20	26	32	38
3	9	15	21	27	33	39
4	10	16	22	28	34	40
5	11	17	23	29	35	41
6	12	18	24	30	36	42

[1] This content was supported in part by a grant from MathWorks.

To compute $\varphi(42)$, we need to determine how many entries in the chart are relatively prime to 42. This is not so difficult because 42 is a fairly small number; in theory, we could just go through the numbers one by one and cross off any number that is not a multiple of 2, 3, or 7. However, if we had chosen larger numbers for n and m, this would not be a fun task. Imagine trying to compute $\varphi(n \cdot m)$ in this way when n and m are both in the thousands — it would probably take an entire afternoon! Fortunately, we have organized our chart in a way that makes it easier to get rid of the entries that are not relatively prime to 42. Namely, the entries in a given row all have the same remainder when divided by 6. As a result, if an entry in the first column is not relatively prime to 6, then all of the other entries in the same row will also fail to be relatively prime to 6. So, we can immediately eliminate all of the entries in several of the rows.

1	7	13	19	25	31	37
~~2~~	~~8~~	~~14~~	~~20~~	~~26~~	~~32~~	~~38~~
~~3~~	~~9~~	~~15~~	~~21~~	~~27~~	~~33~~	~~39~~
~~4~~	~~10~~	~~16~~	~~22~~	~~28~~	~~34~~	~~40~~
5	11	17	23	29	35	41
~~6~~	~~12~~	~~18~~	~~24~~	~~30~~	~~36~~	~~42~~

Next, we examine the remaining rows. The entries of each row are of the form i, $i + 6$, $i + 2 \cdot 6$, $i + 3 \cdot 6$, $i + 4 \cdot 6$, $i + 5 \cdot 6$, and $i + 6 \cdot 6$ (where $i = 1$ or 5). Since i and 6 are necessarily relatively prime (because we eliminated the rows where i and 6 share a common factor!) then all 7 entries in each row are relatively prime to 6. Moreover, if we divide each of the entries in a given row by 7, we will get all of the numbers between 0 and 6 as remainders. So, exactly $\varphi(7) = 6$ of these entries will be relatively prime to 7. Since these 6 entries are relatively prime to both 6 and 7, they will be relatively prime to 42.

We can summarize what we have done in more general terms. We start by writing the numbers 1 through $n \cdot m$ in a rectangular array that has n columns and m rows. We place the 1 in the upper left and then place the numbers sequentially top to bottom then left to right. We have argued that exactly $\varphi(m)$ rows in the chart contain numbers that are relatively prime to $n \cdot m$. Each of these $\varphi(m)$ rows contains exactly $\varphi(n)$ numbers that are relatively prime to n. So, there are $\varphi(n) \cdot \varphi(m)$ numbers in the chart that are relatively prime to $n \cdot m$. But, by the very definition of Euler's totient function, there are $\varphi(n \cdot m)$ integers relatively prime to $n \cdot m$ that are between 1 and $n \cdot m$ (which is exactly what this chart was designed to count in the first place!). So, we have shown that $\varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$.

Recall that if D is relatively prime to N, then the remainders of the numbers

$$M, M + D, M + 2D, \ldots, M + (N - 1)D$$

upon division by N produce a complete set of remainders (where M is any integer). To see this, pick two of the numbers in the list: $M + KD$ and $M + JD$, where $0 \le K < J < N$. Suppose they both leave the same remainder upon division by N. Then N divides their difference, i.e., N divides $(M + JD) - (M + KD)$ or, simplifying, $N \mid (J - K)D$. Since D and N are relatively prime, N must divide evenly into $J - K$. But $0 < J - K < N$, a contradiction. Therefore, the remainders of the numbers are all distinct. Since there are N numbers in the list and N total possible remainders, we must get them all.

The Arithmetic of φ, Part II

We know how to multiply together totients of different numbers (provided that they are relatively prime to one another), but what if we want to add them instead? It turns out that, if we add the totients of just the right numbers, we will discover another neat property of Euler's totient function. First, we will need some new notation. We write $d \mid n$ as shorthand for "d divides n." In other words, $d \mid n$ means that, if we compute n/d, we will get a whole number answer (without a remainder).

One more piece of notation[2] that will be useful to us is

$$\sum_{d|n} \varphi(d),$$

which tells us that we are summing $\varphi(d)$ for each value of d that divides n. For example,

$$\sum_{d|4} \varphi(d) = \varphi(1) + \varphi(2) + \varphi(4),$$

since the divisors of 4 are precisely 1, 2, and 4. Our "neat" property about summing totients can now be stated in the following manner:

$$\sum_{d|n} \varphi(d) = n.$$

In order to show why this is true, we will give an argument in the case where $n = 12$. However, the same argument will work when 12 is replaced with any whole number n.

If $n = 12$, we observe that the (positive) divisors of n are 1, 2, 3, 4, 6, and 12. So, using the notation from above, our goal is to show that

$$\sum_{d|12} \varphi(d) = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) = 12.$$

Of course, we could just compute $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, $\varphi(4)$, $\varphi(6)$, and $\varphi(12)$ and add them together. That said, since we want to write a proof that generalizes to any value of n, we will give a different argument (one that doesn't rely on knowing the specific φ-values!). The first step is to write down all of the fractions with denominator 12 and numerator between 1 and 12:

$$\frac{1}{12} \quad \frac{2}{12} \quad \frac{3}{12} \quad \frac{4}{12} \quad \frac{5}{12} \quad \frac{6}{12} \quad \frac{7}{12} \quad \frac{8}{12} \quad \frac{9}{12} \quad \frac{10}{12} \quad \frac{11}{12} \quad \frac{12}{12}$$

Next, we will go through the list of fractions and rewrite each of them in "lowest terms":

$$\frac{1}{12} \quad \frac{1}{6} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{5}{12} \quad \frac{1}{2} \quad \frac{7}{12} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{5}{6} \quad \frac{11}{12} \quad \frac{1}{1}$$

[2] See Notation Station on page 6 for more information on sigma notation.

Now, we will go through and count the number of fractions with a given denominator, compiling our data into the following list:

d	# of fractions with denominator d
1	1
2	1
3	2
4	2
6	2
12	4

Notice that there are always $\varphi(d)$ fractions with denominator d (for d = 1, 2, 3, 4, 6, and 12) and that the total number of fractions in our list is (still) equal to 12. This pattern will always hold, regardless of the number that we choose for n. As an exercise, try to think about why this argument works for *any* choice of n. (Hint: It has to do with the fact that we are making the numerator and denominator relatively prime when we rewrite each of the fractions in lowest terms.)

The History (and Future) of φ

The Euler totient function dates all the way back to 1760, when Leonhard Euler unveiled it to the world. However, it wasn't until forty years later when Gauss wrote his famous *Disquisitiones Arithmeticae* that the modern-day "φ" notation started to be used. In spite of the fact that mathematicians have studied Euler's totient function for over 250 years, there is still a lot that we do not know about this (somewhat mysterious) function. For example, D. H. Lehmer posed the question, "Are there any composite numbers n such that $\varphi(n) \mid (n - 1)$?" This question has not yet been answered, in spite of the best efforts of many experts in the field.

Another unsolved problem about Euler's totient function is called Carmichael's conjecture, which says that if there is a number n for which $\varphi(n) = m$ then there is at least one other number (call it n') with $\varphi(n') = m$. In other words, if we were to list the totients of all the positive numbers, every totient would appear on our list at least twice. Carmichael's conjecture has been checked for all "small" values of n. Here, "small" is a relative term — with the help of computers, mathematicians have shown that any counterexample to Carmichael's conjecture must be at least $10^{10^{10}}$! However, since there are infinitely many positive whole numbers, we are still infinitely far away from finding a solution. The good news is that there are still plenty of interesting questions about the φ-function that are waiting to be answered. Perhaps some of you will answer them one day!

n	$\varphi(n)$
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4
11	10
12	4
13	12
14	6
15	8
16	8
17	16
18	6
19	18
20	8
21	12
22	10
23	22
24	8
25	20
26	12
27	18
28	12
29	28
30	8

Table of the first 30 φ values.

Eyeballing the Distance

by Ken Fan
edited by Jennifer Silva

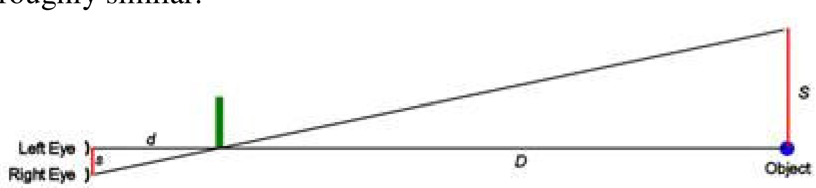
Logo Design by Hana Kitasei

Have you ever looked out the window of an airplane and wondered how far away the buildings you saw were? Judging distance without any method can be difficult. Victims of a boat capsize will often underestimate the distance to shore. And though we can spot the moon quite easily, it took some time before people figured out how far away it is.

But with a little method, we can turn ourselves into a spiffy distance-measuring machine by exploiting the fact that we have two eyes. Because our eyes are in to separate locations, they each receive a different view of the world. These two distinct views enable stereo pair photography to produce a 3D illusion (see the cover). They also explain why perspective paintings (see page 21) work best when viewed from a very specific spot.

The Setup Imagine that you are sitting on an airplane looking out the window at a distant building. Close your right eye and position yourself so that the left edge of the window lines up with the left edge of the building. Now, without moving your head, close your left eye and open your right eye. You will see the building shift away from the left edge of the window. This shift is known as **parallax**. To understand parallax, consider the lines of sight from your two eyes when you are positioned as described above. By design, the line of sight from your left eye continues on to the building. By contrast, the line of sight from your right eye will end up somewhere to the left of the building (see the figure below for a top view of the situation). Take note of how much the building shifts and estimate this distance. It's much easier to estimate left-to-right distances than near-to-far distances. In estimating left-to-right distances, you can often use other objects located at the same distance to assist, such as a football field, truck, or house. Let's call the shifted distance S.

The lines of sight from each eye that pass right by the left edge of the window form the sides of two triangles. The line segment connecting your pupils forms the 3rd side of one of these triangles. The line segment represented by the shifted distance S forms the 3rd side of the other triangle. These two triangles are roughly similar.



Let D be the distance from the left edge of the window to the building. Let d be the distance from your eyes to the left edge of the window. Finally, let s be the distance separating your two pupils.

Similarity means that ratios of corresponding lengths are equal. Hence, $D/S = d/s$. Rearranging terms, we find that $D = dS/s$. With this formula, we can estimate the distances to objects that we see from afar.

Let's put this formula to work in a real life situation.

Transamerica Pyramid Here are the left- and right-eye views of San Francisco from a plane:

Photos by C. Kenneth Fan

Left-eye view. Right-eye view.

The Transamerica Pyramid is a well-known landmark there. In fact, it is the tallest building in San Francisco. Let's figure out about how far away the building was from the plane.

The red line segment in the right image shows how far the Transamerica Pyramid shifts from the left-eye view to the right-eye view. The distance looks like it is about 2 city blocks. If we estimate that a city block is 300 feet, then the distance shifted is approximately 600 feet. In the notation set up on the previous page, we have $S = 600$.

The distance between the left and right pupils of the viewer is about $s = 1/5$ of a foot.

The distance from the viewer's eye to the left edge of the window was about 20 inches or 5/3 feet, so $d = 5/3$.

Substituting these values into our formula, we find that $D = 5 \times 600 \times 5/3 = 5{,}000$ feet, or about a mile.

Take it to Your World

When you use this technique, you don't have to align with the left edge of a window. You could hold up your thumb (as though you were hitchhiking) with outstretched arm and memorize the distance from your eyes to your thumb, as well as the distance between your pupils. (Or, more efficiently, memorize the ratio of these two distances.) You will then have all of the information you need to apply this formula wherever you go!

Why don't you use this method to estimate the distance to the moon, given that the diameter of the moon is about 3,500 km? If you measure s, S, and d carefully, you might be surprised at how accurate your measurement turns out to be. In fact, this is essentially the method that the Greek astronomer Hipparchus used to be the first to compute the lunar distance.

What factors affect the accuracy of this method? How much do errors in your measurements affect the accuracy of the approximation?

Another way to measure distance using your two eyes is triangulation (see "Math in Your World," Volume 5, Number 5). In triangulation, one measures the distance between the pupils and the angles that the lines of sight make from each pupil to the object. This information determines a unique triangle from which desired measurements can be computed using trigonometry. Which method is more practical?

Primitive Roots

by Ken Fan

In this article, all equations are to be interpreted modulo p, where p is a prime number. For instance, if you see $x \neq 0$, it means that x is not divisible by p.

In Part 2 of Robert Donley's series on Fermat's little theorem (see Volume 6, Number 2), readers were challenged to show that there exists x with $o(x) = p - 1$, where $o(x)$ is the **order** of x, i.e., the smallest positive integer n such that $x^n = 1$.

For example, if $p = 5$, then $o(2) = 4$. We can verify this by computing powers of 2:

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8 = 3, \text{ and } 2^4 = 16 = 1.$$

(If this makes no sense to you, be sure you've read the contents of the red box above! Also, review Robert's articles.)

Exercise. When $p = 31$, check that $o(3) = 30$. However, $o(2) = 5$.

A **primitive root** for p is a number x that satisfies $o(x) = p - 1$. So Robert's challenge was to show that every prime has a primitive root. The purpose of this article is to prove this.

We present a well-known proof attributable to the French mathematician Legendre.

Polynomials modulo *p*

Polynomials modulo p share some behavior with regular polynomials over the rational numbers or the real numbers. Over the rational numbers, if you have a polynomial $p(x)$ and r is a root, then $p(r)$ can be written as a product of the form $(x - r)q(x)$, where $q(x)$ is a polynomial of degree 1 less than the degree of $p(x)$. To see this, you can perform polynomial long division.

Modulo p, you can do the same. For example, the polynomial $x^2 + 5x - 1$ has the root 4, modulo 7, since $4^2 + 5(4) - 1 = 0 \pmod 7$. And, as you can verify, $x^2 + 5x - 1 = (x - 4)(x + 5) \pmod 7$.

This implies that a polynomial $p(x)$ of degree d can have at most d roots, modulo a prime p. To see this, we proceed by induction on the degree. A polynomial of degree 1 has exactly one root. Indeed, if $ax + b = 0$ with $a \neq 0$, then the unique root is $-ba^{-1}$ (recall that modulo a prime p, nonzero numbers, i.e., numbers not divisible by p, have multiplicative inverses). Now suppose $p(x)$ has degree $d > 1$. If $p(r) = 0$, then we write $p(x) = (x - r)q(x)$. Modulo a prime p, if a product is equal to 0, then at least one of the factors must be 0 (mod p). So if s is another root of $p(s)$, then either $(s - r) = 0$ or $q(s) = 0$. By induction, $q(s)$ has at most $d - 1$ roots. Therefore, $p(x)$ has a maximum of d roots.

If *d* divides *p* – 1, then $x^d - 1$ has exactly *d* roots

Fermat's little theorem tells us that $x^{p-1} - 1$ has exactly $p - 1$ distinct roots. If d divides $p - 1$, we can factor $x^{p-1} - 1$ like this: $x^{p-1} - 1 = (x^d - 1)(x^{p-1-d} + x^{p-1-2d} + x^{p-1-3d} + \cdots + x^d + 1)$.

(This can be checked directly by multiplying out the right-hand side and collecting like terms.) From the above discussion, the polynomial $x^d - 1$ has at most d roots and the polynomial $x^{p-1-d} + x^{p-1-2d} + x^{p-1-3d} + \cdots + x^d + 1$ has at most $p - 1 - d$ roots. But since their product, $x^{p-1} - 1$, does have $p - 1$ roots, both factors must have the maximal possible number of roots. Thus, $x^d - 1$ must have exactly d roots modulo p.

If $o(a)$ and $o(b)$ are relatively prime, then $o(ab)$ is the product of $o(a)$ and $o(b)$

Suppose $o(a)$ and $o(b)$ are relatively prime. Let s be $o(a)$ and t be $o(b)$. Notice that $(ab)^{st} = 1$. Therefore, $o(ab)$ must divide st. This means we can write $o(ab)$ as a product $s't'$ where s' divides s and t' divides t, and s' and t' are relatively prime. We have

$$1 = (ab)^{s't'} = a^{s't'}b^{s't'}.$$

If we raise both sides of this equation to the s/s' power, we get

$$1 = a^{st'}b^{st'} = b^{st'}.$$

This means that $o(b)$ divides st'. Since $o(b)$ is relatively prime to s, it must be that $o(b)$ divides evenly into t'. Since t' divides evenly into $o(b)$, t' and $o(b)$ must be equal. Similarly, s' and $o(a)$ must be equal, and this shows that $o(ab)$ is, indeed, the product of $o(a)$ and $o(b)$.

There exists a primitive root

A primitive root for the prime p would have order $p - 1$. Let $p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_m^{n_m}$ be a prime factorization of $p - 1$. The above discussion informs us that if we can find x_k with $o(x_k) = p_k^{n_k}$, then we can build a primitive root by taking the product of the x_k.

Consider the polynomial $x^{p_k^{n_k}} - 1$. Any root of this polynomial will have order p_k^m for some nonnegative integer m less than or equal to n_k. Therefore, if r is a root of $x^{p_k^{n_k}} - 1$ but is not a root of $x^{p_k^{n_k - 1}} - 1$, then $o(r)$ would have to be $p_k^{n_k}$.

But we know that $x^{p_k^{n_k}} - 1$ has exactly $p_k^{n_k}$ roots and $x^{p_k^{n_k-1}} - 1$ has exactly $p_k^{n_k - 1}$ roots. Since $p_k^{n_k} > p_k^{n_k - 1}$, there are exactly $p_k^{n_k} - p_k^{n_k - 1}$ roots of $x^{p_k^{n_k}} - 1$ that are not roots of $x^{p_k^{n_k-1}} - 1$.

So we can find x_k with $o(x_k) = p_k^{n_k}$. (In fact, we can find $p_k^{n_k} - p_k^{n_k - 1}$ such numbers.) We multiply these together and obtain a primitive root for p, and we're done!

More to think about

Show that the number of primitive roots for p is equal to $\varphi(p - 1)$.

Use a primitive root to prove Wilson's theorem, which says that when p is a prime number, $(p - 1)! + 1 = 0$. (Hint: Suppose that g is a primitive root for p. This means that the remainders obtained by dividing the numbers $g, g^2, g^3, \ldots$, and g^{p-1} by p will be a complete list of nonzero remainders. Therefore, $(p - 1)! = g^{1+2+3+\ldots+(p-1)} \pmod{p}$.)

The observations we made about polynomials depend on the modulus being a prime number. For example, show that $x^2 - 1$ has more than 2 roots modulo 8.

COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 11

<u>The Egyptian Fraction Algorithm</u>

1. Start with original target fraction, *a*/*b*, between 0 and 1.
2. Find the smallest counting number *c* such that $1/c \leq a/b$.
3. Record 1/*c* as one of the unit fractions that will sum to the original target fraction.
4. Update the target fraction by subtracting 1/*c* from it. Use *a*/*b* to refer to this updated target.
5. If *a*/*b* = 0, stop. Otherwise, continue from step 2.

You: Hi, 3/7. Boy am I glad to see you! I'm stuck.

$\frac{3}{7}$: What? How is this haystack of papers talking to me? And in the voice of my little Egyptian fraction friend!

You: It *is* me. I ended up buried in paper from all the examples I've calculated.

$\frac{3}{7}$: Such industry is always a cheery sight. I've been wondering how your investigation of Fibonacci's Egyptian fraction algorithm has been going. I'll get you unstuck in a jiffy. I'll dash home for my crane so I can extract you.

You: No, I'm not that kind of stuck – I'm climbing out. I'm just not making headway on showing whether the algorithm always stops.

$\frac{3}{7}$: Oh, for that sort of stuck we don't need my crane. Tell me everything.

You: I definitely have the hang of computing the calculations, and in every example I've done, the algorithm stops. So that makes me suspect that it always stops, but I don't know for sure. If my suspicion is correct, I want to know the reason it is so.

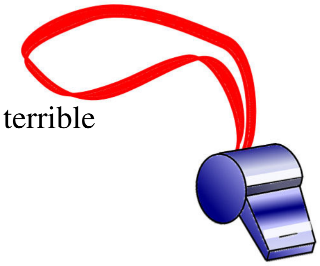
$\frac{3}{7}$: A very noble aspiration. Let's fulfill it.

You: How?

$\frac{3}{7}$: Your calculations are filled with useful information. If the algorithm always stops, there might be clues as to why in all the examples you've done. So let's start by looking through these papers for clues: they may suggest a line of reasoning that helps us establish whether the algorithm always stops.

You: What kind of clues? I've certainly looked at all of my calculations as I've done them, but nothing jumped out at me.

$\frac{3}{7}$: Patterns. To find them, your best bet is to consider your examples together, in an organized way. Can you present your work in a systematic manner, including all of the steps you took along the way? There might be clues in those intermediate steps, and it would be a terrible shame to miss them.



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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Prof. DeMarco, Part 4 of the Stable Marriage Algorithm, and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

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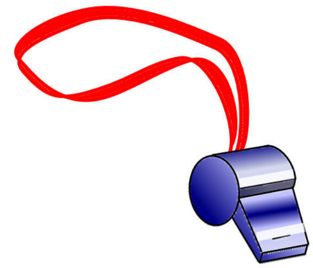
Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls



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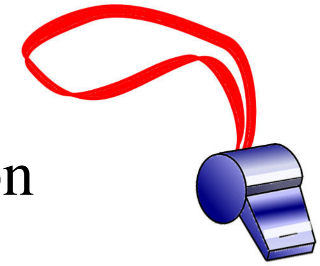


Summer Fun!

The best way to learn math is to do math, so here are the 2013 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems before seeing solutions.

Some of these problems are very challenging and could take several weeks to solve, so please don't approach these problem sets with the idea that you must solve all of them. The main goal of these problems is to give you some interesting things to think about.

If you are stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, feel free to email us.

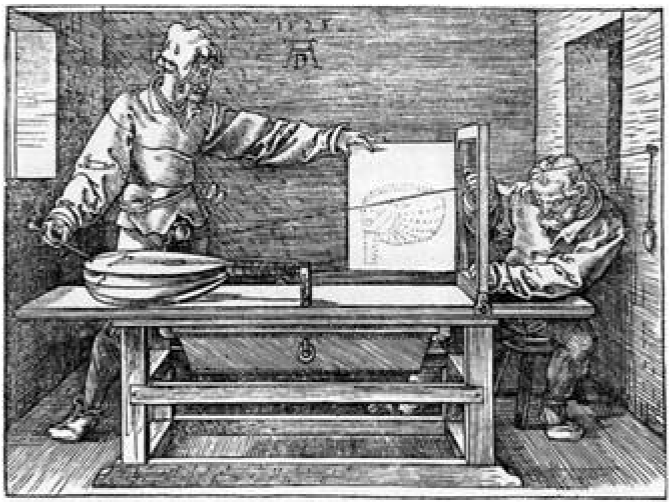
If you are used to solving problems quickly, it can feel frustrating at first to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Perspective Drawing

by Ken Fan



A Man Drawing a Lute by Albrecht Dürer

A perspective drawing is a drawing that gives the viewer the illusion of a 3 dimensional world. This is accomplished by making the drawing represent the view of the 3D world beyond the drawing as if the drawing were a window. Albrecht Dürer showed a way to make a perspective drawing in his wood cut *A Man Drawing a Lute*, shown at right. In this drawing, a man and his assistant are carefully constructing a perspective drawing of a lute, the musical instrument placed on the table. A string is stretched taut between points on the lute and the point that represents the location of the eye of the person looking at the drawing. The place where this string, which represents a line of sight, intersects the plane of the drawing is carefully noted, and a mark is made on the drawing at this location. This is tedious work, but the result is the illusion of a 3D world from a flat drawing.

If you are interested in creating 3D illusions in your own drawings and paintings, perspective is an important concept to master. In this Summer Fun problem set, we'll explore perspective from the ground up. When you've gotten a handle on this material, you will be able to make realistic 3D renditions and fool people into believing in 3D worlds that don't exist.

Here's the setup: we'll call the object(s) being drawn the **model**. The drawing will be on a flat piece of paper, oriented so that its plane is perpendicular to a flat ground. We'll assume that the location of the viewer's eye relative to the drawing is fixed. Points on the model are rendered in the drawing at points corresponding to where the line of sight from the viewer's eye to the point on the model intersects the plane of the drawing.

Okay, let's get started!

1. We'll begin with a very simple model: an idealized dot. An idealized dot is a figment of our imagination. It is a dot that we imagine you can see, but has no physical size. Imagine a dot hovering before you. What would a perspective drawing of this dot look like?

2. A perspective drawing of the dot would be a dot on the paper. Take one of your perspective drawings of the dot. Using only your drawing, is it possible to determine how far away the model dot is supposed to be from the viewer's eye? What are the possible locations of other model dots that would be depicted by the same perspective drawing?

3. Let's move on to line segments. What do perspective drawings of line segments look like? Under what circumstances would a perspective drawing of a line segment look like a dot?

4. A perspective drawing of a line segment could represent many different line segments. Describe all line segments depicted by the same perspective drawing. Because of this, the length of the model line segment cannot be determined from the drawing of it. Can the direction of the model line segment be determined from a perspective drawing of it?

Summer Fun!

5. Let's move from line segments onto lines, which can be thought of as line segments extended forever on both sides. What do perspective drawings of lines look like? Can a perspective drawing of a line look like a dot? If so, which lines would be depicted as a dot? Which lines are depicted by vertical lines in the drawing?

6. Now imagine that the model consists of a whole bunch of parallel lines in space. What will the perspective drawing of this model look like?

7. Notice that parallel lines not parallel to the drawing canvas will appear as radial lines emerging from a common point. How can this point on the canvas be located? This point is called a **vanishing point**.

8. Imagine several sets of lines, each consisting of parallel lines that are not only parallel to each other, but also parallel to the ground (which we are assuming is perpendicular to the drawing canvas). Show that the vanishing points of each set of lines all lie on a single horizontal line. This horizontal line is called the **horizon line**.

9. The model will now be a floor with a checkerboard tiling. In a checkerboard tiling, there are 4 natural sets of parallel lines to consider. Two consist of lines parallel to the sides of the tiles and two consist of lines parallel to the diagonals of the tiles. Use your understanding of vanishing points to construct a proper perspective rendering of a checkered floor. You can see an example of this by the artist Jan Vermeer in his painting *The Art of Oil Painting*.

10. Make a proper perspective drawing of a straight railroad track going off into the distance, lined by evenly spaced telephone poles. Use your understanding of vanishing points to ensure that the drawing of the railroad track slats depict evenly spaced slats and the drawing of the telephone poles depict evenly spaced poles all of the same height.

The following exercises are somewhat more challenging.

11. Make several perspective drawings of *perfect* cubes. Note that the 3 sets of parallel lines defined by the edges of the cube could, potentially, define 3 vanishing points. Think carefully about the placement of these 3 vanishing points. Note that the edges of a rectangular block with faces parallel to or perpendicular to the faces of the cube would define the same vanishing points as those of the edges of the cube. This means that knowing the locations of the vanishing points defined by the edges of the cube *does not determine the drawing*. How can you ensure that the drawing will depict a true cube and not a rectangular block?

12. Make a proper perspective drawing of a cityscape filled with streets and buildings.

13. Explain how the stereo pair on the cover works to produce a 3D illusion.

14. People's faces often imply parallel lines. For example, the lines that pass through the corners of the eye and the corners of the mouth are typically parallel. Think about how these considerations could affect portraiture.

Vieta’s Formulas

by Shravas Rao

If you don’t yet know Vieta’s formulas, do this Summer Fun problem set and you will.

1. For each quadratic, determine the sum of the roots and the product of the roots.
Do you see a pattern?

Quadratic	Sum of Roots	Product of Roots
$x^2 - 6x$		
$x^2 - 4x + 3$		
$x^2 - x - 30$		
$x^2 + 7x + 6$		
$x^2 - 3x - 28$		

2. What is the sum of the roots of the quadratic $x^2 + bx + c$, where b and c are constants? What about the product of the roots? Do these answers agree with the answers in the previous question when you plug in values for b and c? Make a conjecture and try to prove it.

3. For each quadratic, determine the sum of the roots and the product of the roots.
Do you see a pattern?

Quadratic	Sum of Roots	Product of Roots
$2x^2 + 8x + 4$		
$3x^2 - 6x - 18$		
$5x^2 - 15x + 5$		
$2x^2 + 5x + 2$		
$4x^2 - 12x - 16$		

4. What is the sum of the roots of the quadratic $ax^2 + bx + c$ where a, b, and c are constants. What about the product of the roots? Do you see how the coefficient of x^2 affects the answer to these questions?

5. Now find the sum and product of the roots of the quadratics $x^2 + 2x - 4$ and $3x^2 + 4x + 5$, but this time, find them without calculating the roots. Instead use your answer to #2 and #4.

6. Let p and q be the roots of the polynomial $x^2 + 6x + 3$. Can you calculate the value of $\frac{3}{p} + \frac{3}{q}$ without calculating the values of p and q?

7. Let p and q be the roots of the polynomial $2x^2 + 4x + 8$. Can you calculate the value of $p^2 + q^2$ without calculating the values of p and q?

Summer Fun!

8. Let p and q be two different numbers so that $p+\frac{4}{p}=q+\frac{4}{q}$. What is the product of p and q?

9. Let the roots of the polynomial $x^2 - 13x + 25$ be the lengths of the legs of a right triangle. What is the length of the hypotenuse of the triangle? Compute this without finding the roots.

10. Now we're going to move on to cubic polynomials. For each cubic in the table below, compute the sum and the product of the roots.

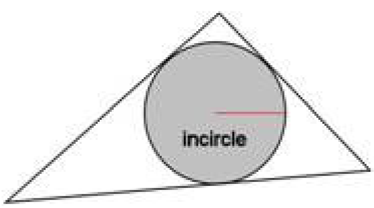
Cubic	Sum of Roots	Product of Roots
x^3		
$x^3 - x$		
$x^3 + 3x^2 + 3x + 1$		
$x^3 + 3x^2 - x - 3$		

11. Let p, q, and r be the roots of a cubic in x where the coefficient of x^3 is 1. What are the coefficients of x^2, x, and the constant term in terms of p, q, and r? Note the coefficient of x in particular.

12. Let p, q, and r, be the roots of the polynomial $x^3 - 4x^2 + 2x - 3$. Can you calculate the value of $(p + q + r)(p + q + r)$ without determining p, q, and r? What about $p^2 + q^2 + r^2$?

13. Let p, q, and r be the roots of a cubic in x where the coefficient of x^3 is a. What are the coefficients of x^2, x, and the constant term in terms of a, p, q, and r?

14. The length, width, and height of a block are the roots of the cubic $x^3 - 20x^2 + 120x - 210$. What are the surface area and volume of this block?



15. Let the side lengths of a triangle be the roots of the polynomial $x^3 - 9x^2 + 26x - 24$. What is the radius of the incenter of the triangle? (Hint: Heron's formula tells us that the area of a triangle with side lengths p, q, and r is $\sqrt{s(s-p)(s-q)(s-r)}$ where $2s = p + q + r$.)

16. Let the roots of the polynomial $x^3 + ax^2 + bx - 1$ be positive real numbers, where a and b are constants. Prove that a must be less than or equal to -3.

17. Let $p_1, p_2, \ldots, p_d$ be the roots of a polynomial in x of degree d, where the coefficient of x^d is a. What are the coefficients of x^k, where $0 \le k < d$, in terms of $p_1, p_2, \ldots, p_d$, and a? These formulas are referred to as Vieta's formulas.

18. Show that the value for any one of the coefficients you found in #16 remains unchanged if you permute the p_k's. Why should this be expected?

Summer Fun!

Z, $Z[\sqrt{-1}\,]$, and $Z[\sqrt{-5}\,]$

by Addie Summer

Kirsten Wickelgren mentioned systems of numbers different from the integers in her interview on page 3. Let's explore some of them in this Summer Fun problem set! We'll begin by reviewing some facts about the integers, which we'll denote by Z.

A set of numbers S that contains the sum of any two of its members, is **closed under addition**. Similarly, if S contains the product of any two of its members, it is **closed under multiplication**. The numbers in S whose multiplicative inverses are also in S are called the **units** in S.

1. The set of integers Z is closed under addition and multiplication. What are the units in Z? That is, for which integers a is $1/a$ also an integer?

2. Some integers can be written as a product of integers where neither factor is a unit. For example, $-6 = -2 \times 3$. Give an example of an integer that is not a unit and cannot be expressed as a product of two integers, neither of which is a unit. Such integers are called **irreducible**.

In Z, the irreducible numbers are the same as the prime numbers. Every number can be written as a product of irreducible numbers in a unique way, if we ignore the order of factors and multiplication by units. For instance, while 6 is 2×3 and $3 \times -1 \times -2$, the expressions differ only by the order of factors and multiplication by units. This is known as **unique factorization**.

Let $Z[i]$ denote the set of complex numbers of the form $a + bi$, where a and b are integers and $i = \sqrt{-1}$. We add and multiply these numbers as complex numbers.

3. Show that $Z[i]$ is closed under addition and multiplication.

4. If $z = a + bi$, show that $z^{-1} = \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2}i$. What are the units in $Z[i]$?

5. Although 5 is irreducible in Z, show that 5 is not irreducible in $Z[i]$.

6. Determine which $a + bi$ in $Z[i]$ with $0 \le a, b \le 6$ are irreducible. Make a picture by circling the irreducible numbers in $Z[i]$ in the complex plane.

7. Do you think that $Z[i]$ has unique factorization?

Let $Z[\sqrt{-5}\,]$ denote the set of complex numbers of the form $a + b\sqrt{-5}$.

8. Do #3 and #4 for $Z[\sqrt{-5}\,]$ in place of $Z[i]$.

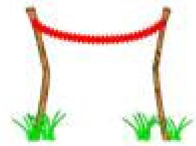
9. Find all irreducible numbers $a + b\sqrt{-5}$ in $Z[\sqrt{-5}\,]$ with $|a|, |b| < 5$.

10. Show that $Z[\sqrt{-5}\,]$ does not have unique factorization by showing two different ways to express 6 as a product of irreducible numbers.

Summer Fun!

The Meddling Gnomes

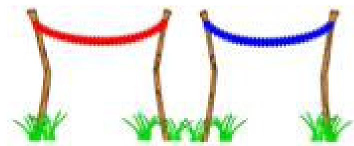
by Lauren McGough



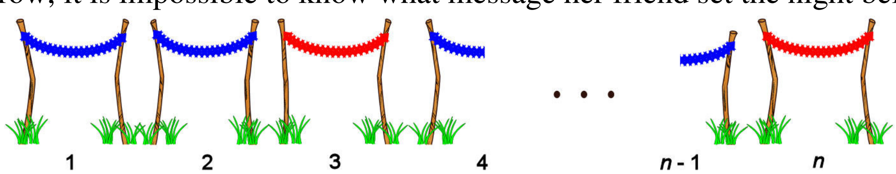
Katie and Sarah are best friends. They are neighbors and like to play together. They also love codes. Every day, each one leaves a secret message for the other by tying colored yarn in between two sticks in the ground, as shown above right. Blue meant "yes, I can play today"; red meant "no, I cannot." We'll call this shape a "goalpost."

But there's a problem! Tiny, pesky gnomes live in their backyards. These gnomes have magical powers. If a gnome touches one of the sticks, any piece of yarn tied to that stick flips its color, red to blue and blue to red! This ruins the girls' messages. But the girls aren't giving up. They have many sticks and lots of yarn! Is there a way to send a message using yarn and sticks even though they can't control the gnomes?

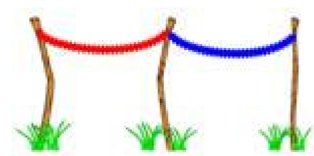
1. At first, the girls wonder if they can fix the problem using two goalposts instead of one, as shown at right.



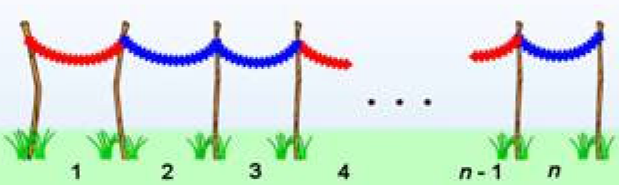
a. If they agree to read the two goalposts in a specific order, how many different sequences of red and blue are there? (For example, one combination could be left- red, right- blue.)
b. Suppose the girls set up the goalposts to be "red, red," and leave the message overnight. The gnomes can touch any of the sticks in any order and any number of times. The girls have no control over that. What possible color combinations could they see when they check the message in the morning, depending on the gnomes' behavior?
c. Would one girl ever be able to look at the yarn in the morning and know anything about which colors the other girl set the night before?
d. Can you answer a, b, and c for 3 goalposts in a row? 4? What happens with *n* goalposts, where *n* can be any number? Show that no matter how many goalposts there are, if they are arranged in a row, it is impossible to know what message her friend set the night before.



2. The girls realize that separate goalposts won't work, but they know that isn't the only possibility! They try stringing together goalposts together in a line, as shown at right.

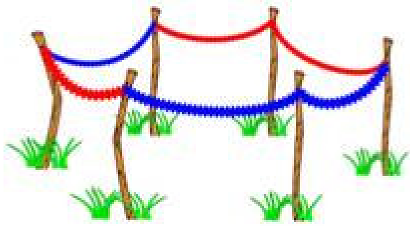


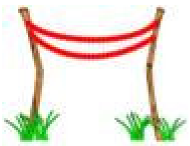
a. Suppose the girls set up the goalposts to be "red, red," and leave the message overnight. As always, the pesky gnomes can meddle with the message, and the girls have no control over them. What possible sequences could they see the next morning? Would one girl ever be able to look at the yarn in the morning and know anything about which colors the other girl set up the night before?
b. Can you answer part a for a line of 3 attached goalposts? What about *n* attached goalposts (see below)? Show that it's still impossible to know what message was set the night before.



Summer Fun!

3. Neither disconnected goalposts nor a line of goalposts will work... but the girls still aren't giving up! They sit and think until suddenly, they have an idea! What if they make a line, and then connect the first and last sticks? That is, what if they make a loop, as shown at right, for example.



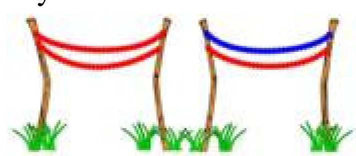


a. Let's first analyze the case where there are just two sticks and two pieces of yarn going between them (shown at left). They set the yarn's colors to be "red, red." They leave the message overnight. After the gnomes' mischief, what possible sequences could they see in the morning?
b. If Katie observes "red, red" in the morning, what could Sarah have set the night before?
c. Show that now there *is* a way for the girls to communicate "yes" and "no."
d. What if the girls wanted to add a set of "maybe" color patterns that represent different levels of sureness? Is it possible? If so, how many levels between "yes" and "no" are possible?
e. Analyze a, b, c, and d for a loop made with *n* sticks? (The figure above right shows such a loop made with 6 sticks.)

Notice what we just found! Even though the gnomes can make changes at any stick – and even though this is completely out of the girls' control – the girls can still define a system where they can communicate information to each other with 100% certainty.

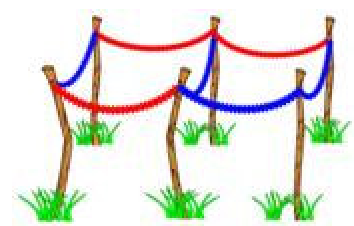
Let's go further. Can the girls communicate more information if they add more loops? In order for the girls to send a greater number of distinct messages, they must collect all the possible configurations of colors into groups such that if a girl sets the configuration to be in one group, it is sure that when she checks in the morning, the other girl will see a message that is also in that group no matter how hard the gnomes try to change the message during the night! The number of groups they can define is the number of distinct messages they can send.

4. Suppose the girls use two loops, as shown at right. How many different messages can they send to each other?



5. Analyze the situation with *n* loops, where the loops can involve any number of sticks. How many messages can they send? How can they quickly check the meaning of a given message? How do these answers depend on how many sticks are in each loop?

6. Now suppose the girls connect loops by making a 2 by 3 grid of sticks and connecting them with yarn as shown at right. Now how many different messages can they send to each other? What if they extend this to a 2 by *n* rectangular grid of sticks, connecting the yarn between nearest neighbors? How can they quickly check the meaning of a given color configuration?



7. Finally, what happens if the girls create an even bigger rectangular network with an *n* by *m* grid work of sticks with yarn strung between nearest neighbors? How many different messages can they send? How does this compare to making $(n - 1)(m - 1)$ separate loops?

8. Invent your own configuration of yarn and sticks. Analyze it for its gnome-proof message sending properties. Share your designs with us!

Summer Fun!

The Gauss-Wilson Theorem

by Robert Donley

Fix $n > 2$. Let U_n be the set of all integers k, with $1 \leq k < n$, that are relatively prime to n, and denote by $U_n!$ the product of all the numbers in U_n.

In this Summer Fun problem set, you'll prove the

Gauss-Wilson Theorem: If $n = 4$, or n is a power of an odd prime, or n is twice a power of an odd prime, then $U_n! = -1 \pmod{n}$. Otherwise, $U_n! = 1 \pmod{n}$.

1. Verify the theorem for $n = 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18$, and 20.

2. Prove that $x^2 = 1 \pmod{n}$ if and only if $x = x^{-1} \pmod{n}$.

3. Let I_n be the set of all elements s in U_n such that $s^2 = 1 \pmod{n}$, and let $I_n!$ be the product of all elements in I_n. Show that $U_n! = I_n! \pmod{n}$ and $(I_n!)^2 = 1 \pmod{n}$. (Hint: Seek pairs that can be cancelled because their product is 1 and use #2.)

4. For each of the cases in #1, find all x in U_n such that $x^2 = 1 \pmod{n}$. Verify #3 in these cases.

5. Suppose that there exists an a in U_n such that every number in U_n may be written as a power of $a \pmod{n}$. When $n > 2$, show that I_n has exactly 2 elements. What are they?

6. Challenge: Let p be a prime number. Show that U_p enjoys the property in #5, i.e., there is an a in U_p such that the powers of a yield all elements of U_p. Using #5, we deduce that $(p - 1)! = -1 \pmod{p}$. This proves part of the Gauss-Wilson theorem and is known as Wilson's theorem.

7. Suppose a and b are distinct elements in I_n, neither equal to $1 \pmod{n}$. Show that ab is another element of I_n not equal to 1, a, or $b \pmod{n}$.

8. Show that I_n has 2^k elements for some $k \geq 0$. If I_n has more than 1 element, show that there is an element $s \neq 1 \pmod{n}$ and a subset I' of I_n such that $I_n = I' \cup sI'$ as a disjoint union. (Hint: Build I_n stepwise in the following way: add a new element, take all products, count, and repeat. Use #7 to relate your new set of elements to the old set.)

9. Using the second half of #8, prove that $I_n! = -1 \pmod{n}$ if and only if there are exactly 2 elements in U_n that satisfy $x^2 = 1 \pmod{n}$. Otherwise, $I_n! = 1 \pmod{n}$.

10. Cliffhanger/Challenge: To finish, show that the first part of #9 occurs precisely when $n = 4$, or n is a power of an odd prime, or n is twice a power of an odd prime. For this, we need
    1. if m and n are relatively prime, then $|I_{mn}| = |I_m||I_n|$,
    2. if p is an odd prime and $n = p^k$ $(k > 1)$, then U_n is cyclic, and
    3. if $n = 2^k$ and $k > 2$, then I_n has exactly 4 elements.

Here $|X|$ denotes the number of elements in X.
Verify for all n in #1.

Summer Fun!

Calendar

Session 12: (all dates in 2013)

January	31	Start of the twelfth session!
February	7	
	14	
	21	No meet
	28	
March	7	
	14	Iris Ortiz, Cambridge Systematics, Inc.
	21	No meet
	28	
April	4	Crystal Fantry, Wolfram Research
	11	
	18	No meet
	25	Ashlee Ford Versypt, MIT Dept. of Chemical Eng.
May	2	Emily Riehl, Harvard University
	9	

Session 13: (all dates in 2014)

September	12	Start of the thirteenth session!
	19	No meet
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	
	12	

Here are answers to some of the *Notation Station* exercises on page 7.

1. There are many answers. One way is $\sum_{k=1}^{50} 2k$.

2. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}+\frac{1}{110}$.

3. Again, there are many answers. One way is to write $\sum_{p=0}^{n-1} ar^p$.

Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, custom content production including our magazine, the Girls' Angle Bulletin, and various outreach activities such as our Math Treasure Hunts and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The print version (beginning with volume 3, number 1) costs $36 for an annual subscription and brings with it access to our mentors through email. Subscribers may send us their solutions, questions, and content suggestions, and expect a response. The Bulletin targets girls roughly the age of current members. Each issue contains a variety of content at different levels of difficulty extending all the way to the very challenging indeed.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We also aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club where they work directly with our mentors exploring mathematics. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a **Remote Membership** which comes with a year-long subscription to the Bulletin. Remote members may email us math questions (although we won't do people's homework!), send us problem solutions for constructive comment, and suggest content for the Bulletin. To become a remote member, you can simply subscribe to the print version of the Bulletin.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls

Membership Application

Applicant's Name: (last) ______________________________ (first) ______________________________

Applying For (please circle): Membership Remote Membership/Bulletin Subscription

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: ________________ Email: ______________________

For **membership applicants only**, please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: ____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________
(Parent/Guardian Signature)

Membership-Applicant Signature: ____

- ☐ Enclosed is a check for (indicate one) (prorate as necessary)
  - ☐ $216 for a one session Membership (which includes 12 two-hour club meets)
  - ☐ $36 for a one year Remote Membership (which includes 1-year subscription to Bulletin)
  - ☐ I am making a tax free charitable donation.
- ☐ I will pay on a per meet basis at $20/meet. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Girls' Angle: A Math Club for Girls
Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: _______________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: __________________________________

Girls' Angle

A Math Club for Girls