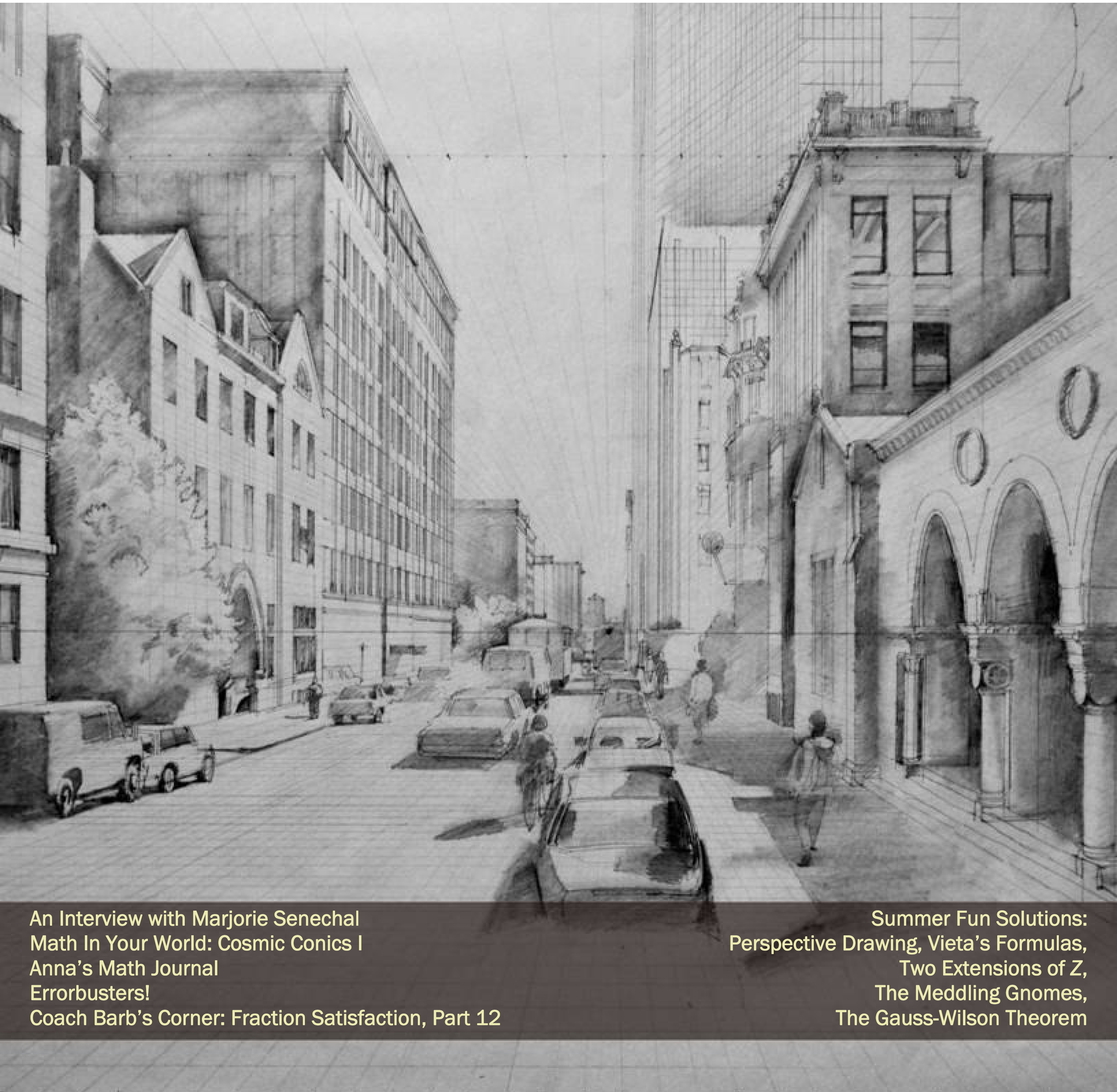
Girls' Angle Bulletin

August 2013 • Volume 6 • Number 6

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Logic underpins everything. Therefore, the thinking skills you develop by practicing mathematics will prove valuable in all that you do. In a certain sense, mathematical problems are much simpler than other problems you will encounter in life. In a typical math problem, the assumptions are clear and few. But in life, problems tend to be enormously complex and intermingle people with radically different value systems that are rich in subtle detail. Math can be used like a gym to develop your thinking ability in isolation so that you can tackle life's problems more effectively.

- Ken Fan, President and Founder

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Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: The Back Bay sketched by master painter and local artist Joel Babb. To convey a sense of 3D, Mr. Babb carefully observes principles of perspective drawing. Used with permission from Joel Babb.

An Interview with Marjorie Senechal

Marjorie Senechal is the Louise Wolff Kahn Professor Emerita in Mathematics and History of Science and Technology at Smith College. She is also editor-in-chief of *The Mathematical Intelligencer*.

The truth-beauty dialogue is important. Both sides are needed, one to ask "what-if?", the other to say "let's see whether." This sparring is an engine of science, and of mathematics too.

Ken: What has and what does mathematics mean to you?

Marjorie: Mathematics is a lens on the world. It's not crystal-clear: the real world and math are very different things. But charting the world through pattern-recognition and logical reasoning and searching for simplicity in complexity is, I think, a clearer lens than most. Another thing math means to me is constant challenge. I decided to major in math in college because there would always be more, and harder, problems to solve, and new (and old) ideas to understand, or understand better. I knew I would never get bored with math. And I would never know all there was to know. I would always have adventures of the mind. And so I have – more than I could have guessed at the time. That's because math itself has changed and expanded. The computer revolution has transformed the subject.

Ken: What was the first mathematical thought you had that excited you?

Marjorie: It wasn't an original thought, it was handed to me. But it certainly excited me. I went to the University of Chicago without finishing high school, which meant (among other things) that I had to take the introductory college courses. That was a good thing, as they were much better than the high school courses I missed. At UC at that time, the semester grade for each course was entirely determined by a single three-hour final. As part of the math course final, we were given some axioms and a list of theorems. These axioms weren't self-evident like Euclid's. They seemed to have been invented out of the blue. Nor did I know what the theorems were about. Still, it wasn't hard to prove them. I found it thrilling, and could hardly wait to share my excitement with the other students afterwards. But when the exam was over, I found they weren't smiling. Some were even in tears. I realized then, for the first time, that I loved math, and that I might even be good at it.

Ken: You've done a lot of research on polyhedra, tilings, and crystals. How did you become interested in these subjects? What fascinates you about them?

Marjorie: It started with crystals. Soon after I started teaching math at Smith, I came across a book on crystals in the library and was hooked by their beautiful geometrical shapes and the atomic patterns that give rise to them. Polyhedra and tilings are mathematical models for crystals, so it was natural that I would gravitate to them. What fascinates me, now more than ever, is how quickly and profoundly our understanding of crystals, polyhedra, and tilings is changing. Despite the metaphor "crystal clear" (yes, I use it too, see above), some things about crystals aren't as clear as they seem. For example, the atomic patterns that first drew me in were periodic arrangements, like three-dimensional wallpaper. Back then, everyone "knew" that the atoms in crystals had to arrange themselves that way. But then it turned out that there are aperiodic crystals, whose atomic patterns don't repeat in any direction.

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Dr. Senechal, and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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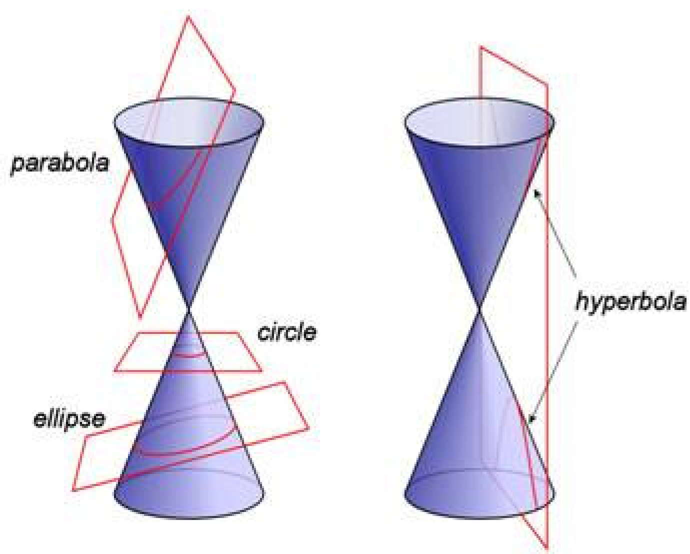
Cosmic Conics I[1]

by Aaron Lee
edited by Jennifer Silva

In sophomore year high school math, when I was introduced to the conic sections – the four curves that result from slicing through a circular cone – I remember thinking, "Wow, how cool!" During junior year my reaction turned into, "That's kinda neat." And by senior year I was practically screaming, "Why on earth is this useful? Who goes around slicing through cones these days?"

The Greeks, through an exercise of cone-carving, were the first to discover the fact that the circle, ellipse, parabola, and hyperbola are all related to each other. They had their reasons for chopping through cones: geometry was their crown jewel of science. And because of this unifying observation, these curves forever became known as the "conic sections." Still, I couldn't help but wonder whether there is something more profound going on. Are these four curves related only through the cone? Or are there other instances, perhaps more familiar ones, in which these four curves present themselves in tandem? We will see that the cosmos gladly provides us an answer, and we will not need to look further than our own Solar System.



As you read this, a newborn comet, Comet ISON, is hurtling toward the Sun at speeds around 60,000 mph. Originating in the **Kuiper belt**, an asteroid-like belt outside of Neptune's orbit, this enormous ice ball about 3 miles in diameter will hurl itself dangerously close to the surface of the Sun, coming closest at the end of November. After that, it will fly around the Sun and begin a trip back to the edges of the Solar System. Observable with binoculars starting in September, this comet should be visible to the naked eye between November and January. It is speculated that at its brightest, this comet may outshine even the full Moon! A comet's brightness and visibility to the naked eye are determined by several factors, such as its size and its composition. But the most important factor is the orbit on which the comet travels. Comets that get closer to the Sun reflect more light, and thus appear to be brighter.

Determining the orbits of objects around the Sun is a classic problem in astronomy and mathematics. Even today, so much of astronomy involves knowing how one object orbits another: comets and planets around the Sun, exoplanets around other stars, and moons around

[1] This content supported in part by a grant from MathWorks.

planets are a few examples. Originally, astronomers were interested only in modeling the orbits of planets in the Solar System, because they assumed that nature and all its laws, phenomena, and foibles were bestowed upon the Earth by divine intervention. The precise, logical way of deriving scientific results from mathematics, and the idea that the laws of science could be discovered through reason, would not come about until the 1600s with the scientific revolution of Galileo Galilei and Isaac Newton.

In a flash of insight, Newton connected the motion of the celestial bodies with the observations that apples fall from trees and balls thrown in the air always return to the ground. He reasoned that massive bodies exert a far-reaching attractive force on every other mass in the Universe, a force he called gravity. He came up with his theory of universal gravitation while on spring break from Cambridge University (well, it wasn't actually spring break – the plague had forced the school to close for 18 months). He also invented calculus during this break. He showed that for any object dominated by the Sun's gravitational pull, the object's path is exactly one of the four conic sections, with the Sun located at one of the foci. Said another way, every planet or comet in the Solar System traces out a conic section. And since gravity is universal, the Solar System is not special: the path of any object orbiting another can be described by one of the four conic sections!

Even though the result is general, let's stick with the Solar System. In this case, Newton's result can be concisely summarized as the following: with the Sun at the origin (0,0), the path traced out by any object dominated by the Sun's gravity takes the following form:

$$r(\theta) = a \cdot \left(\frac{L(e)}{1 + e \cdot \cos(\theta)} \right), \text{ where } L(e) = \begin{cases} 1 - e^2 \text{ if } 0 \le e < 1 \\ 2 \text{ if } e = 1 \\ e^2 - 1 \text{ if } e > 1 \end{cases}. \qquad (1)$$

There are two constants, a and e (both ≥ 0), which are usually called the **semi-major axis** (a) and the **eccentricity** (e) of the trajectory, and a function $L(e)$ which depends on the value of e. We will see what a and e represent below. Note that the equation gives us $r(\theta)$ instead of $y = f(x)$. This equation is called a **polar equation**, because it gives the distance from the origin as a function of the angle measured counterclockwise from the positive x-axis. If you want to plot the function on a Cartesian plane, you can get the x and y coordinates by the transformations

$$x = r(\theta)\cos\theta,$$
$$y = r(\theta)\sin\theta,$$
$$\text{and } r(\theta)^2 = x^2 + y^2.$$

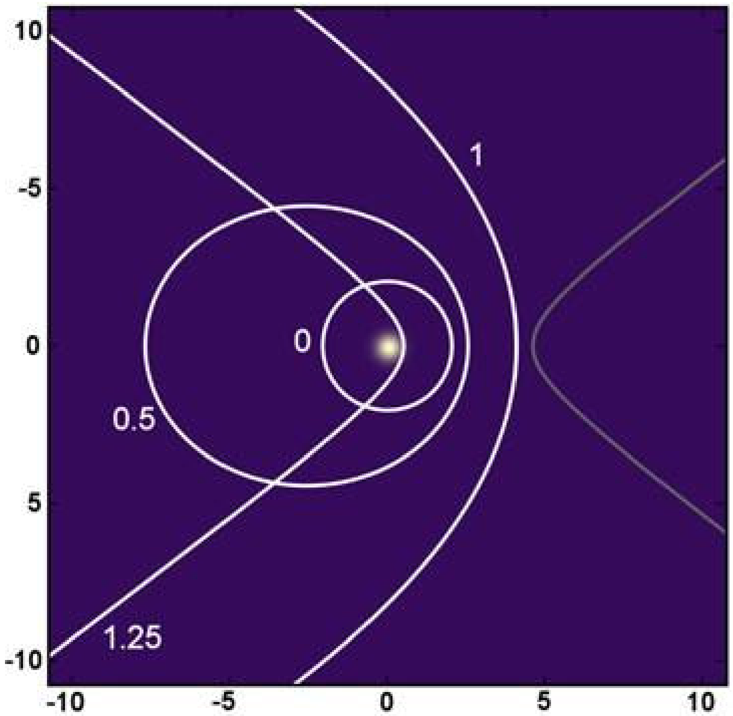
These may look familiar to anyone who knows SOH-CAH-TOA or has dabbled with the unit circle.

Polar Plots

If you've never played with polar plots before, here's your chance! Graph the following polar equations. Remember that $r(\theta)$ tells you how far out you must go along the ray that makes an angle of θ with the positive x-axis (measured in the counterclockwise direction). (In these equations, we are assuming that you are measuring angles in radians.)

(a) $r(\theta) = 2$
(b) $r(\theta) = \theta/\pi$
(c) $r(\theta) = -\sin(\theta)$

Compare your answers with mine on the next page.



One of the best parts of being a physicist is that when you have a new equation, you get to play around with it and learn all its secrets. A picture is worth a thousand words, but an equation yields infinite possibilities. Recall that I claimed equation (1) (on the previous page) could be used for all four conic sections. This equation has two constants, a and e, and their values must somehow determine which conic section is made. Let's try to reason out whether a or e does this.

First, let's think about the semi-major axis a. It appears as an overall multiplicative factor. Changing a makes the entire curve larger or smaller, so a does not determine which type of conic section is made (making an ellipse bigger will not suddenly turn it into a hyperbola). So the eccentricity e must be what determines the shape of the curve. The plot above and to the left shows equation (1) for four different values of e (written next to the curves), for θ between 0 and 2π. Aha! When I set $e = 0$, I get what looks like a circle; for $e = 0.5$, an ellipse; for $e = 1$, a parabola; and for $e = 1.25$, a hyperbola (the second half of the hyperbola is shown in gray). So, the eccentricity e determines which conic section is made.

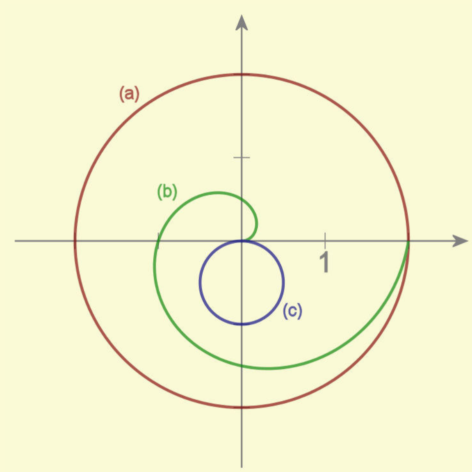
What about other eccentricities besides the four values I chose? For every single $e \geq 0$, we need to assign a conic section to its value. Make various polar plots of equation (1) with different values of e to come up with your own theory. Here are some hints to help you along:

- The circle and ellipse are closed curves – they loop back on themselves – while the parabola and hyperbola soar off to infinity. If a fraction is diverging to infinity, it often means that the denominator is going to zero. For what values of e is it possible for the denominator to vanish when θ is between 0 and 2π?
- The function L is defined piecewise, and the 3 pieces provide a big clue.
- Parabolas are peculiar. Only one value of e will give a parabola.
- A circle is just one special case of an ellipse.

Make your theory, and tune in next time to compare it to mine!

Polar Plots

Here are my answers to the 3 polar plot exercises from the previous page:



(a) $r(\theta) = 2$
(b) $r(\theta) = \theta/\pi$
(c) $r(\theta) = -\sin(\theta)$

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna is asked to show that the product of 5 consecutive positive integers can never be a perfect square. Help her resolve this question! Send in your thoughts and suggestions.

Hm. Why 5 consecutive numbers? I'll start by thinking about 1 instead. Actually, 1 number is either a square or not, so I'll start with 2 consecutive integers. Can the product of 2 consecutive positive integers be a perfect square?

2 consecutive integers:

$$n(n+1) = n^2+n$$

$$n^2 < n^2+n$$

If n and $n + 1$ are the 2 consecutive positive integers, then their product is definitely greater than n squared.

$$(n+1)^2 = n^2+2n+1$$

$$n^2 < n^2+n < n^2+2n+1 = (n+1)^2 \qquad n \cdot n < n(n+1) < (n+1)(n+1)$$

And it's also less than $n + 1$ squared...so it's sandwiched between 2 consecutive perfect squares...that came out nicely!

I should have been able to see that more simply, because if one increases a factor in a product of positive numbers, the product will get bigger.

I can say more...that the product of 2 consecutive positive odd, or even, integers is never a perfect square.

In fact $\quad n^2 < n(n+2) < (n+1)^2$

Now, I'll try the product of 3 consecutive positive integers. Hm. I'm not sure what to do... it isn't so clear what square is nearest to this product...

3 consecutive integers:

$$n(n+1)(n+2)$$

$$x^2 < n(n+1)(n+2) < (x+1)^2 \quad ?$$

What is the nearest square to $n(n+1)(n+2)$?

...but I might be able to make headway for a product of 4 consecutive positive integers, because it might be near the square of a quadratic.

4 consecutive integers:

$$n(n+1)(n+2)(n+3) = n(n+1)(n^2+5n+6)$$
$$= n(n^3+6n^2+11n+6)$$
$$= n^4+6n^3+11n^2+6n$$

I'll square this and see what to make a and b to get close to the product of 4 consecutive numbers.

$$(n^2+an+b)^2 = n^4+2an^3+(a^2+2b)n^2+2abn+b^2$$

Comparing coefficients suggests setting $a = 3$ and $b = 1$...

$$\text{Set } a=3: \quad n^4+6n^3+(9+2b)n^2+6bn+b^2$$

$$\text{Set } b=1: \quad n^4+6n^3+11n^2+6n+1$$

$$\Rightarrow \boxed{n(n+1)(n+2)(n+3) = (n^2+3n+1)^2 - 1}$$

...hey, amazing! The coefficients are all the same except for the constant term, which is just 1 off... so the product of 4 consecutive integers is always 1 less than a perfect square!

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Sandwiching between two squares worked for 2 and 4 numbers, but not 3. I don't see how to do 3, so I'll revisit 2 and try to prove it in a different way.

2 consecutive integers:

$n, n+1$ one must be even, and the even one must be divisibly by an even power of 2, that is, must be of the form $2^{2m}x$ where x is odd.

I'll think about factors, since the problem involves products and squares. I'll start with factors of 2.

Actually, the same reasoning works for any prime p... consecutive numbers cannot both be divisible by p.

If p is prime, p can only divide 1 of n or $n+1$, and it must appear in the prime factorization an even number of times. If $n(n+1)$ is a perfect square, then both n and $n+1$ are perfect squares.

Since every prime must appear with even exponent in the prime factorization of any perfect square, it must appear with even exponent in each factor n and $n+1$. But that means both n and $n+1$ must also be perfect squares. But positive perfect squares cannot differ by just 1.

Only 0, 1 are consecutive numbers both perfect squares.

<u>Consecutive numbers are relatively prime.</u>

Let's see if this kind of reasoning can be used to resolve the 3 case.

3 consecutive numbers: $n, n+1, n+2$.

If $p \mid n+1$ then $p \nmid n$ and $p \nmid n+2$

and if $p \mid n$ or $p \mid n+2$ then $p \nmid n$.

$\Rightarrow$ $n+1$ must be a perfect square

and $n(n+2)$ must be a perfect square. $\Leftarrow$ impossible!

It does!

This is encouraging. But before tackling 5 numbers, I'll try to reprove the case of 4 using this kind of reasoning.

4 consecutive numbers:

$n, n+1, n+2, n+3$

Case 1. n odd.

I'm not sure, but it might help to break this into cases depending on whether n is even or odd.

Then $n, n+2$ odd, $n+1, n+3$ even.

$\Rightarrow$ $n, n+2$ are relatively prime.

If n and $n+2$ are odd, then since all odd primes exceed 2, these two numbers must be relatively prime.

If p prime > 3, then only appears as a factor in at most one of $n, n+1, n+2, n+3$, so must appear with even exponent in prime factorization.

If $p = 3$, could have n and $n+3$ divisible by 3.

If 3 doesn't divide n, then all the prime factors of n and of $n+2$ must appear with even exponent in their prime factorizations, that is n and $n+2$ would both be odd perfect squares. That can't happen because they're too close together.

If $3 \nmid n$ then n and $n+2$ are perfect odd squares $\rightarrow$ impossible.

So $3 \mid n$. Must have $n+2 = x^2$, x odd.

Also, I know that an odd squared is 1 mod 4, so $n+3$ must be 2 mod 4, which means that w must be odd.

I'll summarize what I've deduced so far.

n	$n+1$	$n+2$	$n+3$
$3y^2$	$2z^2$	x^2	$6w^2$
y odd	z even	x odd	w odd

Case 2. n even

n	$n+1$	$n+2$	$n+3$
$6z^2$	x^2	$2w^2$	$3y^2$
z even	x odd	w odd	y odd

Hm. Using the same kind of reasoning, I can see this for the case where n is even, but in both cases, I cannot find a contradiction. But there must be one, since I've already proven this in another way!

ABB 8.27.13

Can you help Anna find an argument along these lines of reasoning for 4 consecutive positive integers? And what about the original question for a product of 5 consecutive positive integers?

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Errorbusters!

by Cammie Smith Barnes / edited by Jennifer Silva

Last week, a middle school student whom I tutor was studying what her teacher called "two-step equations," algebraic equations that take two steps to solve. At first, the student sometimes made a type of error that I'll call "forgetting to distribute." By this, I mean that the student would take the equation

$$3x + 6 = 21$$

and try to solve it by first dividing the $3x$ by 3 and the 21 by 3 to get

$$\frac{3x}{3} + 6 = \frac{21}{3},$$

or

$$x + 6 = 7.$$

There's no error in choosing to divide the equation by 3 as the first step, but one must carry out the operation properly. The *entire equation*, including the 6, must be divided by 3 to get

$$\frac{3x+6}{3} = \frac{21}{3},$$

then

$$\frac{3x}{3} + \frac{6}{3} = \frac{21}{3},$$

or

$$x + 2 = 7.$$

Next, one could subtract 2 from both sides to get $x = 5$. It is crucial to divide the entire expression on each side of the equation by 3.

Another approach to solving the equation would be to reverse the operation of addition first, rather than the operation of multiplication. In other words, starting with the original equation of $3x + 6 = 21$, one could subtract 6 from both sides to get $3x = 15$, then divide both sides by 3 to get $x = 5$, as before.

Let's look at another two-step equation:

$$\frac{x}{5} - 4 = 16.$$

It's a fine approach to first multiply both sides of the equation by 5, but one must be sure to distribute properly. It would be incorrect to multiply just the $x/5$ and 16 by 5 to get

$$5\left(\frac{x}{5}\right) - 4 = 5(16),$$

or $x - 4 = 80$. Instead, one must distribute the factor of 5 throughout the equation. The correct continuation is

$$5\left(\frac{x}{5} - 4\right) = 5(16),$$

or

$$x - 20 = 80.$$

Adding 20 to both sides yields $x = 100$, which is the correct answer.

Alternatively, we could begin by adding 4 to each side of $x/5 - 4 = 16$ to get $x/5 = 20$. Then, we can multiply both sides by 5 to get $x = 100$.

Let's try solving one more equation:

$$4(x + 3) = 48.$$

What should our first step be?

There are actually many choices. The general principle at work here is that if you have two equal quantities and you change both *in the same way*, then the resulting quantities will also be equal. To *solve* the equation, our goal is to obtain an equation that has x on one side and a number on the other.

So suppose we decide that we would like to manipulate the equation so that the left-hand side becomes $4x$. In terms of solving the equation, achieving such an equation (with $4x$ on one side and no x on the other) would represent progress, because we would then be one step away from isolating x. To accomplish this, however, it would be incorrect to think that we can subtract 3 from both sides to arrive at the unrelated equation

$$4x = 45.$$

What went wrong? The mistake is that $4(x + 3) - 3$ is not equal to $4x$. In the original equation, the factor of 4 applies to the entire expression $x + 3$, not just the x. So to transform the left-hand side to $4x$, we really need to subtract by 4(3) or 12 from both sides:

$$\begin{aligned} 4(x + 3) - 4(3) &= 48 - 4(3) \\ 4x &= 36 \end{aligned}$$

Now we can divide both sides by 4 to obtain the correct answer of $x = 9$.

Alternatively, we could first divide both sides of the equation by 4. This would lead to the equation $x + 3 = 12$. From there, we could subtract 3 from both sides to obtain $x = 9$. Notice that we don't have to worry about distribution when we solve the equation in this order (divide by 4 then subtract 3). Why is that? It is because the left-hand side of the original equation shows an implied *order of operations* in which the addition of 3 takes place first and the multiplication by 4 happens last. It is natural to unravel things in reverse. So if multiplication by 4 is done last, it would be natural to unravel by first dividing throughout by 4.

Is it always best to unravel an expression by reversing the order of operations?

The answer is no. It depends on what you are trying to do with the equation and what your own personal strengths are. It would be really great if you can train yourself to be comfortable solving problems in multiple ways.

Let's revisit the first equation we looked at:

$$3x + 6 = 21.$$

According to the order of operations, multiplication is done before addition, so in $3x + 6$ the addition is done last. If we insist on reversing the order of operations to unravel, we will then subtract 6 from both sides first. But we have seen that doing so is not absolutely necessary. As long as you manipulate equations properly, you will obtain a valid solution.

Sometimes, it may even seem more practical to deliberately *not* undo the order of operations by reversing the order. For example, consider the equation

$$101x - 2929 = 9999.$$

Below are two different ways to solve this equation. Which one do you find to be simpler?

Method 1
(undo subtraction, then multiplication)

$$\begin{aligned} 101x - 2929 &= 9999 \\ 101x - 2929 + 2929 &= 9999 + 2929 \\ 101x &= 12928 \\ \frac{101x}{101} &= \frac{12928}{101} \\ x &= 128 \end{aligned}$$

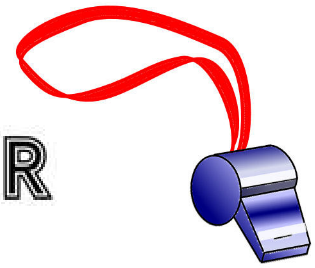
Method 2
(undo multiplication, then subtraction)

$$\begin{aligned} 101x - 2929 &= 9999 \\ \frac{101x - 2929}{101} &= \frac{9999}{101} \\ x - 29 &= 99 \\ x - 29 + 29 &= 99 + 29 \\ x &= 128 \end{aligned}$$

Now let's take the equation $125x + 125 = 500$. How would you solve it?

For practice, try to solve each equation in more than one way. If you use multiple valid approaches for an equation, they should result in the same answer. Answers can be found on page 29.

1. $2x + 8 = 18$
2. $8x + 2 = 18$
3. $\frac{x}{3} - 6 = 15$
4. $\frac{x}{6} - 3 = 15$
5. $4(x - 6) = 24$
6. $6(x - 4) = 24$
7. $\frac{x + 4}{5} = 20$
8. $\frac{x + 5}{4} = 20$



COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

Owning it: Fraction Satisfaction, Part 12

There's 3/7 sitting on a park bench, out of breath and staring dully. Something is not right.

You: Hi, 3/7. What's going on?

$\mathbf{\frac{3}{7}}$: I have been bored almost to death. Please amuse me.

You: Well, I learned quite a bit about fractions and decimals. Want to hear about that?

$\mathbf{\frac{3}{7}}$: Desperately.

You: First, I thought it was going to be dull, since we started off with things I knew. I tried to pay attention, because I figured at some point the teacher was going to jump to something new. I didn't want to miss the jump, since that's the point where the old stuff helps makes sense of the new stuff.

$\mathbf{\frac{3}{7}}$: Well said. I must say that you are admirably sensible. Tell me about the jump.

You: Well, we started with how numbers with decimal points are an extension of our base 10 system. In that system, the rightmost digit would have a place value of 1, and the place value would increase by a factor of 10 as you moved to the left. Moving over to the right made the place value decrease by a factor of 10, but we'd stop when the place value was 1. However, instead of stopping at the ones digit, we can put a decimal point – a dot – to its right and keep on going. So the digit just to the right of the decimal point has a place value of 1/10. A second digit to the right of the decimal point has place value 1/100, and so on.

$\mathbf{\frac{3}{7}}$: Are we at the jump yet?

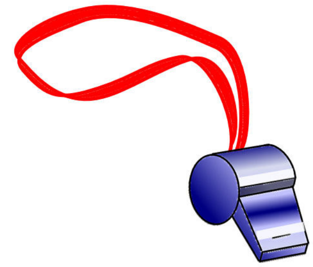
You: No, but hold on. That was enough to make it clear that 0.1 = 1/10, that 0.7 = 7/10, and that 0.27 = 2/10 + 7/100 = 27/100. At that point, I was thinking I could translate any decimal into a fraction, and I was trying to resist tuning out.

$\mathbf{\frac{3}{7}}$: I hope you didn't tune out, darling, because then you would have missed two fascinating things, the first of which is ...

You: Repeating decimals!

$\mathbf{\frac{3}{7}}$: I do wish you would not interrupt, darling, even in the face of your invigorating enthusiasm. You may redeem yourself by explaining them.

You: Sorry. Well, then my teacher talked about 0.33333333333333333333333333333333..., with the 3s going on forever.



$\frac{3}{7}$: How does one arrive at that decimal representation of a fraction?

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Thank you and best wishes,
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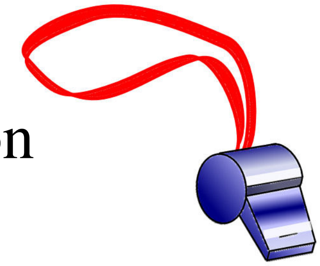


Summer Fun!

In the last issue, we invited members to submit solutions to a batch of Summer Fun problem sets.

In this issue, we give solutions to many of the problems. These solutions will sometimes be rather terse and, in some cases, are more of a hint than a solution. We prefer not to give completely detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil and move your mind! You might discover something new to people.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members and Subscribers: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Perspective Drawing

by Ken Fan

1. A perspective drawing of a dot is a dot on the paper. If we want to be really technical, we can never really make an actual drawing of such an idealized dot because any mark we make on paper is going to have some size. We have to imagine being able to draw an idealized dot. Drawings of idealized dots all look the same, except for the location of the dot on the canvas.

2. The position of the dot on the canvas will indicate to the person looking at the picture that the model dot is somewhere along the ray that originates at the observer's eye and passes through the dot on the canvas. Points along this ray could be any distance from the observer. Therefore, it's impossible to determine how far away the model dot is from the observer. All the points along this ray are possible locations for the model dot. In fact, there might even be several model dots on this ray. You wouldn't be able to tell from the drawing.

3. Perspective drawings of line segments are line segments or dots. If the model line segment is part of a ray that emanates from the viewer's eye, then the drawing of the model line segment will appear as a dot on the canvas. Otherwise, the drawing of the model line segment will be a line segment on the canvas.

4. Consider a model line segment that does appear as a line segment on the canvas. Imagine the two rays that begin at the eye of the viewer and pass through the endpoints of the model line segment. The drawn line segment on the canvas that depicts the model line segment has endpoints where these two rays intersect the plane of the canvas. Avoiding the feet of these rays, if you take any other point on one of these rays and connect it to any other point on the other ray, you will have constructed another model line segment that is depicted by the same drawing. Since these constructed line segments could have any length and could point in many directions, there is no way to tell how long the model line segment is or in what direction it points from the drawing alone.

5. Perspective drawings of lines can look line lines or dots. If a line happens to pass through the eye of the viewer, then the drawing of the line will appear as a dot on the canvas. Otherwise, the drawing will consist of a line on the canvas. Any line that is contained in a vertical plane in front of the viewer's eye will appear as a vertical line on the canvas. (Recall that we are assuming that the drawing canvas is perpendicular to a horizontal ground.)

6. If the parallel lines are not parallel to the drawing canvas, then a drawing of them will look like rays emanating from a single point. This single point may not actually be depicted on the drawing canvas because the canvas may be too small. Specifically, this point of convergence can be found by looking at the plane of the canvas in the direction of the parallel lines.

Notice that a model consisting of rays emanating from a single point (such as sunrays) will also be depicted on the canvas by rays emanating from a single point. So if you have a drawing that shows a bunch of rays emanating from a single point, how can you know whether it is a drawing of parallel lines or not? You can't tell! There's not enough information.

If the parallel lines are also parallel to the drawing canvas, then a drawing of them will look like parallel lines on the canvas.

Summer Fun!

Bonus Question

If you make a drawing of the ocean on a calm day, the edge where the water and sky meet does not actually correspond to the "horizon line" of perspective drawing. The earth is not a flat plane. In fact, a careful drawing of this edge is not even a straight line. What curve is it? For a hint, see this issue's *Math In Your World*.

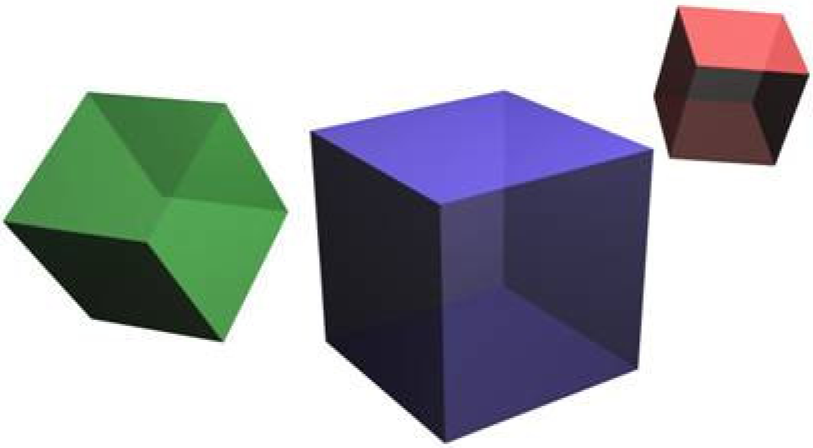
7. See the solution to #6.

8. The vanishing points all lie along the line that is the intersection of the plane of the drawing canvas with the horizontal plane that contains the viewer's eye. This line is called the **horizontal line** because drawings of horizons look like them. (Although, see the box at left.)

9. To make a realistic drawing of a checkered floor, you cannot pick the vanishing points for the 4 sets of parallel lines (described in the problem statement) willy-nilly. The 2 rays from the viewer's eye to the vanishing points corresponding to the sides of the tiles must create a right angle, and the rays associated with the diagonals must make 45° angles with the "side" rays.

10. Hint: Consecutive slats in the railroad track, together with the railing, create a series of rectangles. The diagonals of these rectangles are all parallel to each other and so converge in a common vanishing point.

11. The 3 cubes at right are renderings of perfect cubes in the same scene. That is, all 3 are designed to give the illusion of perfect cubes from a specific vantage point. For another example, see **Rowena**'s article in *Member's Thoughts* and the cover of Volume 3, Number 4 of this Bulletin.



12. See Joel Babb's drawing on the cover and the painting by Canaletto on page 28.

13. The stereo pair works by presenting the two slightly different images that each eye would perceive when looking at a 3D world. By crossing your eyes so that your left eye receives the right image and your right eye receives the left image, your mind will combine the two images into the illusory perception of a 3D world.

14. There are several implications that artists must be aware of in order to render realistic portraits. The closer the subject is to the viewer, the more pronounced the effects.

Summer Fun!

Vieta's Formulas

by Shravas Rao

1.

Quadratic	Sum of Roots	Product of Roots
$x^2 - 6x$	6	0
$x^2 - 4x + 3$	4	3
$x^2 - x - 30$	1	-30
$x^2 + 7x + 6$	-7	6
$x^2 - 3x - 28$	3	-28

2. We could find the roots by applying the quadratic formula. But we'll use a different method. Let the roots of the quadratic be p and q. Then the polynomial must be $(x - p)(x - q)$, since the coefficient of x^2 is 1. (Why?) This expands to $x^2 - (p + q)x + pq$. Therefore, the sum of the roots is $-b$ and the product of the roots is c.

3.

Quadratic	Sum of Roots	Product of Roots
$2x^2 + 8x + 4$	-4	2
$3x^2 - 6x - 18$	2	-6
$5x^2 - 15x + 5$	3	1
$2x^2 + 5x + 2$	-5/2	1
$4x^2 - 12x - 16$	3	-4

4. We will use the same technique as that used in #2. If the polynomial has roots p and q, then the polynomial must be $a(x - p)(x - q)$, since the coefficient of x^2 is a. This multiplies out to $ax^2 - a(p + q)x + apq$. Therefore, the sum of the roots is $-b/a$ and the product of the roots is c/a.

5. This time, we won't explicitly find the roots. Using #2, we can find that the sum of the roots of the quadratics $x^2 + 2x - 4$ is -2, and the product of its roots is -4. Using our solution to #4, the sum of the roots of the quadratic $3x^2 + 4x + 5$ is -4/3 and the product of its roots is 5/3.

6. You could solve for the roots and then substitute these into the expression $3/p + 3/q$, but instead, we'll exploit what we've learned so far. Notice that $3/p + 3/q = 3(p + q)/(pq)$. From #2 (or #4), we know that $p + q = -6$ and $pq = 3$. Therefore $3/p + 3/q = 3(-6)/3 = -6$.

7. We can write $p^2 + q^2 = (p + q)^2 - 2pq$, then use #4 to find $p + q = -2$ and $pq = 4$. Substituting, we find $p^2 + q^2 = -4$. Note that the sum of the squares of the roots is negative, which means that p and q are non-real complex numbers, yet we were still able to answer the question without even working with complex numbers!

8. Let $k = p + 4/p = q + 4/q$. Then $p^2 - kp + 4 = 0$ and $q^2 - kp + 4 = 0$. In other words, p and q are roots of the quadratic equation $x^2 - kx + 4 = 0$. Therefore, $pq = 4$.

9. If p and q are the lengths of the legs of the right triangle, then the length of its hypotenuse is $\sqrt{p^2 + q^2}$. We know that $p + q = 13$ and $pq = 25$. Thus, (see #7), $p^2 + q^2 = 13^2 - 2(25) = 119$. The length of the hypotenuse is then $\sqrt{119}$.

Summer Fun!

10.

Cubic	Sum of Roots	Product of Roots
x^3	0	0
$x^3 - x$	0	0
$x^3 + 3x^2 + 3x + 1$	-3	-1
$x^3 + 3x^2 - x - 3$	-3	3

11. If the roots of a cubic in x are p, q, and r, and the coefficient of x^3 is 1, then the polynomial must be $(x - p)(x - q)(x - r)$. This expands to $x^3 + (-p - q - r)x^2 + (pq + qr + rp)x - pqr$. Therefore, the coefficient of x^2 is $-p - q - r$, the coefficient of x is $pq + qr + rp$, and the constant term is $-pqr$.

12. We can calculate $(p + q + r)(p + q + r)$ by squaring the sum of the roots. Using #11, we know that $p + q + r = 4$, so $(p + q + r)(p + q + r) = 16$. Now we will calculate $p^2 + q^2 + r^2$. Note that $(p + q + r)^2 = p^2 + q^2 + r^2 + 2pq + 2qr + 2rp$. Using #11, we know $pq + qr + rp = 2$. Hence $p^2 + q^2 + r^2 = 16 - 2(2) = 12$.

13. The coefficient of x^2 is $-a(p + q + r)$. The coefficient of x is $a(pq + qr + rp)$. The constant term is $-apqr$.

14. Let p, q, and r be the length, width, and height of the block. The surface area is $2(pq + qr + rp)$, and from #11, we know $pq + qr + rp = 120$. Thus, the surface area is 240. The volume is pqr, which from #11, we know is 210.

15. If you connect the center of the incircle to each of the vertices, you split the triangle into 3 triangles. These 3 triangles have bases equal to the side lengths p, q, and r of the triangle and heights with respect to these bases equal to the radius of the incircle. Therefore the area of the triangle is equal to sr. Hence $sr = \sqrt{s(s-p)(s-q)(s-r)}$. Using #11, we know that $s = 9/2$. Also, $(s - p)(s - q)(s - r)$ is just the value of the given cubic evaluated at s. Thus, we find $(9/2)r = \sqrt{(9/2)((9/2)^3 - 9(9/2)^2 + 26(9/2) - 24)}$. The rest is simplification.

16. Let p, q, and r be the roots of the cubic. By the arithmetic-geometric mean inequality, we have $(p + q + r)/3 \geq \sqrt[3]{pqr}$. By #11, we know that $pqr = 1$. Hence $-a = p + q + r \geq 3$, or $a \leq -3$.

17. The coefficient of x^k is $(-1)^{d-k}aS$, where S is the sum of all possible products obtained by multiplying k of the roots together. Using sigma notation, we can write this as

$$(-1)^{d-k} a \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq d} p_{i_1} p_{i_2} \cdots p_{i_d} .$$

18. All of the formulas for the coefficients in #17 are **symmetric** with respect to the roots. Another way to see this is that permuting the roots changes the order of factors in the product $a(x - p_1)(x - p_2) \cdots (x - p_d)$, but multiplication is commutative.

Summer Fun!

Z, $Z[\sqrt{-1}]$, and $Z[\sqrt{-5}]$

by Addie Summer

1. The units in Z are 1 and -1.

2. The number 17 is not a unit and cannot be expressed as a product of two integers, neither of which is a unit. Other examples are -101, -5, 7, and 666,667.

Recall that $Z[i]$ consists of the complex numbers $a + bi$ where a and b are integers.

3. Suppose $a + bi$ and $c + di$ are both in $Z[i]$. In other words, a, b, c, and d are integers. Their sum is $a + bi + c + di = (a + c) + (b + d)i$. Since $a + c$ and $b + d$ are integers too, this sum is in $Z[i]$. Their product is $(a + bi)(c + di) = ac + adi + bci + bdi^2 = ac - bd + (ad + bc)i$. Since Z is closed under addition and multiplication, both $ac - bd$ and $ad + bc$ are integers. Hence the product is in $Z[i]$.

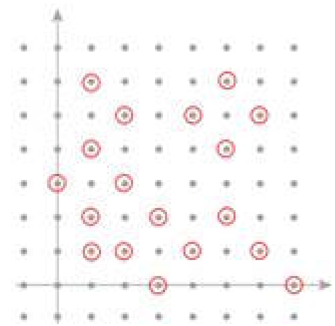
4. We have to check that $(a+bi)\left(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\right) = 1$. We multiply this out:

$$\begin{aligned}
(a+bi)\left(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\right) &= a\frac{a}{a^2+b^2}-a\frac{b}{a^2+b^2}i+b\frac{a}{a^2+b^2}i-b\frac{b}{a^2+b^2}i^2 \\
&= \frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}-\frac{ab}{a^2+b^2}i+\frac{ba}{a^2+b^2}i \\
&= \frac{a^2+b^2}{a^2+b^2} \\
&= 1
\end{aligned}$$

The units in $Z[i]$ are 1, -1, i, and $-i$.

5. In $Z[i]$, we can write $5 = (2 + i)(2 - i)$, and neither $2 + i$ nor $2 - i$ is a unit in $Z[i]$.

6. See figure at right. (Also, note the watermark on this page!)



7. Yes, $Z[i]$ does have unique factorization.

Let $Z[\sqrt{-5}]$ denote the set of complex numbers of the form $a + b\sqrt{-5}$.

8. The units in $Z[\sqrt{-5}]$ are 1 and -1.

9. They are $\sqrt{-5}$, $1-3\sqrt{-5}$, $1-2\sqrt{-5}$, $1-\sqrt{-5}$, $1+\sqrt{-5}$, $1+2\sqrt{-5}$, $1+3\sqrt{-5}$, $2-3\sqrt{-5}$, $2-\sqrt{-5}$, 2, $2+\sqrt{-5}$, $2+3\sqrt{-5}$, $3-4\sqrt{-5}$, $3-2\sqrt{-5}$, $3-\sqrt{-5}$, 3, $3+\sqrt{-5}$, $3+\sqrt{-5}$, $3+4\sqrt{-5}$, $4-3\sqrt{-5}$, $4-\sqrt{-5}$, $4+\sqrt{-5}$, $4+3\sqrt{-5}$, and their additive inverses.

10. We can write $6 = 2 \times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

Summer Fun!

The Meddling Gnomes

by Lauren McGough

1. a. 4.
b. The girls could see any color combination in the morning!
c. No.
d. The gnomes can control the color of each stick without affecting the other sticks. Therefore, they can set the colors of the strings to any combination they wish. Therefore, no communication is possible.

2. a. As in problem 1, there are four color sequences: RR, RB, BR, and BB. The situation is a little bit different from that of problem 1 because if a gnome touches the middle stick, then both of the strings flip colors! But the end result is the same: any color combination could appear in the morning.
b. Still, no communication is possible no matter how many sticks are used. Suppose there are n sticks in a row, and a string between each consecutive pair of sticks. A group of $n - 1$ gnomes decides on a sequence of reds and blues that they want to create. The gnomes figure out a way to do this: one gnome stands next to each stick *except* the leftmost stick. Then, each gnome turns to face the left. They take turns one by one from left to right. First, the leftmost gnome looks at the string directly to its left. If that string is the correct color, the gnome does nothing; otherwise, the gnome touches its stick once. The next gnome does the same thing: it looks at the string to its left. If it's the right color, the gnome does nothing; otherwise, it touches its stick once. Notice that when a gnome changes a string to its left, it will also change a string to its right, unless it happens to be the rightmost gnome. It seems like the gnomes might be messing each other up. However, the key is that even if one gnome changes the "wrong" string, *the next gnome fixes it,* and it *never* gets changed again.

3. a. If the gnomes do not touch either stick, the girls will see RR. If the gnomes touch one stick exactly once, the girls will see BB – no matter which stick it is. Touching a stick a second time will bring the colors back to RR. The pattern continues: an odd number of touches creates BB; an even number creates RR. These are the only possible sequences the girls could see in the morning.
b. Sarah could have set either BB or RR the night before.
c. Say that an even number of reds means "yes," and an odd number of reds means "no." Since the gnomes cannot change whether the number of reds is even or odd (this is called the *parity* of the number of reds) by touching the sticks, the gnomes cannot change a "yes" to a "no" or vice versa in this scheme.
d. There is no "maybe" level possible! If the girls start with RR, they can end with either RR or BB in the morning; if they start with RB, they can end with either RB or BR in the morning. This leaves only two groups of possibilities – not three. So, "yes" and "no" are the only options in this design.
e. No matter how many sticks there are, there are only two messages possible. One way to see this is to consider a variant of the strategies used in #1 and #2. We create a loop with n sticks by taking a line, like in #2, and connecting the first and last sticks with a string. Call this string the "connecting" string. By pretending that the connecting string is not there, the gnomes can start with any sequence of n colors, choose any sequence of $n - 1$ colors for the non-connecting strings, and carry out the strategy used in #2 to make the non-connecting sequence follow the chosen color pattern. The only string they are not able to control by this method is the connecting string. The question becomes: is it possible for the gnomes to come up with a more clever strategy that will allow them to control the color of *all* the strings? (Think about this for a moment before reading ahead!)

Given the work you've already done, you immediately have reason to believe that it shouldn't be possible – they did not have full control over the color of all strings in the two-string case! In fact, this is still true in the case of a larger loop, and for essentially the same reason: the gnomes cannot change the parity of the number of red strings (or, equivalently, blue strings) by touching the sticks. Why is this true? Consider what happens when a gnome touches a single stick. The gnome changes the colors of exactly two strings, namely, the two strings attached to this stick. There are 4 cases to consider. The 2 strings can be BB, RR, BR, or RB. When the gnome touches the stick, BB becomes RR, RR becomes BB, BR becomes RB, and RB becomes BR. In each case, one can check that the total number of blue strings changes by an even number (including, possibly, 0). The total number of red strings can also only change by an even number. Therefore, parity is conserved.

It follows that no matter how big the loop is, there is one thing that determines whether or not the gnomes can transform one sequence of colors to another: whether the number of red strings is even or odd. (Actually, we could have said "blue" instead of "red" – either color works as long as we choose one. From now on we will choose red, but this is arbitrary.)

Summer Fun!

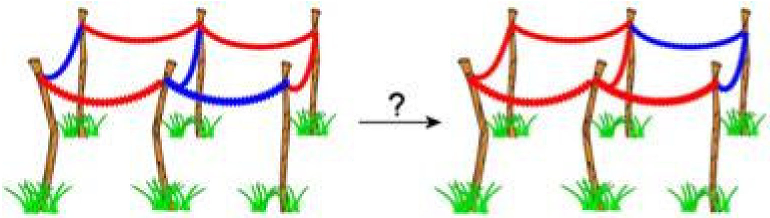
4. The girls can send 4 different messages. Call a loop "0" if it has an even number of reds going around, and "1" if it has an odd number of reds. Remember that if a loop is 0 at night, it will still be 0 in the morning, and vice versa. Then the 4 possible messages are given by "00," "01," "10," and "11."

5. Now, the girls can choose either "0" or "1" for each loop. Every time they add a new loop, the number of possible messages multiplies by 2: an old message M (a sequence of 0s and 1s) gives rise to two new messages, M0 and M1, where the last digit is the "value" of the new loop. So, an n-loop setup can communicate 2^n different meanings. Remember that the labeling system of "0" and "1" works for any loop size, since for any loop size, the girls can only control the parity of the number of red strings. So within each loop, having 2 strings is just as effective as having 200!

6. We will build up the $2 \times n$ grid one loop at a time in order to figure this out. Let's start with 2×2. That's just a loop with 4 strings, and we know the answer for that: there are two possible messages and we can label them as "0" and "1" depending on whether there are an even or odd number of red strings, respectively.

Now, let's add on two more sticks to make a 2×3 grid.

First, we can apply what we know about single loops to this situation: The gnomes can't change the parity of the number of red strings around any loop. So if we think of our grid as 2 loops, side by side (sharing a common string), we already know that the gnomes can't change a "00" configuration to a "01" configuration. And, since it is possible color the strings to represent any of the 4 possible configurations "00," "01," "10," or "11," we know that there are *at least* 4 messages possible.



But we're not done! The 2 loops share a common string, and this gives the gnomes less control. That opens up the possibility that there are more messages possible. For example, it may be that there is a "00" configuration that the gnomes cannot change to another "00" configuration. Suppose the girls set a "00" configuration, and the gnomes pick some *other* "00" configuration. Is it always possible for them to change the girls' configuration to theirs? Suppose they set a gnome at each stick, as before. We already know that they can go from any "0" loop to any other "0" loop, so if they focus on the left loop and forget about the right loop for now, they can make it match their configuration.

Now, the question is: can they fix the other three strings without changing the four strings they've fixed? Consider the two gnomes all the way on the right edge. The top one looks at the top string. If it is the right color, the top one does nothing during its turn; otherwise it touches its stick. Then, it is the bottom gnome's turn. The bottom gnome looks at the rightmost string oriented vertically. If it is already the right color, it does nothing; otherwise it touches its stick.

Hmm... now there seems to be a problem. There is no gnome to fix the last string without potentially messing up the left loop! What to do? It seems we are stuck, but we have to consider all the possibilities. In particular... does the last string ever *need* fixing? Fortunately, it does not! We *assumed* the gnomes picked a "00" configuration – they had to, otherwise they would have been doomed from the start. But since we *know* they have a "00" configuration, and we know that three of the colors going around the loop are correct, the fourth one is *automatically* correct; otherwise, we wouldn't have a "0" loop! So, in fact, all "00" configurations are equivalent. But nothing about the argument depended on the values being "0" *per se*; "01," "10," or "11" would all have worked just as well. Therefore, there are exactly 4 possible messages.

The exact same argument works to show that the number of classes for a $2 \times n$ grid equals 2^{n-1} – that is, each "new loop" multiplies the number of possible messages by 2. Can you see why the same argument applies?

7. In our solution for #6, we analyzed the situation by carefully building up the grid loop by loop and found that each time we added a loop, the number of possible messages doubled. Although in that argument, the new loops were all obtained by adding 3 more strings, a similar kind of argument can be constructed to analyze what happens if you build up an $m \times n$ grid loop by loop, even if a new loop is formed by adding 2 more strings instead of 3. One can show that adding a loop doubles the number of possible messages. (Check this assertion carefully!) Since it takes $(m-1)(n-1)$ loops to build an $m \times n$ grid, there are $2^{(m-1)(n-1)}$ possible messages.

8. No right answer! We would love to see your designs and analysis.

Summer Fun!

The Gauss-Wilson Theorem

by Robert Donley

Gauss-Wilson Theorem: If $n = 4$, or n is a power of an odd prime, or n is twice a power of an odd prime, then $U_n! = -1 \pmod{n}$. Otherwise, $U_n! = 1 \pmod{n}$.

1. For example, when $n = 10$, we have $U_n = \{1, 3, 7, 9\}$ and $1 \times 3 \times 7 \times 9 = 189 = -1 \pmod{10}$.

2. Recall that x^{-1} is the unique element in U_n such that $xx^{-1} = 1 \pmod{n}$. If $x^2 = 1 \pmod{n}$ then multiplying both sides of the equation by x^{-1} gives $x = x^{-1} \pmod{n}$. In the other direction, if $x = x^{-1} \pmod{n}$, multiply both sides by x to get $x^2 = 1 \pmod{n}$.

3. From #2, if an element x in U_n does not satisfy $x^2 = 1 \pmod{n}$, then x and x^{-1} are distinct elements in U_n. Thus, we may split the product $U_n!$ into two parts. One part is over all elements with $x \neq x^{-1} \pmod{n}$; these cancel in pairs to give product 1. The product of the remaining terms is $I_n!$, by definition.

4. For example, when $n = 10$, we have $I_n = \{1, 9\}$ and $U_n! = I_n! = 1 \times 9 = -1 \pmod{10}$.

5. There are at least 2 elements with $b^2 = 1 \pmod{n}$: 1 and -1. (Recall that $n > 2$.) On the other hand, every element b in U_n may be written uniquely as $b = a^k$ for some $0 \leq k < C$, where C is the number of elements in U_n. If $b^2 = 1 \pmod{n}$, then $a^{2k} = 1 \pmod{n}$, and this implies that C divides $2k$. Since $0 \leq k < C$, we must have $k = 0$ or $C/2$. Hence, 1 and -1 are exactly the elements in I_n.

6. See *Primitive Roots* in the previous issue of this Bulletin.

7. By #2, we know $b = b^{-1} \pmod{n}$. If $ab = 1 \pmod{n}$, then $a = b^{-1} = b \pmod{n}$. If $ab = a \pmod{n}$, then $b = 1 \pmod{n}$, and likewise if $ab = b \pmod{n}$, then $a = 1 \pmod{n}$.

8. Suppose I_n has more than 2 elements. We build I_n using induction. First choose any nonidentity element s_1 in I_n and define $S_1 = \{1, s_1\}$. Note that S_1 is closed under multiplication. Next, choose any element in I_n not in S_1, say s_2. Then by #7, the set $S_2 = \{1, s_1, s_2, s_1s_2\}$ consists of 4 distinct elements. Also, by checking all cases, we see that S_2 is closed under multiplication. If $S_2 = I_n$, we have the desired result and $I_n = S_1 \cup s_2S_1$.

Otherwise, we repeat the process: at the k-th stage, S_k has 2^k elements and is closed under multiplication. We choose s_{k+1} in I_n but not in S_k. Note that $s_{k+1}S_k$ is disjoint from S_k, for if $s_{k+1}a = b$ where a and b are both in S_k, then $s_{k+1} = ba^{-1}$, which would be in S_k since S_k is closed under multiplication. Define $S_{k+1} = S_k \cup s_{k+1}S_k$. By #7, we know S_{k+1} is contained in I_n. Since $s_{k+1}S_k$ and S_k have the same number of elements (why?), S_{k+1} has 2^{k+1} elements. As I_n has only finitely many elements, this process eventually ends, and $I_n = S_{k'} \cup s_{k'+1}S_{k'}$ for some k'.

9. If I_n has only 2 elements, then $I_n! = -1 \pmod{n}$ by #5. Otherwise, #8 shows that $I_n! = (s_{k'+1}{}^2)^{k'} = 1 \pmod{n}$.

Summer Fun!

This could be a solution to problem 12 of the Perspective Summer Fun problem set on page 20! This is an oil painting of the Plaza at San Marco in Venice, Italy, by Canaletto. You can see the original painting at the Fogg Art Museum at Harvard University. Is the plaza a rectangle? Verify your answer using the satellite view on Google Maps.

10. For part 1, suppose m and n are relatively prime. If r is relatively prime to mn, then r is relatively prime to both m and n. Thus, we can associate each r in U_{mn} with a unique ordered pair (r_1, r_2) with r_1 in U_m and r_2 in U_n by taking $r_1 = r \pmod{m}$ and $r_2 = r \pmod{n}$. Also, $|U_n| = \varphi(n)$ and $\varphi(mn) = \varphi(m)\varphi(n)$, since m and n are relatively prime. For a proof of these facts, see *An Euler φ-for-All* in the previous issue of this Bulletin. Thus every ordered pair (r_1, r_2) with r_1 in U_m and r_2 in U_n can be obtained from some r in U_{mn} in this manner. Observe that $r^2 = 1 \pmod{mn}$ if and only if $r_1^2 = 1 \pmod{m}$ and $r_2^2 = 1 \pmod{n}$. Hence $|I_{mn}| = |I_m||I_n|$.

We shall sketch a solution for parts 2 and 3. For details, see Ireland and Rosen's *A Classical Introduction to Modern Number Theory*, pp. 41-44.

For part 2, let g be a primitive root for the odd prime p. We claim $g^{p-1} \neq 1 \pmod{p^2}$ or $(g+p)^{p-1} \neq 1 \pmod{p^2}$. To see this, note that $(g+p)^{p-1} = g^{p-1} + (p-1)g^{p-2}p \pmod{p^2}$. That is, $(g+p)^{p-1} - g^{p-1} = (p-1)g^{p-2}p \pmod{p^2}$. Since p does not divide $(p-1)g^{p-2}$, this says that $(g+p)^{p-1} - g^{p-1} \neq 0 \pmod{p^2}$. Hence, we cannot have $g^{p-1} = (g+p)^{p-1} = 1 \pmod{p^2}$. On the other hand, both g and $g + p$ are primitive roots modulo p. Thus, we have a primitive root h modulo p that satisfies $h^{p-1} \neq 1 \pmod{p^2}$. We claim that h generates U_{p^t} for all $t > 1$.

For part 3, show that the numbers $(-1)^a 5^b$ where $0 \leq a \leq 1$ and $0 \leq b < 2^{t-2}$ constitute all of the elements if U_{2^t}, when $t > 2$. From this, one can proceed by computing directly.

Summer Fun!

Calendar

Session 13: (all dates in 2013)

September	12	Start of the thirteenth session!
	19	No meet
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	
	12	

Here are answers to the Errobusters! questions on page 14.

1. $x = 5$
2. $x = 2$
3. $x = 63$
4. $x = 108$
5. $x = 12$
6. $x = 8$
7. $x = 96$
8. $x = 75$

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Girls’ Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don’t like math, what don’t you like? If you love math, what do you love? What would you like to get out of a Girls’ Angle Membership?

The $36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $36 for a 1-year Girls’ Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.

- ☐ Enclosed is $216 for one session (12 meets)
- ☐ I will pay on a per meet basis at $20/meet.

Nonmembers: Please choose one.

- ☐ I will pay on a per meet basis at $30/meet.
- ☐ I'm including $36 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____