# Girls' Angle Bulletin

October 2013 • Volume 7 • Number 1

*To Foster and Nurture Girls' Interest in Mathematics*

# From the Founder

It's been a while since we had a *Member's Thoughts* column, but this time, we do, and I want to draw special attention to it. One of our members asked a very natural question about the spiral of squares in the golden rectangle: What are the side lengths and areas of the successive rectangles? She ended up rediscovering some beautiful facts about the Fibonacci numbers. Her exploration exemplifies the process of math research in microcosm: get curious about something, ask questions, try to answer them …

- Ken Fan, President and Founder

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## Girls' Angle Bulletin

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## Girls' Angle:
### A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*



On the cover: Romanesco broccoli by Toshia McCabe. This is an example of self-similarity in nature. For more on self-similarity see this issue's *Math Buffet*.

# An Interview with Anne Shiu

Anne Shiu is an L. E. Dickson Instructor and a National Science Foundation Postdoctoral Fellow in the Department of Mathematics at the University of Chicago. She received her doctoral degree in mathematics with a designated emphasis in genomic and computational biology from the University of California, Berkeley under the supervision of Lior Pachter and Bernd Sturmfels. This interview was conducted by Sisi Liu, a high school summer intern at Girls' Angle and a student at Groton-Dunstable Regional High School in Groton, MA.

**Sisi**: How and when did you know you wanted to be a mathematician?

*If possible, I try first to understand a special or small case of the problem, and then ask if I can generalize what I've learned.*

**Anne**: Early in life I didn't like math, but a turning point for me was in fifth grade. It came as a great surprise to both me and my teacher when I got the highest score in the class on a standardized math exam – it was a boost of confidence for me, especially as I wasn't one of the two students previously selected for the top math track. Then, throughout junior high and high school, I discovered that I like learning math and solving math problems – plus, I enjoyed explaining math to fellow students. When I arrived at the University of Chicago for college, I knew I wanted to be a math major. During those four years, I found how rewarding a study group for math can be – in each of my math classes, a few of us students would meet weekly to work on the homework together. From my professors and teaching assistants, I started to find out what a mathematician is and was encouraged to pursue a Ph.D. in mathematics. Graduate school, which for me was at the University of California Berkeley, is like an apprenticeship program for mathematicians. It is where you learn more mathematics, and then begin doing research with guidance from a professor with the goal of writing a dissertation. Along the way, you improve at communicating mathematics (through teaching, reading, writing, and giving talks), and learn from older students about how to navigate the upcoming years – passing qualifying exams, writing the dissertation, attending conferences, applying for jobs, and also looking ahead to pursuing the goal of becoming a professor. Of course there are many opportunities for graduates with math degrees, but in graduate school I decided that I wanted to be a math professor – I liked pursuing research, especially in joint projects with other researchers, and I liked the community of mathematicians I was joining.

**Sisi**: What is mathematical biology?

**Anne**: Math has many applications in the real world. Mathematical biology is any use of mathematics to understand the biological world – anything from DNA to how animals move. One biological process I have studied is somitogenesis. In vertebrates, somites are precursors to the backbone segments and related tissues, and somitogenesis refers to the creation of somites – periodically, one at a time, from the anterior (head) end of an organism to the posterior (tail) end – during early embryonic development. The data we were analyzing were from mice; we were looking for genes whose expression levels (how much protein was being made) had a periodic pattern – for instance, something that looks like the graph of a sine or cosine function. We inferred that these genes might be related to somitogenesis, and the biologists could then examine them in further experiments.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Dr. Shiu, and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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# 3-D Movie Technologies[1]

by Antony Orth / edited by Jennifer Silva

Scientists and engineers have been studying light for a very long time. The study and manipulation of light – a field called optics – is arguably one of the oldest branches of physics. Newton was among the first to offer a rigorous framework of (classical) optics in his early 18th century book *Opticks*. The field of optics has expanded a lot since the days of Newton, but the understanding of ray optics, which is mostly what Newton dealt with, is largely unchanged. What has changed, however, is our ability to manipulate optical information on a computer, something that Newton never had the opportunity to do. With a regular desktop computer, we can rearrange light rays in ways that Newton could have only dreamed of. This new emerging field of combining optics with computers is called computational optics.

The most prevalent piece of computational optics equipment is the digital camera. A digital camera consists of a lens that bends light rays into an image on the image sensor. The image sensor is made up of millions of pixels arranged in a grid. Each pixel is like a miniature bucket that catches photons instead of raindrops. The camera records how many photons fell on each pixel; this information is the digital image itself. Each pixel takes on a value between 0 and 255, depending on how much light it captured. The higher the number, the more light was collected by the pixel. When you display the picture on a screen, each pixel on the screen emits an amount of light proportional to the amount of light that the equivalent camera pixel absorbed.

The immense power of digital photography lies in what happens between image capture and display. The image is represented by a grid of numbers, which makes it incredibly easy to rearrange and alter the image in almost any way you can imagine. In this article, we're going to explore 3-D movies and the simple manner in which they can be captured by exploiting computers.

**3-D Movies**

Humans perceive depth through parallax, the apparent movement of an object when viewed from one eye versus the other. The amount of parallax for a given object depends on how far away it is: the farther away, the larger the apparent movement when you switch eyes. Moviemakers have exploited this effect in order to fool the audience into thinking they are seeing parallax, when in fact they're just looking at a flat screen that has no real parallax. How is this done?


Source: John Alan Elson/Wikipedia

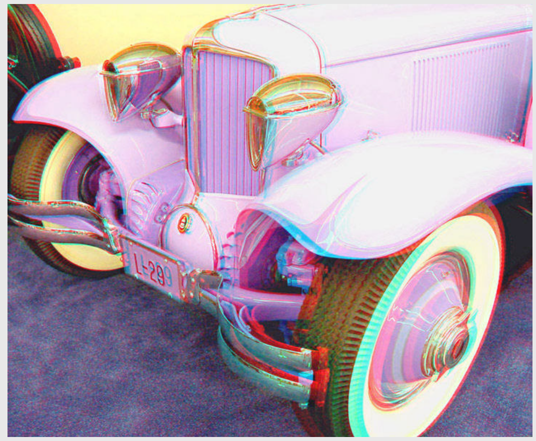
Figure 1. Stereo 3-D camera by Fujifilm. Note that there is a pair of camera lenses at the top – each lens with its own image sensor.

First, the movie must be filmed with a camera that in actuality is comprised of a pair of cameras, called a 3-D stereo camera. This camera consists of two lenses spaced by a distance equal to the separation between our eyes. The camera records a single image through each of the lenses, with a separate image sensor collecting each perspective. In order to trick your brain into thinking that it's seeing something in 3-D on a 2-D screen, a different picture needs to be sent to each of your eyes. Specifically, the left lens image of the 3-D camera must be sent to your left eye, and the right lens image needs to be sent to your right eye. This technique makes it seem as if your eyes were actually placed right where the two camera lenses were.

[1] This content was supported in part by a grant from MathWorks.

The easiest way to send different images to each eye is by using color-filtered glasses. The left lens of these glasses looks red because it only lets red light through, and the right lens looks blue because it only lets blue light through. The left eye image is shown on the screen with a red tint, and the right eye image with a blue tint. As a result, your left eye only sees the image that was captured by the left lens of the 3-D camera, and likewise for the right eye. Your eyes see exactly the same parallax that the camera captured, even though the actual 3-D object isn't there!




Source: Allan Silliphant/Wikipedia.

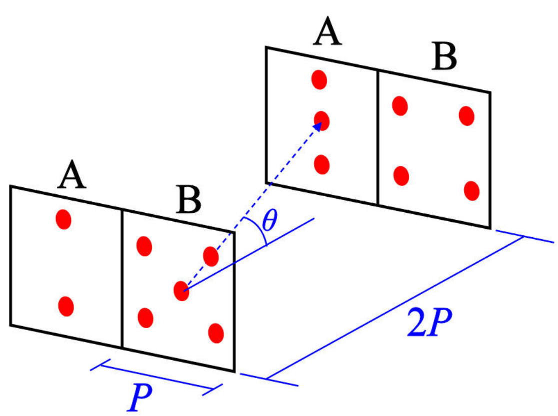
Figure 2. 3-D image of a car. Note that there appears to be a double image of the bumper, one in red and one in blue. The displacement between the red and blue images is a manifestation of parallax. For 3-D effect, view with Red/Cyan 3-D glasses.

**The Continuity Equation**

At the Harvard School of Engineering and Applied Sciences, we're working on a way to turn any regular camera into a 3-D camera. In order to do this, we consider the physics of the system and use it to our advantage.

The first thing a physicist does when presented with a physical system is to check whether there are any conserved quantities involved. A conserved quantity is something that is neither created nor destroyed. For example, if you start out with 5 red marbles at the beginning of your experiment and there are 5 red marbles at the end, then the number of marbles has been conserved. When you apply this conservation principle, you can often construct a so-called "continuity equation" for your system.

Let's think about packets of light, called photons, as the conserved quantity. Suppose we take a picture with a really small image sensor comprised of only two pixels (we'll call them pixel A and pixel B) spaced by a distance $P$. Pixel A received 2 photons, whereas pixel B received 5. Now let's take another picture, but move the image sensor back by a distance $2P$.




Courtesy of the Author

Figure 3. Pixels A and B initially have 2 and 5 photons, respectively. After defocusing by $2P$, one photon has traveled laterally from pixel B to pixel A. This photon travels at an angle $\theta$.

This is equivalent to changing the focus on a camera. In this second picture, pixel A received 3 photons and pixel B received 4. The total number of photons (7) stayed the same from picture to picture, but the distribution changed: one photon traveled from pixel B to pixel A as the image sensor moved back, while the rest of the photons stayed at their initial pixels. Let's denote the angle of propagation of this photon that went from B to A by $\theta$. This one photon traveled laterally a distance $P$ over a longitudinal distance $2P$. Therefore, we can write

$$\tan\theta = \frac{P}{2P} = \frac{1}{2}.$$

Let's take the average of all the tangents of all of the photons initially in pixel B. We had 4 photons that traveled in a straight line ($\tan\theta = 0$), and one photon with $\tan\theta = ½$. Therefore, the average value of $\tan\theta$ for all 5 photons originally in pixel B is $\frac{1/2}{5}$, or $\frac{1}{10}$. Finally, let's solve for the average ray angle:

$$\tan^{-1}\frac{1}{10} \approx 5.71°.$$

Now imagine doing this over an entire image sensor with millions of pixels. The principle is the same, except now you have to take into account photons going from one pixel to any one of its four neighbors. You wouldn't want to do this by hand, but with a computer it's straightforward; computers don't get bored. At the end of this analysis, we get the average ray angle at each pixel in the image. With this ray angle information, we can go back and actually change the viewpoint from which we are looking at a scene. If we want to look at the scene from a perspective toward the right, then we make an image that only includes pixels that had a positive average ray angle (these rays would hit your right eye). To view from the left, we include only pixels that had a negative ray angle (these would hit your left eye). We can also adjust this for the up/down directions (something you cannot do with a regular stereo camera). Now we've essentially achieved what a 3-D camera does, but with only one lens instead of two! All we needed to do was to take two pictures – one of them in focus, and the other at a slight defocus.

**Back to Newton**

3-D movies and cameras with one lens are surely concepts that Isaac Newton would have understood easily. However, cameras did not exist. Newton therefore couldn't record images even in 2-D, let alone play them back with a screen and color-coded glasses. Newton invented the math that makes the continuity equation analysis possible, but he could never have applied it over millions of pixels. Thankfully, it's a snap with computers nowadays. Physics has been around a long time, but if you incorporate computers into your experiments, you can compete with the big names in the field!

Antony Orth is a graduate student in Applied Physics at Harvard's School of Engineering and Applied Sciences.

# Cosmic Conics II[1]

by Aaron Lee
edited by Jennifer Silva

Last time, we left you with the problem of understanding how the parameter $e$ in

$$r(\theta) = a \cdot \left( \frac{L(e)}{1 + e \cdot \cos(\theta)} \right), \qquad (1)$$

where $L(e) = \begin{cases} 1 - e^2 \text{ if } 0 \le e < 1 \\ 2 \text{ if } e = 1 \\ e^2 - 1 \text{ if } e > 1 \end{cases}$, relates to

the type of conic section. How does your theory compare with mine?

| | | |
|---|---|---|
| $e = 0$ | $\rightarrow$ | Circle |
| $0 < e < 1$ | $\rightarrow$ | Ellipse |
| $e = 1$ | $\rightarrow$ | Parabola |
| $e > 1$ | $\rightarrow$ | Hyperbola |

Of course, a theory is only a theory, but having one is a great first step. The next step is to prove it.

Let's start by checking that $e = 0$ gives us the circle. If $e = 0$, then $L(e) = 1$ and the function in equation (1) becomes $r = a$, which is indeed the polar equation for a circle with radius $a$.

Next, let's verify that values of $e$ between 0 and 1 correspond to an ellipse. First substitute $L(e) = 1 - e^2$ and $\cos \theta = x/r$, to get

$$r = a \cdot \frac{1 - e^2}{1 + e \cdot (x/r)}.$$

Now it gets trickier. Solve this equation for $r$, then square both sides. After substituting $x^2 + y^2$ for $r^2$ and some algebraic manipulation, you can get

$$x^2 + 2aex + \frac{y^2}{1 - e^2} = a^2(1 - e^2).$$

Complete the square with the two terms involving $x$, then rearrange to finally arrive at

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

[1] This content supported in part by a grant from MathWorks.

which is the equation for an ellipse centered at (-$ae$, 0) with major axis length $2a$ and minor axis length $2a\sqrt{1-e^2}$. Phew! Notice that if you plug in $e = 0$, you get back the circle (the circle is a special case of an ellipse), and the minor axis does not allow you to have $e \geq 1$ (do you see why?).

You try the remaining two, then look at my results below.

For $e = 1$, I get a parabola

$$x = a - \frac{y^2}{4a}.$$

Here, $a$ represents the distance from the focus to the vertex of the parabola.

Finally, $e > 1$ gives me the hyperbola

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1.$$

Once again, $a$ represents the semi-major axis length, just as in the case of the ellipse.

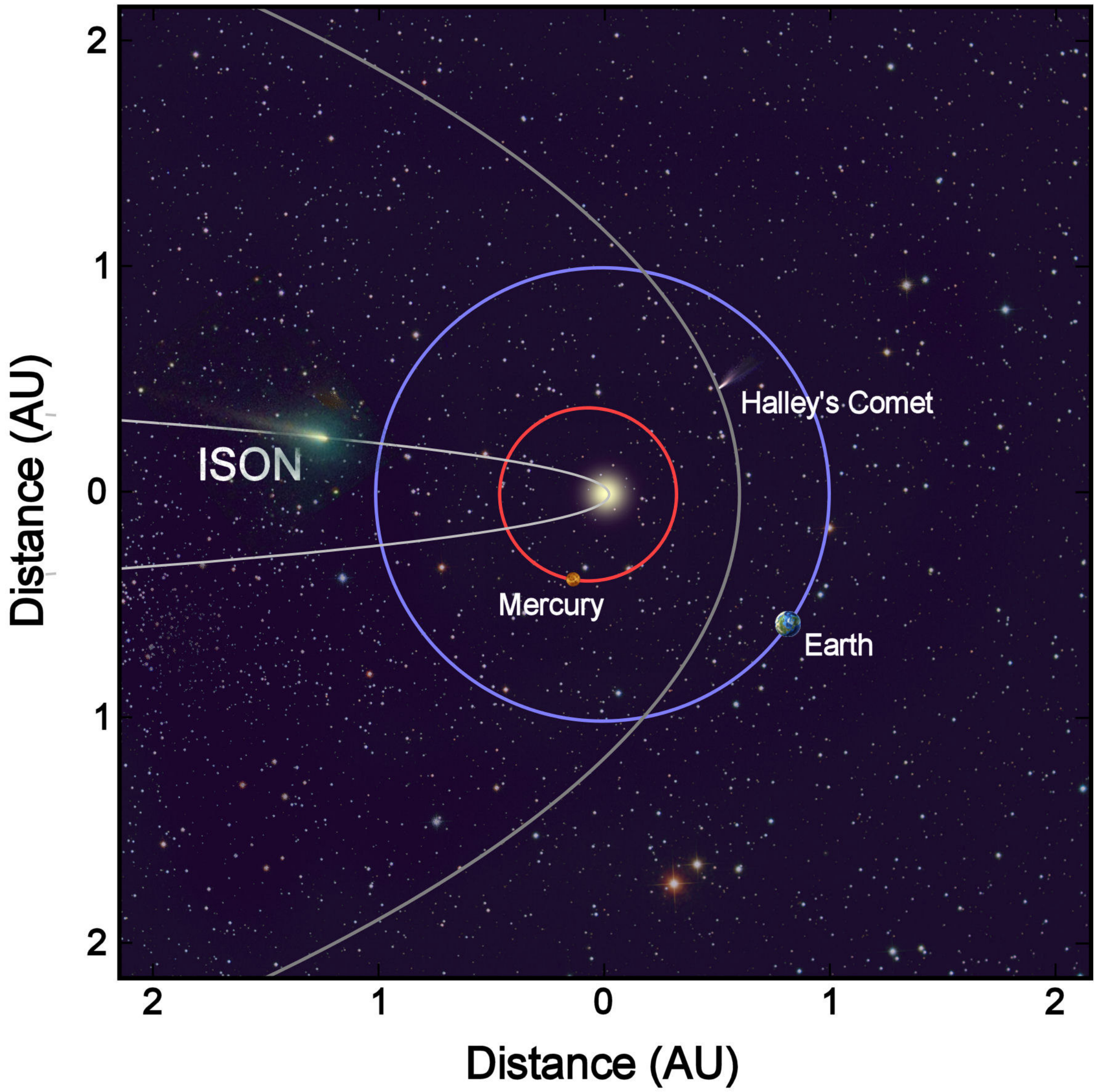
All you need to know is the value of $e$, and equation (1) gives you one of the four conics! Then you adjust its size by changing $a$.

We have proven that equation (1) represents the four conic sections by reasoning out or deriving all of our claims. Now let's use this knowledge to solve some real-world (or, more accurately, out-of-this-world!) problems. Equation (1) is a great equation because it immediately tells us interesting information about the orbit of the comet or planet. The way the equation is set up, the value of $r$ at $\theta = 0$ is the closest distance the object ever comes to the Sun. This location is called **perihelion**. For the ellipse, the farthest distance in the orbit occurs at $\theta = \pi$; this is called **aphelion**. For a circle, perihelion is the same as aphelion (can you explain why?). If an object is on a parabolic or hyperbolic orbit, talking about aphelion does not make sense (it's ∞); these objects are on their way to being ejected from the Solar System!

Let's start with the Earth. Its orbit is nearly circular with an eccentricity of $e = 0.017$, and it has a semi-major axis length of 93 million miles. This distance is such a convenient unit of measurement in the Solar System that astronomers have given it its own name, the **astronomical unit** (the **AU**). So the Earth has a semi-major axis of $a = 1$ AU. How close is the Earth to the Sun at perihelion (which occurred on January 2nd this year)? Plugging in $\theta = 0$, I get about 0.9833 AU, which is about 91.5 million miles. What about aphelion (which occurred July 5th)? Plugging in $\theta = \pi$ gives me 1.02 AU, or 94.5 million miles. That's a difference of 3 million miles: the Earth is 3 million miles farther away from the Sun at aphelion than it is at perihelion! Based on what I have told you, can this be the reason we have different seasons? (Hint: when was the Earth closest to the Sun?)

What about perihelion and aphelion for the famous Halley's Comet, whose eccentricity is 0.967 and whose semi-major axis is 17.8 AU? Recall that comets originate in the Kuiper belt beyond Neptune's orbit (which is nearly circular and 30 AU away from the Sun). Does your result for Halley's Comet's aphelion corroborate this statement? Try this with some other planets in the Solar System.

Finally, let's understand why Comet ISON might outshine the Moon. ISON is on a nearly parabolic orbit with an eccentricity only slightly above 1 (that means you'd better not miss it, because it is probably not coming back!). Let's approximate its orbit as parabolic ($e = 1$). When the comet passes the Earth's orbit (that is, when $r = 1$ AU), $\theta = 3.31$ radians. Using this, can you calculate $a$ and determine how close the comet will get to the Sun? How does this compare to the radius of the Sun (432,000 miles) or the orbit of Mercury (0.39 AU)? You should find that ISON's closest approach is about 650,000 miles, only ~1.5 times the radius of the Sun. Thus, it makes sense that the comet is going to be bright. Let's hope it doesn't completely melt!




**Further Reading**

For an introduction to Newton's work on gravity, see

*Newton and Gravity: The Big Idea*, by Paul Strathern.

For more detailed treatment of Newton's work on gravity, see

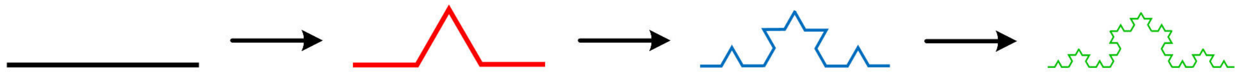
*Magnificent Principia: Exploring Isaac Newton's Masterpiece*, by Colin Pask.

*Feynman's Lost Lecture*, by David and Judith Goodstein.

# Mathematical Buffet

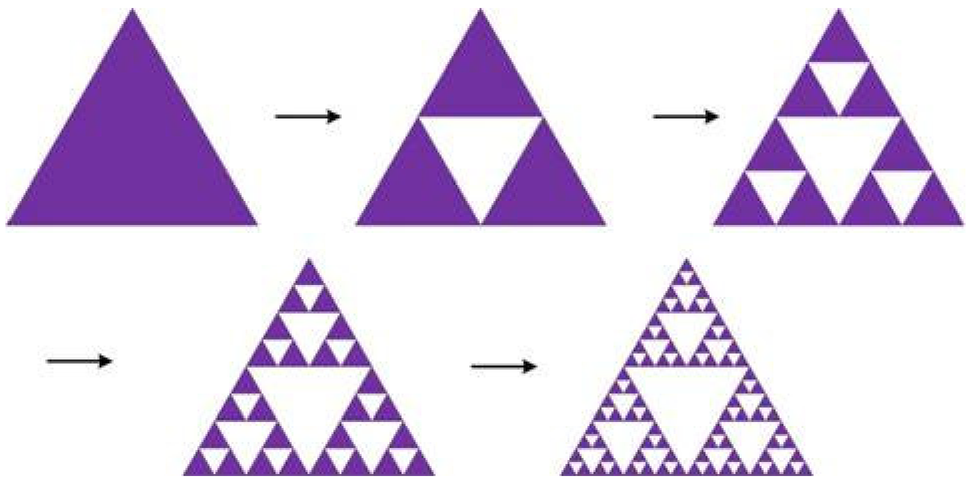
Self-Similarity by **Jesse Lee**

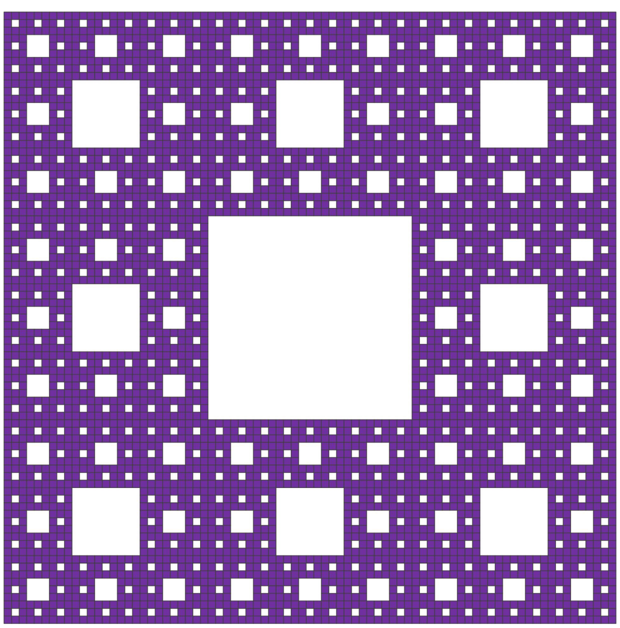
An object is **self-similar** if it is similar to, or close to being similar to, a part of itself.



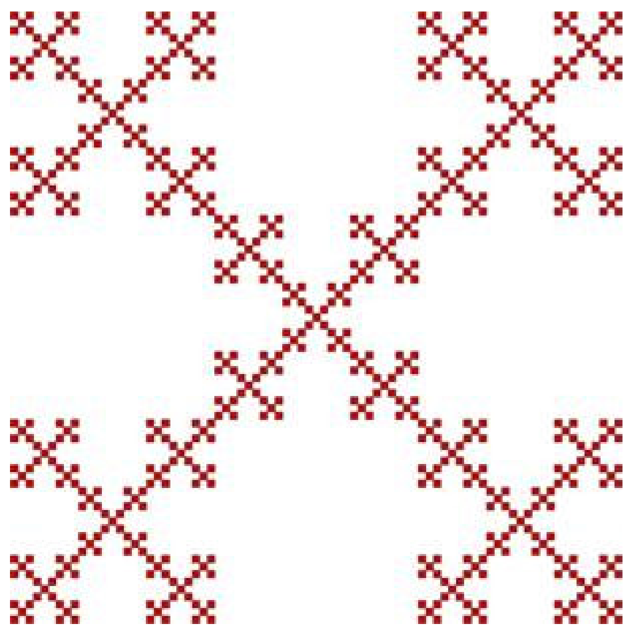
To get the next figure in this sequence, the middle third of each segment is replaced with two segments of the same length as the removed middle third. When this process is repeated ad infinitum, the result is an exactly self-similar shape known as the **Koch curve**.

The **Sierpinski triangle** is another example of exact self-similarity. The *n*th figure in the sequence consists of $3^n$ triangles, each of which is $2^{-n}$ times the size of the first triangle (which we are indexing by zero).





The analog of the Sierpinski triangle using squares is the **Sierpinski carpet**.

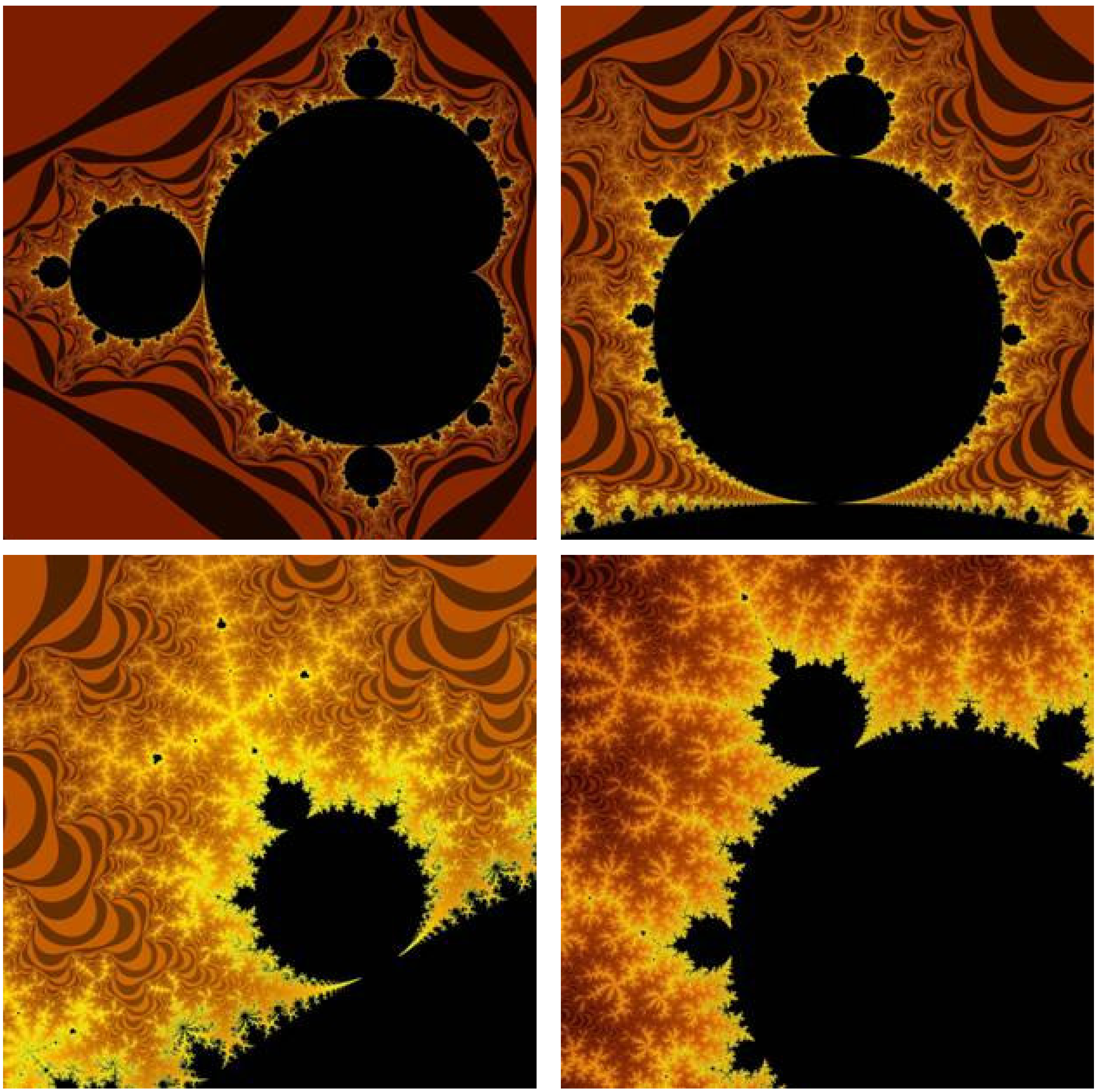


This is the **box fractal**. Can you realize it as the limiting figure of a sequence of figures that are constructed in a manner similar to the Sierpinski triangle?

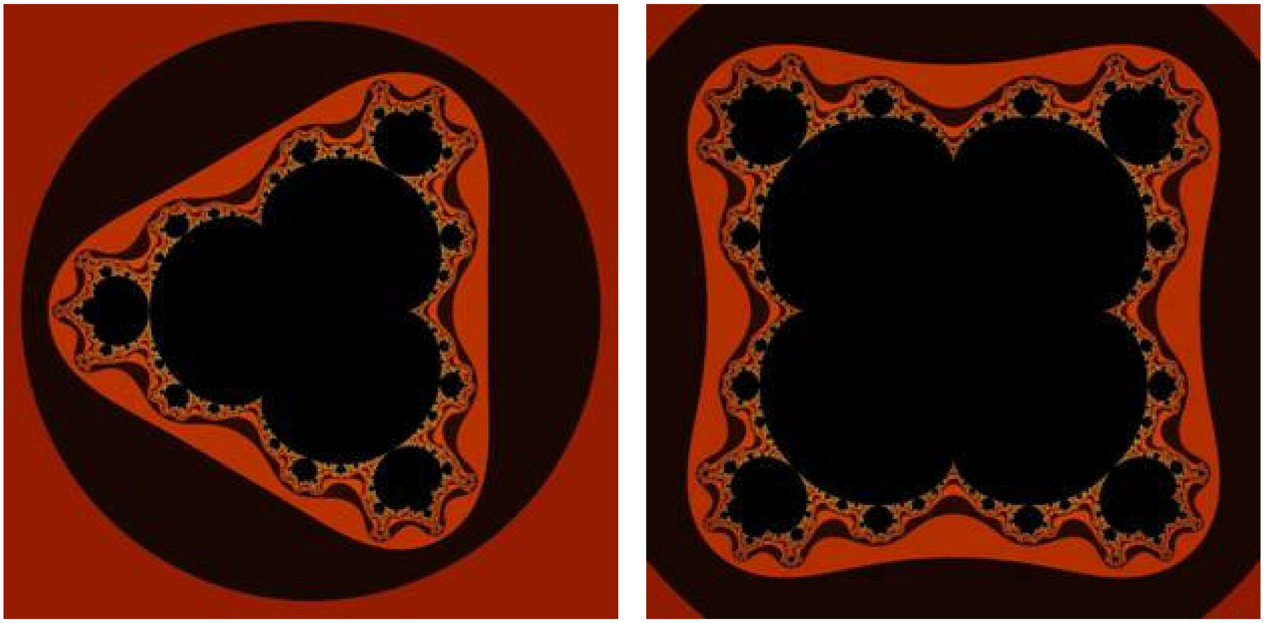
All images on this page were created by the author.

A classic example of self-similarity is the **Cantor middle thirds set**. Beginning with the unit interval, at each step remove the middle third of the remaining intervals. The first 7 iterations are shown. (Image by the author.)

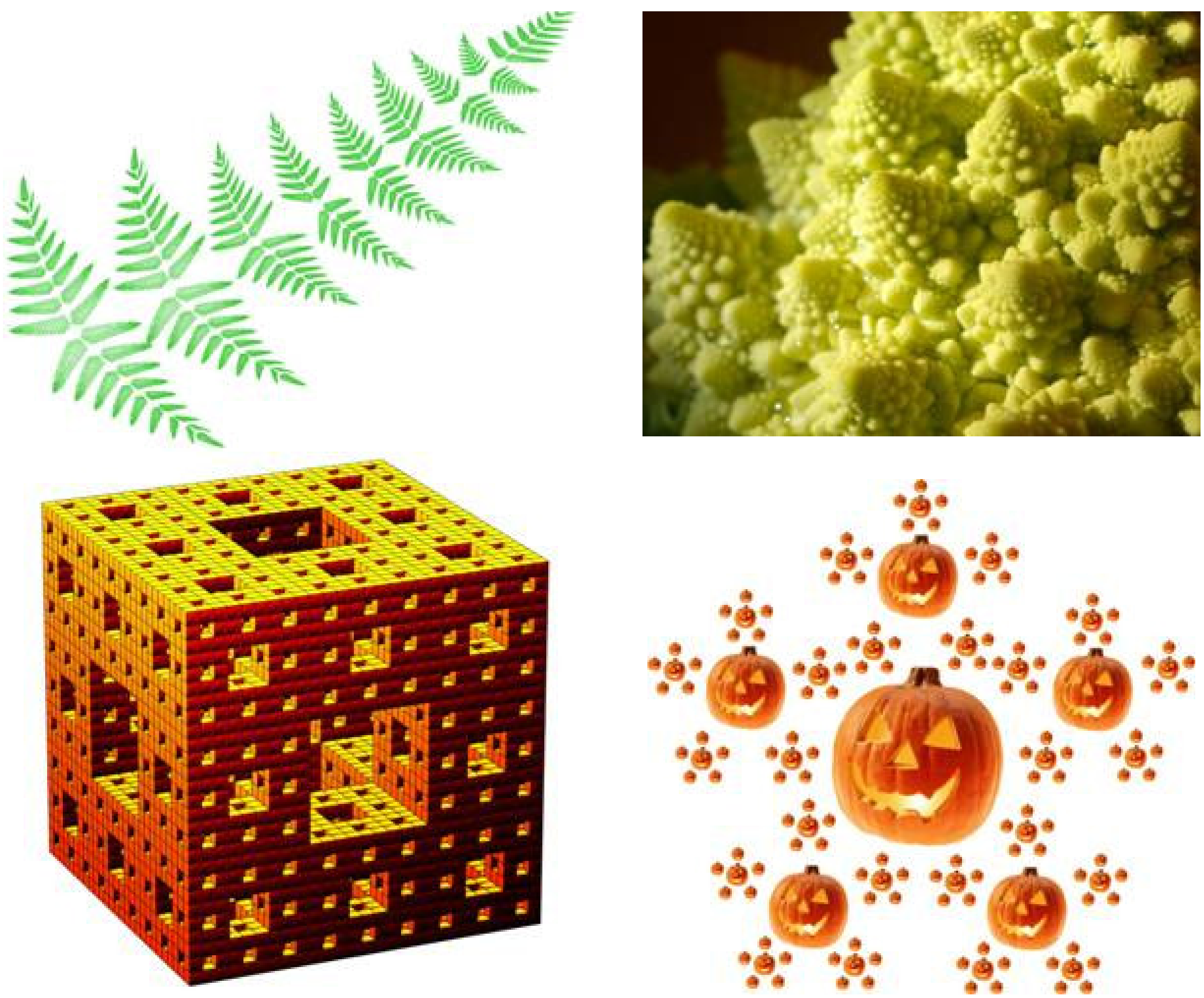
An example of non-exact self-similarity is furnished by the Mandelbrot set (see Volume 6, Number 4 of this Bulletin for more on the Mandelbrot set). The Mandelbrot set (indicated by the blackest points in the 4 figures below) consists of complex numbers $c$ such that $\{ f_c^{(n)}(0) \}_{n = 1, 2, 3, \ldots}$ is bounded, where $f_c(z) = z^2 + c$.



The 4 figures above were generated using MATLAB. To indicate the region of the complex plane depicted, we shall give the complex numbers in the lower left and upper right corners. Top left: -1.5 – $i$, 0.5 + $i$, top right: -0.27 + 0.63$i$, 0.02 + 0.92$i$, bottom left: -0.19 + 0.82$i$, -0.165 + 0.845$i$, and bottom right: -0.1785 + 0.8325$i$, -0.1760 + 0.8350$i$. The lower right image is magnified 1000 times compared to the upper left image.



Replacing $f_c(z)$ with $f_c(z) = z^4 + c$ (above left) and $f_c(z) = z^5 + c$ (above right). Images made with MATLAB.



Top left: Ferns (Girls' Angle staff). Top right: Romanesco broccoli (Photo by Toshia McCabe).
Bottom left: Level 3 Menger sponge (Girls' Angle staff). Bottom right: Jack O'Lanterns (author).

# Anna's Math Journal

By Anna B.

*Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.*

Anna continues to investigate whether products of consecutive numbers can be perfect squares.

I had an idea about how to get what I was trying to do last time. If I can compare products of two numbers, I might be able to get a contradiction. But first, I'll summarize what I found from last time.

4 consecutive numbers: $n, n+1, n+2, n+3$.

Case $n$ odd:

| $n$ | $n+1$ | $n+2$ | $n+3$ |
|---|---|---|---|
| $3y^2$ | $2z^2$ | $x^2$ | $6w^2$ |
| $y$ odd | $z$ even | $x$ odd | $w$ odd |

Case $n$ even:

| $n$ | $n+1$ | $n+2$ | $n+3$ |
|---|---|---|---|
| $6z^2$ | $x^2$ | $2w^2$ | $3y^2$ |
| $z$ even | $x$ odd | $w$ odd | $y$ odd |

□ stands for a perfect square.

So I don't get confused later, I better record the fact that I'm using a small square to denote a perfect square.

Here's the idea...the product of the first and last number is close to the product of the middle two numbers...

I'll draw a small square to mean "perfect square."

Both cases:

$$n(n+3) = 18\,\square = n^2+3n$$

$$(n+1)(n+2) = 2\,\square = n^2+3n+2$$

$$\Rightarrow \quad 9\,\square = \frac{n^2+3n}{2}$$

$$\square = \frac{n^2+3n}{2} + 1$$

These differ by 1.

Impossible

Since 9 is a square, 9 times a perfect square is a perfect square. So if the product of these 4 consecutive numbers is a perfect square, we would have to have two positive perfect squares that differ by 1. Impossible!

These proofs seem so ad hoc. I'm going to take a moment to try to extract some general principles that I've been using.

General Principles.

1. I can ignore square factors, in a way. That is, if $x$ is a □, then $yx$ is a □ iff $y$ is a □.

Hm. In a way, square factors don't concern me, because they won't cause a product to not be a square. I can focus on non-square factors.

2. Consecutive positive perfect squares cannot differ by 1. For any constant $c$, $c$ odd, there is only one pair of consecutive perfect squares that differ by $c$. If $c$ is even, no such pair exists.

And several times I've used the fact that as they get bigger, consecutive squares differ by ever increasing amounts.

3. Numbers close to each other cannot share large prime factors. Specifically, if $p$ is prime and $p|x$ and $p|y$, $x \neq y$, then $|x-y| \geq p$.

And I've used the fact that nearby integers cannot share large prime factors.

Key:

Anna's thoughts

Anna's afterthoughts

*Editor's comments*

I'll do 5 consecutive numbers now, and try to use the general principles as much as I can.

5 consecutive numbers:

General principles 1 & 3: If $p \geq 5$ is prime, then $p$ can divide at most one of the 5 consecutive numbers. If the product of the 5 numbers is a $\square$, then $p$ must appear with even exponent.

First, I know that primes greater than or equal to 5 can only appear in one of the 5 consecutive numbers as a factor.

General principle 1 suggests that I might factor each number as $ab^2$, where $a$ is not divisible by the square of any prime number. This kind of factorization is always possible.

*Such a number* **a** *where all prime factors appear with exponent at most 1 is called a "square-free" number.*

$n = ab^2$

↑ prime factors of $a$ appear in prime factorization of $a$ 0 or 1 time.

With 5 consecutive numbers, factor each in this way:

$a_1 b_1^2$, $a_2 b_2^2$, $a_3 b_3^2$, $a_4 b_4^2$, $a_5 b_5^2$

where $a_1, a_2, a_3, a_4, a_5$ have no prime factors that appear with exponent greater than 1.

So I'll factor all 5 consecutive numbers like that. I'll label so that the subscript $k$ stands for the $k$th consecutive number.

If the product of the 5 consecutive numbers is a perfect square, then any prime factor greater than or equal to 5 will appear with even exponent as a factor of at most one of the numbers. So the $a_k$'s can only have 2 and 3 as prime factors.

Only prime factors of $a_k$ possible are 2 and 3.

4 possible values: 1, 2, 3, and 6.

Only 1, 2, 3, and 6 have prime factors of 2 or 3 only, and with exponent at most 1.

So we have 5 numbers that each must equal 1 of 4 numbers, 1, 2, 3, or 6... that means there must be two that are equal...

That's the pigeonhole principle!

$\Rightarrow$ 2 must be equal, say $a_k = a_j$, $1 \leq k < j \leq 5$.

$$a_k b_k^2 + j - k = a_j b_j^2$$

$$b_k^2 + \frac{j-k}{a_k} = b_j^2$$

...I'm sensing that this will give me two perfect squares that are too close to exist...

$\Rightarrow$ $b_k^2$ and $b_j^2$ are $\square$s that differ by less than 5... so 3 or 1.

Yes...I get two perfect squares that must differ by 3... that is they must be 1 and 4.

1 not possible, if 3, then $b_k^2$ and $b_j^2$ are 1 and 4.

So one of the numbers must be 1, 2, 3, or 6.

But that means one of the numbers must be 1, 2, 3, or 6...so I only have to check a few sets of 5 consecutive numbers.

$\{1, 2, 3, 4, 5\}$, $\{2 \to 6\}$, $\{3 \to 7\}$ ... $\{5 \to 9\}$ all have just one factor of 5.

$\{6, 7, 8, 9, 10\}$ — one factor of 7. Impossible.

None work...the products of the numbers in these sets of 5 consecutive numbers will all have just one factor of 5...or one factor of 7...and so cannot be perfect squares!

6 consecutive numbers: $a_k b_k^2$ possible prime factors of the $a_k$ 2, 3, 5.

I think I'll press on and see if a product of 6 consecutive numbers can be a perfect square or not.

5 must divide one of the numbers, and if the product is a perfect square, then 5 must divide the first and last...

Must have $5 | a_1$ and $5 | a_6$.

Either even indexed $a_k$ are even or odd indexed $a_k$ are even. 3 cases for divisibility by 3 $\Rightarrow$ 6 cases to consider!

I'll make a table covering all possible cases for the values of the $a_k$.

must redo!

| Case \ k | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 3 5 | | 2 | 3 | 2 | 5 |
| 2 | 2 5 | 3 | 2 | | 2 3 | 5 |
| 3 | 2 5 | | 2 3 | | 2 | 3 5 |
| 4 | 3 5 | 2 | | 2 3 | | 2 5 |
| 5 | 5 | 2 3 | | 2 | 3 | 2 5 |
| 6 | 5 | 2 | 3 | 2 | | 2 3 5 |

ABB 10.26.13

Oh, wait a sec! This is all wrong! It could be that the first and last are not divisible by 5... if one of the middle numbers is divisible by 25 or some even power of 5... Oh dear... wait, I have to worry about numbers divisible by 4 too... and surely at least one of the numbers will be divisible by 4... scratch this table...it's all wrong!

*Can you fix up this table? Can the product of 6 consecutive numbers be a perfect square?*

Key:

Anna's thoughts

Anna's afterthoughts

*Editor's comments*

# Zeroing In On Zeroes

> Two Useful Facts About Zero
>
> 1. If a product is equal to zero, then at least one of the factors must be equal to zero.
>
> 2. If a sum of nonnegative (real) numbers is equal to zero, then all the numbers must be equal to zero.

If you're having trouble with *Heroes for Zeroes* on page 18, these problems are designed to help.

**Absolute Value**

The absolute value of a real number $x$ is denoted by $| x |$. It is equal to $x$ if $x \geq 0$, and it is equal to $-x$ if $x < 0$. You can think of $| x |$ as the distance of $x$ from 0 on the number line.

1. Make a graph of the absolute value function. That is, plot the point $(x, y)$ in the Cartesian $xy$-coordinate plane where $y = | x |$.

2. Graph the function $f(x) = | x - 1 |$ and the function $g(x) = | x + 1 |$. (In other words, plot the points $(x, y)$ where $y = f(x)$ and $y = g(x)$.)

So far, the 3 graphs you have plotted in problems #1 and #2 should look almost exactly like each other. Each is a horizontal shift of the others.

3. Now graph the function $f(x) + g(x)$. It looks like 2 rays and a horizontal line segment. What is the smallest $f(x) + g(x)$ can be? For what values of $x$ is $f(x) + g(x)$ smallest?

4. Plot the points on the number line where $f(x) + g(x) = 2$.

5. Plot the points $(x, y)$ in the Cartesian plane where $y^2 + | x - 1 | + | x + 1 | = 2$.

Here's a hint for problem #5: If two nonnegative numbers are added together, can the result ever be negative?

**Circles**

6. Using the Pythagorean theorem, show that the distance of the point $(x, y)$ from the origin in the Cartesian plane is given by $\sqrt{x^2 + y^2}$ .

7. Use problem #6 to deduce that the set of points $(x, y)$ whose coordinates satisfy the equation

$$x^2 + y^2 = 4$$

is a circle of radius 2 centered at the origin.

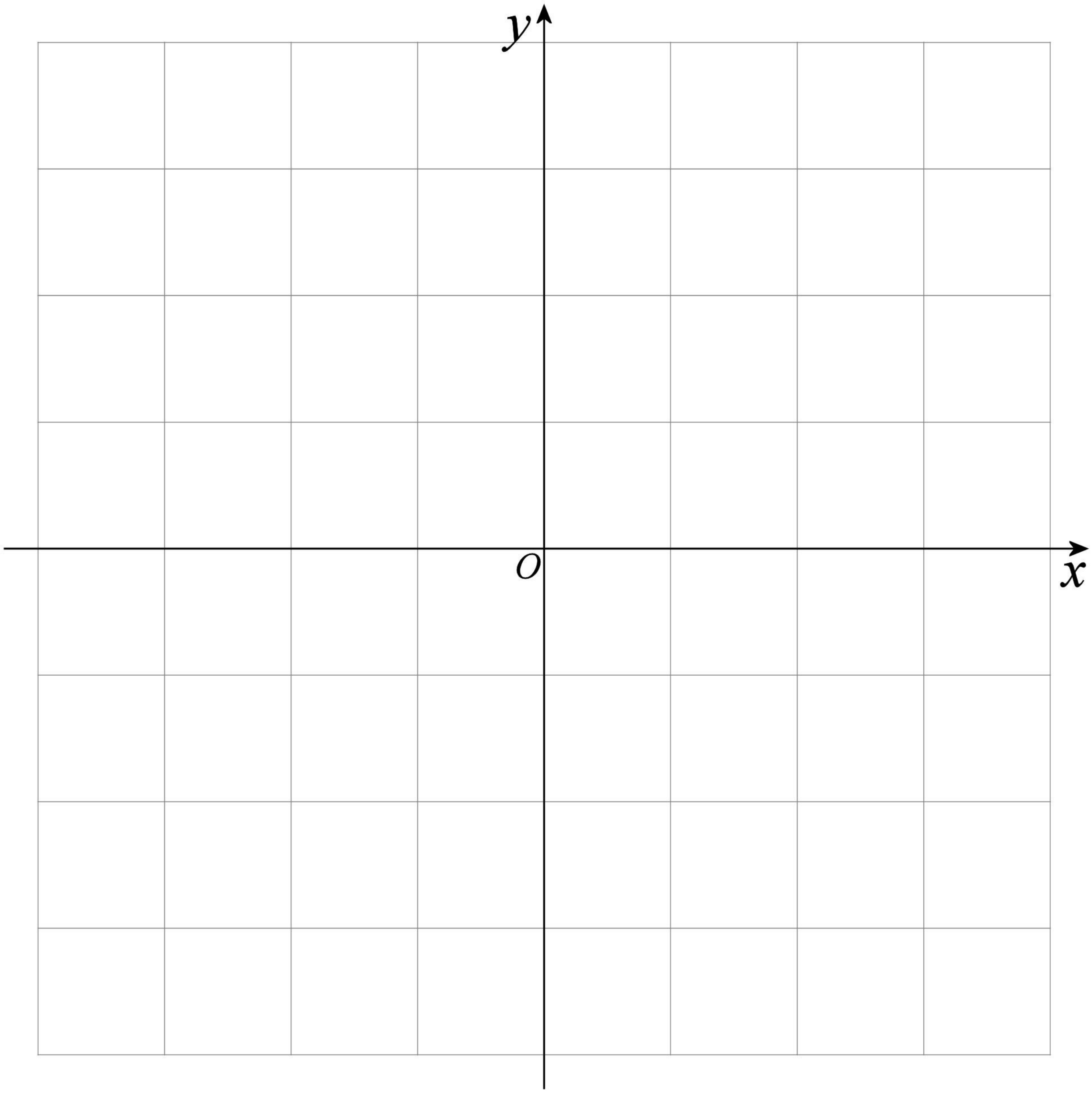
8. What is the set of points $(x, y)$ whose coordinates satisfy the equation $(x - 3)^2 + (y - 1)^2 = 9$?

**Miscellany**

9. Plot the points $(x, y)$ such that $| x | + | y | = 2$.

10. What 4 points $(x, y)$ satisfy $| | x | + | y | - 1 | + | x^2 + y^2 - 1 | = 0$?

# Heroes for Zeroes




$$|x+y|+|x-y|-6$$

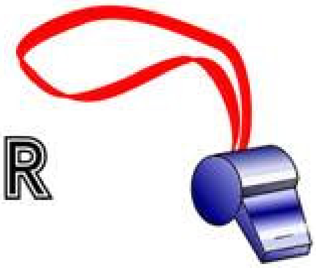
$$|x+y-2|+|x-y-1|-1$$

$$|x+y+1|+|x-y+2|-1$$

$$|y+1|+|y+2|+|x^2+y^2+2y|-1$$

On the Cartesian plane, plot the points $(x, y)$ where each of the expressions evaluate to zero.

(For warm-up problems, see page 17.)



# COACH BARB'S CORNER

by Barbara Remmers | edited by Jennifer Silva

## Owning it: Fraction Satisfaction, Part 13

Walking to school, you notice something surprising about a trash can. It's overflowing with photos of 3/7 – autographed photos. You grab one because the whole situation is so bizarre that you must have some proof. Also, it's nice to have a memento of your longtime fraction friend. Then you notice that on the next corner, there is the dame herself. Inquiries are in order!

**You**: Hi 3/7, what's going on?

$\mathbf{\frac{3}{7}}$: Oh, I am doing my bit to make the world a better place.

**You:** By handing out pictures of yourself?

$\mathbf{\frac{3}{7}}$: Yes. You see, I read a newspaper story entitled "Fear of Fractions." Apparently it's quite common. I said to myself, "I can rectify the situation." With these photos, I am going to dispel all fraction fears and replace them with admiration, adoration, veneration, and so on and so forth.

**You**: I see. How come your autograph is $0.3_7$?

$\mathbf{\frac{3}{7}}$: Well, the margin on the bottom of my photos is too small to easily fit "3/7." Also, as we saw during our last chat, the decimal version of my name goes on forever. Since six digits repeat, even with the "repeating forever" bar, that meant too much writing to autograph box upon box of photos. So I decided to use base 7, which produces a tidy, short, one-line version of my quantity.

**You**: Base 7… will you tell me what that means?

$\mathbf{\frac{3}{7}}$: How about you tell me what base 10 – our usual way of writing numbers – means?

**You**: Okay. We have a ones digit which is located just to the left of the decimal point. That tells us how many ones are in the quantity represented. The digit to the left of the ones digit tells how many tens are in the quantity represented; we say it has place value 10. The digit to the left of the tens digit tells how many hundreds are in the quantity represented; we say it has place value 100. Each time you move one digit to the left, the new digit's place value is 10 times the previous place value. Each digit you move to the right has place value equal to one tenth that of the place value to its left. So you can keep going to the right of the decimal point. The digit just to the right of the decimal point has a place value of 1/10, the digit to the right of that has place value 1/100, and so on.

$\mathbf{\frac{3}{7}}$: Ending an explanation with an example is like icing on the cake.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

---

Session 13 - Meet 1 Mentors: Asra Ali, Meg Doucette, Ruthi Hortsch,
September 12, 2013 Jennifer Matthews, Wangui Mbuguiro

We played "One Minute!" to break the ice. The girls had to accomplish 8 different tasks. Each task had to completed within 1 minute. A specific task would be picked randomly. If the girls accomplished the task successfully in a minute, the task would be marked completed. Otherwise, it would be put back into a grab bag of tasks to accomplish. The 8 tasks were:

| | |
|---|---|
| **Math Charades** | 3 math charades. (Yes, all 3 in 1 minute!) |
| **Math Telephone** | The telephone game using a mathematical definition as the message. |
| **Clock Arithmetic** | A number *N* between 1 and 6 is selected using a die roll. Starting at 12 o'clock noon, the girls go around adding *N* hours to the previously stated time, and state the new time. They have to make it all the way around the group without anyone committing an error. |
| **Binary Race** | 5 girls count from 0 to 31 in binary, with each girl representing a digit. Sitting is 0 and standing is 1. Girls not in the binary line up can assist with the count. |
| **Water Algorithm** | Each girl, in turn, gives a single instruction consisting of fewer than 8 words to get a mentor to successfully drink a cup of water. Synonyms of the words "drink," "pour," "eat," and "pick up" are not allowed. |
| **Clone the Drawing** | A girl is given a geometric figure. Using verbal cues only she must get at least half the others to successfully draw a scale copy. |
| **Ode to Pi** | Girls must create a poem that is an ode to pi. Each participant must, in turn, come up with a line following the rhyme scheme ABAB. Every 4 line stanza must clearly be related to pi. |
| **Super Sort** | Participants must sort themselves according to some order, such as alphabetically by first name, by height, or by distance of their home from the club. |

This game affords countless variations and adaptations to groups of different sizes and mathematical abilities.

---

Session 13 - Meet 2 Mentors: Asra Ali, Elenna Capote, Jordan Downey,
September 26, 2013 Jennifer Matthews, Wangui Mbuguiro

One of our member's has been reading through the book *Charming Proofs: A Journey into Elegant Mathematics* by Claudi Alsina and Roger Nelson. Good math books can be used as a stimulant to mathematical exploration. Armed with ample scratch paper and a pencil, this member has been spelunking through the book. When she finds something that intrigues her and induces her to ask questions, she will put the book aside and try to resolve her questions. When you have the time, this is a great way to read through any math book. One of her mathematical explorations is showcased in this issue's *Member's Thoughts* on page 26. The more active thinking you engage in, the better you will learn the math. Get into the habit of reading math books with scratch paper and pencil at the ready, and always question what you read!

*Charming Proofs* is part of the Dolciani Mathematical Expositions series of books published by the Mathematical Association of America. The Mathematical Association of America offers a treasure trove of excellent math books covering a wide range of topics. Check them out at www.maa.org/publications/books.

Other members began an open-ended exploration of planar shapes. They articulated various features of different shapes that they had created and began categorizing shapes according to these features. Next, questions began arising about how the various features interrelated. Soon, math evolved.

---

Session 13 - Meet 3 | Mentors: Jordan Downey, Meg Doucette, Jennifer Matthews,
October 3, 2013 | Wangui Mbuguiro, Isabel Vogt

Some members engaged in *extreme* thinking. That is, they were presented with various configurations to which they had to identify extreme instances of those configurations. For example, each member's right big toe pointed out the location of a vertex of a polygon in the floor. This polygon has a perimeter. Find configurations that minimize and maximize this perimeter. Another example: 100 people write down the day of the week on which they were born. They note how many people were born on the most common day. Determine circumstances that minimize and maximize this number.

Here are a few math questions that came up at the club. What's your answer to each?

1. What is the largest prime number less than 900?

2. How many ways can you roll a total of 10 using 3 dice?

3. Given a polygon, what billiard ball paths loop back on themselves?

---

Session 13 - Meet 4 | Mentors: Asra Ali, Jennifer Matthews, Wangui Mbuguiro
October 10, 2013

Some members sought all points $(x, y)$ in the Cartesian plane where $f(x, y) = 0$ for various functions $f$. To try your hand at this, see *Heroes for Zeroes* on page 18.

---

Session 13 - Meet 5 | Mentors: Elenna Capote, Jennifer Matthews, Wangui Mbuguiro
October 17, 2013

Visitor: Anna Frebel, Department of Physics, MIT

Dr. Anna Frebel was born in Göttingen, Germany and is a professor in the department of physics at the Massachusetts Institute of Technology. She received her doctorate from the Australian National University's Mt. Stromlo Observatory and held postdoctoral positions at the University of Texas, Austin and at Harvard before joining the faculty of MIT 2 years ago.

At the club, Dr. Frebel explained some ways that astronomers learn about stars by analyzing the light they emit. For example, different elements in the star absorb different wavelengths of light, so when you pass light from a star through a prism, you will see downward spikes in intensity at these absorption frequencies. The area of the spike gives an indication of the abundance of the corresponding element.

She also explained the Doppler shift. She engaged the girls in an activity where 2 girls took on the roles of stars in a binary star system and 2 girls took on the roles of observing astronomers. The observers were to step left or right depending on whether the star assigned to them was moving toward them or away from them. As the 2 "star" girls rotated around each other, the 2 "astronomers" oscillated back and forth. This back and forth oscillation modeled Doppler shift. When a light emitting object moves away from you, it's spectrum shifts to the red, whereas when it is moving toward you, it's spectrum shifts to the blue. Quantitatively,

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

where $\lambda$ is the rest wavelength of emitted light, $\Delta\lambda$ is the change in wavelength due to the Doppler shift, $v$ is speed of the emitting object toward or away from the observer, and $c$ is the speed of light. In a binary star system, astronomers can observe an oscillating Doppler shift as the stars orbit around each other.

In the same way, oscillations due to Doppler shift in the spectrum of light from a star can suggest the existence of orbiting planets. If the planet is massive enough compared to its star, the wobble will be significant enough to produce a measurable oscillating Doppler shift.

Dr. Frebel also showed us two videos. One showed a simulation of the formation of our galaxy. The other showed a time lapse view of the night sky over her telescope in Chile. Because of the Earth's orientation relative to the Milky Way, those in the Southern hemisphere get a view of our galaxy backlit by the galactic core (see page 28). You can view these videos on Dr. Frebel's YouTube channel: www.youtube.com/channel/UC3cyRVDoePNf_rLQlwKpdeg.

---

Session 13 - Meet 6 Mentors: Elenna Capote, Jordan Downey
October 24, 2013

Some members built a large regular icosahedron. After building it, we found the 3 mutually perpendicular golden rectangles defined by its vertices. In other words, let $\varphi$ be the golden mean (see *Member's Thoughts* on page 26). The following 3 groups of 4 points each in 3D Cartesian coordinates describe the corners of 3 mutually perpendicular golden rectangles:

Rectangle 1: $(\varphi, 1, 0)$, $(\varphi, -1, 0)$, $(-\varphi, -1, 0)$, $(-\varphi, 1, 0)$
Rectangle 2: $(1, 0, \varphi)$, $(-1, 0, \varphi)$, $(-1, 0, -\varphi)$, $(1, 0, -\varphi)$
Rectangle 3: $(0, \varphi, 1)$, $(0, \varphi, -1)$, $(0, -\varphi, -1)$, $(0, -\varphi, 1)$

Show that all 12 vertices taken together form the vertices of a regular icosahedron.

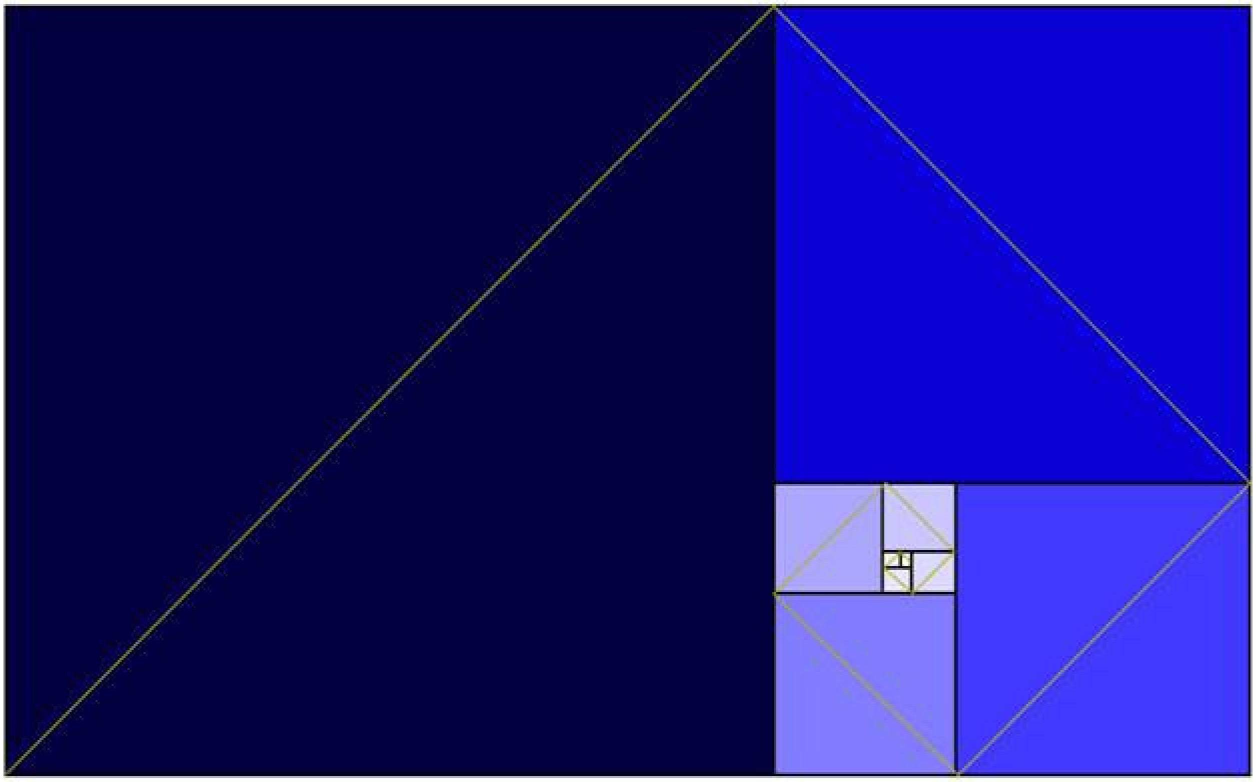
---

Session 13 - Meet 7 Mentors: Wangui Mbuguiro
October 31, 2013

We celebrated Halloween with a spooky math puzzle. Congratulations to all the participants for solving all the math problems and breaking into the treasure chest!

# Area and the Golden Rectangle

by Isabel Macenka



The golden rectangle is a rectangle with sides of length 1 and $\varphi$ where $\varphi = (1+\sqrt{5})/2$, which is approximately equal to 1.618. The amazing defining property of the golden rectangle is that when one cuts a square off one side, the remaining rectangle has the same proportions as the first. This process can be continued, each time producing a new rectangle with sides in the same ratio as the previous rectangles, as shown in the figure above. The defining property means that $\varphi/1 = 1/(\varphi-1)$. This property led me to ask the question: How can one find the area of each successive rectangle in the spiral of golden rectangles? In my first approach, I decided that to find the area, I would first try to determine the side lengths. Let us begin by creating a chart (at left) with the side lengths and areas of the golden rectangles. Each side length can be found by subtracting the previous side length from the one before that. The expressions in the area column can be found by multiplying the length and width in the same row.

**Chart A**

| Length | Width | Area |
|---|---|---|
| $\varphi$ | 1 | $\varphi$ |
| 1 | $\varphi - 1$ | $\varphi - 1$ |
| $\varphi - 1$ | $-\varphi + 2$ | $-\varphi^2 + 3\varphi - 2$ |
| $-\varphi + 2$ | $2\varphi - 3$ | $-2\varphi^2 + 7\varphi - 6$ |
| $2\varphi - 3$ | $-3\varphi + 5$ | $-6\varphi^2 + 19\varphi - 15$ |
| $-3\varphi + 5$ | $5\varphi - 8$ | $-15\varphi^2 + 49\varphi - 40$ |
| $5\varphi - 8$ | $-8\varphi + 13$ | $-40\varphi^2 + 129\varphi - 104$ |

Although there is no obvious pattern in the areas so far, a pattern is emerging in the side lengths. Every side length takes the form $\pm a\varphi \pm b$ where $a$ and $b$ are Fibonacci numbers. The Fibonacci numbers are numbers from the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … We will represent the Fibonacci numbers as $F_n$, so $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, etc. The sequence begins with 0, 1 and then the next term is found by adding the previous two terms together. In mathematical terms, we have $F_n + F_{n+1} = F_{n+2}$, for all $n \geq 0$. Now we'll name the side lengths. I will use $L_n$ to represent the width of the $n$th rectangle, so $L_0 = 1$, $L_1 = \varphi - 1$, $L_2 = -\varphi + 2$, … Notice that the coefficient of $\varphi$ and the constant term in each $L_n$ changes sign with $n$. This suggests that I can write $L_n = -(-1)^n F_n\varphi + (-1)^n F_{n+1}$. To verify that this is true, I only need to check that it is correct for $n = 0$ and 1, and that when I subtract $L_{n+1}$ from $L_n$, I get $L_{n+2}$. The latter part is verified by the following computation:

$$
\begin{aligned}
L_n - L_{n+1} &= (-(-1)^n F_n\varphi + (-1)^n F_{n+1}) - (-(-1)^{n+1} F_{n+1}\varphi + (-1)^{n+1} F_{n+2}) \\
&= -(-1)^n F_n\varphi + (-1)^n F_{n+1} + (-1)^{n+1} F_{n+1}\varphi - (-1)^{n+1} F_{n+2} \\
&= -(-1)^{n+2}(F_n + F_{n+1})\varphi + (-1)^{n+2}(F_{n+1} + F_{n+2}) \\
&= -(-1)^{n+2}(F_{n+2})\varphi + (-1)^{n+2}(F_{n+3}) \\
&= L_{n+2}
\end{aligned}
$$

as desired. Using this as our inductive step, to show that our formula for $L_n$ is valid, all that remains is to check that $L_0 = 1$ and $L_1 = \varphi - 1$, a check we leave to the reader.

Now we can move on to answer our original question. To do this we multiply together the length of the two sides of each rectangle to get its area, as we did in the chart:

$$L_nL_{n+1} = (-1)^{2n+1} F_nF_{n+1}\varphi^2 - (-1)^{2n+1} F_nF_{n+2}\varphi - (-1)^{2n+1} F_{n+1}^2\varphi + (-1)^{2n+1} F_{n+1}F_{n+2}.$$

Since $2n + 1$ is an odd number, we know that $(-1)^{2n+1} = -1$, so we can rewrite this equation as $L_nL_{n+1} = -F_nF_{n+1}\varphi^2 + (F_nF_{n+2} + F_{n+1}^2)\varphi - F_{n+1}F_{n+2}$.

There's another way to compute the lengths, widths, and areas of successive rectangles in the spiral of golden rectangles. Because of the similarity property of the golden rectangle, we know that each side length can be divided by $\varphi$ to get the next side length. We can also express this as $L_{n+1}/L_n = \varphi$. This means that we can take the chart of lengths and areas and instead of subtracting the two previous side lengths to get the next one, we can divide the lengths by $\varphi$ (see Chart B at right).

**Chart B**

| Length | Width | Area |
|---|---|---|
| $\varphi$ | 1 | $\varphi = \varphi$ |
| 1 | $1/\varphi$ | $1/\varphi = \varphi - 1$ |
| $1/\varphi$ | $1/\varphi^2$ | $1/\varphi^3 = -\varphi^2 + 3\varphi - 2$ |
| $1/\varphi^2$ | $1/\varphi^3$ | $1/\varphi^5 = -2\varphi^2 + 7\varphi - 6$ |
| $1/\varphi^3$ | $1/\varphi^4$ | $1/\varphi^7 = -6\varphi^2 + 19\varphi - 15$ |
| $1/\varphi^4$ | $1/\varphi^5$ | $1/\varphi^9 = -15\varphi^2 + 49\varphi - 40$ |
| $1/\varphi^5$ | $1/\varphi^6$ | $1/\varphi^{11} = -40\varphi^2 + 129\varphi - 104$ |

Comparing Charts A and B, we learn that $L_n = 1/\varphi^n = -(-1)^n F_n\varphi + (-1)^n F_{n+1}$. But wait a second! This formula shows us that $1/\varphi^{2n+1}$ is equal to both $F_{2n+1}\varphi - F_{2n+2}$ (which corresponds to our formula for $L_{2n+1}$) *and* $L_nL_{n+1} = -F_nF_{n+1}\varphi^2 + (F_nF_{n+2} + F_{n+1}^2)\varphi - F_{n+1}F_{n+2}$. The defining property of the golden rectangle tells us that $\varphi/1 = 1/(\varphi-1)$, or, rearranging terms, that $\varphi^2 = \varphi + 1$. If we replace $\varphi^2$ with $\varphi + 1$ in our expression for $L_nL_{n+1}$, we get:

$$\begin{aligned} L_nL_{n+1} &= -F_nF_{n+1}\varphi^2 + (F_nF_{n+2} + F_{n+1}^2)\varphi - F_{n+1}F_{n+2} \\ &= -F_nF_{n+1}(\varphi + 1) + (F_nF_{n+2} + F_{n+1}^2)\varphi - F_{n+1}F_{n+2} \\ &= (-F_nF_{n+1} + F_nF_{n+2} + F_{n+1}^2)\varphi - F_nF_{n+1} - F_{n+1}F_{n+2} \\ &= (F_n(F_{n+2} - F_{n+1}) + F_{n+1}^2)\varphi - F_{n+1}(F_n + F_{n+2}) \\ &= (F_n^2 + F_{n+1}^2)\varphi - F_{n+1}(F_n + F_{n+2}) \end{aligned}$$

Thus, we find that $(F_n^2 + F_{n+1}^2)\varphi - F_{n+1}(F_n + F_{n+2}) = F_{2n+1}\varphi - F_{2n+2}$. In other words,

$$(F_n^2 + F_{n+1}^2 - F_{2n+1})\varphi - (F_{n+1}(F_n + F_{n+2}) - F_{2n+2}) = 0.$$

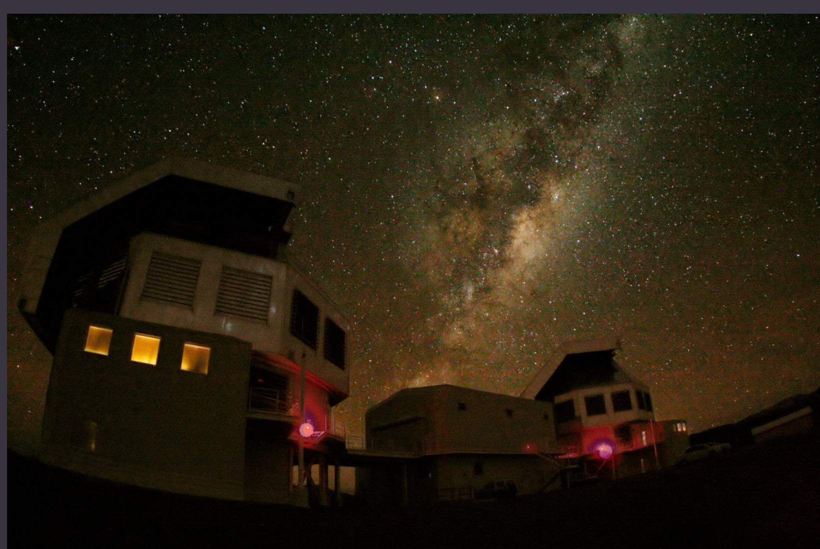
Because $\varphi$ is an irrational number, this is only possible if we separately have

$$F_n^2 + F_{n+1}^2 - F_{2n+1} = 0 \qquad \text{and} \qquad F_{n+1}(F_n + F_{n+2}) - F_{2n+2} = 0.$$

That is, we've just discovered the identities $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $F_{2n+2} = F_{n+1}(F_n + F_{n+2})$.

Chart B also shows that our formula for finding the side lengths of the golden rectangle also gives us a formula for $1/\varphi^n$, specifically, $L_n = -(-1)^nF_n\varphi + (-1)^nF_{n+1} = 1/\varphi^n$. As $n$ gets bigger and bigger, the fraction $1/\varphi^n$ will get smaller and smaller because $\varphi > 1$. Eventually the value will get very, very close to 0, so close that if one chooses an arbitrarily small positive number, the value of $1/\varphi^n$ will be even smaller than that. In other words, $\lim_{n\to\infty} 1/\varphi^n = 0$. Since $1/\varphi^n = L_n$, we can also say that $\lim_{n\to\infty} L_n = 0$. Accordingly, $\lim_{n\to\infty} -(-1)^n F_n\varphi + (-1)^n F_{n+1} = 0$, which implies that $\lim_{n\to\infty}(-\varphi + F_{n+1}/F_n) = 0$, or $\lim_{n\to\infty} F_{n+1}/F_n = \varphi$.

By studying the lengths of the rectangles in the spiral of golden rectangles, we've discovered the fact that the ratio of successive Fibonacci numbers tends to $\varphi$.



The night sky over Magellan, Dr. Anna Frebel's telescope in Chile. This stunning view of the Milky Way is quite different from the view in the Northern hemisphere. Photo courtesy of Dr. Anna Frebel.

# Calendar

Session 13: (all dates in 2014)

| | | |
|---|---|---|
| September | 12 | Start of the thirteenth session! |
| | 19 | No meet |
| | 26 | |
| October | 3 | |
| | 10 | |
| | 17 | Anna Frebel, Department of Physics, MIT |
| | 24 | |
| | 31 | |
| November | 7 | |
| | 14 | Kate Jenkins, Akamai Technologies |
| | 21 | |
| | 28 | Thanksgiving - No meet |
| December | 5 | |
| | 12 | |

Session 14: (all dates in 2014)

| | | |
|---|---|---|
| January | 30 | Start of the fourteenth session! |
| February | 6 | |
| | 13 | |
| | 20 | No meet |
| | 27 | |
| March | 6 | |
| | 13 | |
| | 20 | |
| | 27 | No meet |
| April | 3 | |
| | 10 | |
| | 17 | |
| | 24 | No meet |
| May | 1 | |
| | 8 | |

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

# Girls' Angle: A Math Club for Girls

Membership Application

**Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.**

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ______________________________________________________________________

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

______________________________________________________________________________________-

The $36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $36 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

**A Math Club for Girls**

# Girls' Angle
# Club Enrollment

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

**How do I enroll?** You can enroll by filling out and returning the Club Enrollment form.

**How do I pay?** The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

**Where is Girls' Angle located?** Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Lauren Williams, assistant professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ______________________________________________________________________

Address: ____________________________________________________________ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

**Emergency contact name and number**: ______________________________________________________________________

**Pick Up Info**: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____________________________________________________________________________________________________

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____________________________________________________________________________________________________

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

**Personal Statement (optional, but strongly encouraged!):** We encourage the participant to fill out the optional personal statement on the next page.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

___________________________________________________ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ___________________________________________________________________

| Members: Please choose one. | Nonmembers: Please choose one. |
|---|---|
| ☐ Enclosed is $216 for one session (12 meets) | ☐ I will pay on a per meet basis at $30/meet. |
| ☐ I will pay on a per meet basis at $20/meet. | ☐ I'm including $36 to become a member, and I have selected an item from the left. |

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

**Personal Statement (optional, but strongly encouraged!):** This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

## Girls' Angle: A Math Club for Girls

**Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

_____________________________________________________________________________________,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ___________________________________________________Date: ____________________

Print name of applicant/parent: ___________________________________________________

Print name(s) of child(ren) in program: ______________________________________________