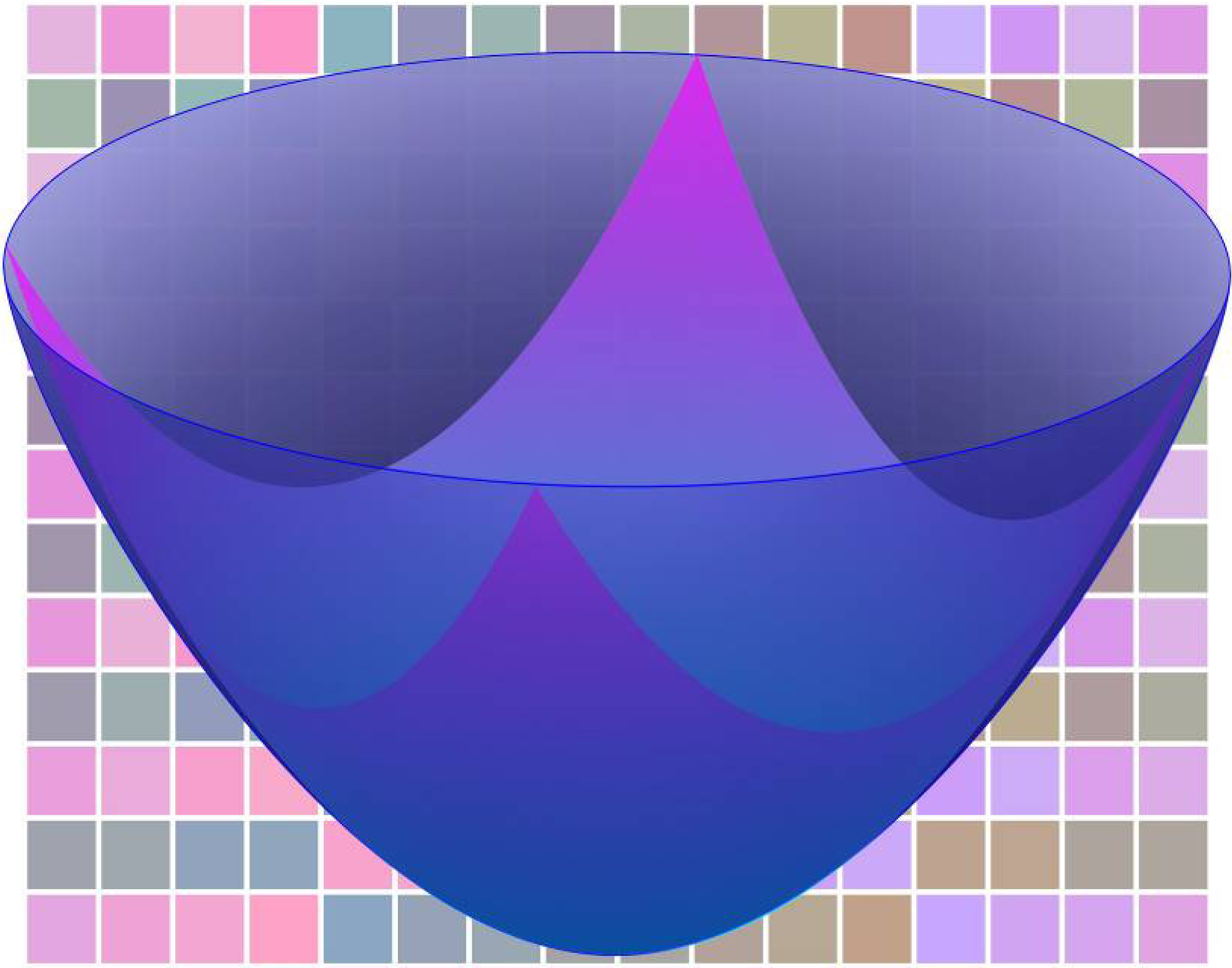
Girls' Angle Bulletin

June 2014 • Volume 7 • Number 5

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Members and subscribers, this Bulletin is yours. That means you have control over its contents. Much content is developed directly to address matters at the Girls' Angle club. And all members and subscribers are encouraged to suggest content. For example, Vida John requested an article on the "Chicken McNugget" problem. The result is this issue's *Math In Your World*. Don't be shy! If you'd like to see something addressed here, email us!

- Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Paraboloid of revolution. Image made with the use of MATLAB, a powerful suite of mathematical software produced by MathWorks.

An Interview with Marie Vitulli, Part 2

This is the concluding half of our interview with University of Oregon Professor Emeritus Marie Vitulli.

Ken: Will you please describe a mathematical result that you proved?

Marie: Together with my late colleague David Harrison, I developed a valuation theory for commutative rings that generalized both Krull valuations and Archimedean valuations (also known as absolute values) for fields. Valuations, particularly Archimedean valuations, measure the size of the objects in a field. The usual notion of absolute value on the field of real numbers is an Archimedean valuation.

A **field** is a commutative ring in which the equation $ax = 1$ has a solution whenever $a \neq 0$; we also say that every nonzero element in a field is **invertible**. A **valuation** on a field is a special function from the field to another mathematical structure called a **totally ordered Abelian group**. A group is Abelian if the operation is commutative. A totally ordered Abelian group is an Abelian group together with an ordering of the objects of the group that is compatible with the group operation and in which any two objects may be compared. Totally ordered Abelian groups are better behaved and understood than fields and we can use them and valuations to put more structure on fields with the aim of better understanding them.

Ken: How did you find the result and how did you prove it?

Marie: Dave Harrison asked me to co-advise his doctoral student Ken Valente when Dave left Eugene to work in Germany on a Humboldt Research Fellowship. The area that Ken was working on was new to me so I had to do some reading to be able to advise him. As I began to reflect on what they were doing I realized that if you replaced a totally ordered Abelian group by a more general object that we later called a *V*-monoid and replaced a field by a commutative ring you could describe some of what they were doing in terms of valuations. Dave and I later developed an entire theory and demonstrated the usefulness of this theory.

Ken: What are the outstanding mysteries in math that you find alluring?

Marie: There are many questions that are still unanswered that I would like to know how to solve. Many are not easy to state without a lot of background. One outstanding problem that I can easily describe is the Jacobian Conjecture. The conjecture was initially posed for polynomials in two variables by Ott-Heinrich Keller in 1939. The conjecture was named later by the late mathematician Shreeram Abhyankar who used it to illustrate a deep problem in algebraic geometry that could be stated and understood by anyone with some knowledge of calculus. Given a polynomial function F from complex n-space $\mathbf{C}^n$ to $\mathbf{C}^n$, where

$$F(c_1, \ldots, c_n) = (f_1(c_1, \ldots, c_n), \ldots, f_n(c_1, \ldots, c_n))$$

and $f_1, \ldots, f_n$ are polynomials in n variables $x_1, \ldots, x_n$, we define the **Jacobian determinant** of F, denoted by J_F, to be the determinant of the matrix of partial derivatives $(\partial f_i / \partial x_j)$. Notice that the Jacobian determinant is a polynomial function. It is straightforward to show that if the polynomial function $F(c_1, \ldots, c_n)$ has an inverse function that is also a polynomial function, then

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the complete interview with Prof. Vitulli, and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Finding the Maximum Subsequence, Part 1[1]

by Kate Jenkins | edited by Jennifer Silva

I'm writing today to introduce you to algorithms – to give you some idea of what they are, why they are interesting to think about, and why good algorithms are important for solving big, hard problems. My name is Kate Jenkins, and I'm a Principal Software Architect at Akamai Technologies. I have spent most of my career developing algorithms to make the internet work better and more efficiently. Sophisticated algorithms tell computers how to solve all kinds of problems that we often take for granted. For example, they are used for searching the web, analyzing DNA sequences, identifying possible new treatments for serious diseases, recognizing faces in photos, playing chess, driving cars, and in computational geometry. And that's just scratching the surface.

An algorithm is like a recipe, but instead of making food, it makes the answer to a problem. Like a recipe, you care about whether it works (makes the thing you wanted it to make) and how long it takes! We use math to analyze algorithms, and we also use algorithms to solve mathematical problems. So you can think of algorithms as being at the intersection of math and computer science.

At the Girls' Angle club, we talked about Dijsktra's shortest path algorithm, which finds the shortest path to take between two nodes in a network. Today we'll look at another problem and consider algorithms that can be used to solve it.

Suppose you have a sequence of numbers, which can include both negative and positive numbers, such as the following one consisting of 10 members:

10	-8	4	2	-9	8	4	-10	-1	12
1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}	9^{th}	10^{th}

The numbers below the line show the position of each number in the sequence. For instance, 10 is the 1^{st} number in the sequence, while 12 is the 10^{th}.

The **sum of the sequence** is the sum of all numbers in the sequence, in this case

$$10 + (-8) + 4 + 2 + (-9) + 8 + 4 + (-10) + (-1) + 12 = 12.$$

A **subsequence** is a sequence of numbers between some particular start and end position within the sequence. For example, the subsequence from the 3^{rd} to 6^{th} position is 4, 2, -9, 8, and it has a sum of 5.

Here's the problem we'll be considering today:

> Of all of the possible subsequences, which one has the biggest sum, and what is that sum?

Before continuing, take some time to think about the answer to this question for the sample sequence above.

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[1] This content supported in part by a grant from MathWorks.

Now think about how you came up with your answer. How would you write down instructions that someone (or something) could follow to find the right answer no matter what sequence was given?

You might say, "Look at all of the possible choices, compute the sum for each one, and remember which one was the biggest."

Let's try that on a simpler example:

1	**-2**	**2**	**-1**	**4**
1st	2nd	3rd	4th	5th

To come up with the answer using this algorithm, we would look at all of the possible start and end positions and compute the sum for each one. Here's a table with a column for each start position and a row for each end position. The entries represent each possible subsequence, with each entry showing the sum for that particular subsequence.

End \ Start	1st	2nd	3rd	4th	5th
1st	1 = **1**				
2nd	1 + (-2) = **-1**	-2 = **-2**			
3rd	1 + (-2) + 2 = **1**	-2 + 2 = **0**	2 = **2**		
4th	1 + (-2) + 2 + (-1) = **0**	-2 + 2 + (-1) = **-1**	2 + (-1) = **1**	-1 = **-1**	
5th	1 + (-2) + 2 + (-1) + 4 = **4**	-2 + 2 + (-1) + 4 = **3**	2 + (-1) + 4 = **5**	-1 + 4 = **3**	4 = **4**

The largest sum here is 5, which is the sum of the subsequence that starts at the 3rd position and ends at the 5th position in the sequence, so that is the answer this algorithm would provide.

The good news is that this gives the right answer! We know it does because it considers every possible answer and finds the best. This is often referred to as a "brute force" approach, because every possibility is examined.

The not-so-good news is that carrying out this algorithm is a lot of work. How many plus signs do you see in the above table? Whoever or whatever is following this algorithm does that many addition problems to come up with the right answer. (By "addition problem," I mean adding two numbers together.)

How many plus signs would you expect to see if there were 6 numbers in the sequence instead of 5? What if there were 10, 100, or 1000 numbers? Or any length N?

The glib answer is "a lot." I'll omit the details, but it turns out that the number of addition problems needed for this brute force approach is $(N^3 - N)/6$. For large N, the N^3 term dominates, so the computer scientist's answer is that the number is "on the order of N^3." This is written as $O(N^3)$ and means that N^3 is the biggest term in the answer, ignoring any constant factors.

If we study the above table carefully, we see that there are a lot of numbers that get added together over and over again. Perhaps we can exploit this observation and modify our algorithm to still get the right answer, but with a lot less work!

Looking at any column, we see that each row involves simply adding one new number to the sum in the row above it. If you already know that the sum of the subsequence starting at position i and ending at position j is S, then the sum of the subsequence starting in the same place but ending at $j + 1$ is just S plus the number in the sequence at position $j + 1$.

Utilizing this insight, we can modify the original algorithm to compute the sum for each subsequence more quickly, using what it already knows from previous subsequences. In the table below, '…' denotes the sum from the entry above.

1	-2	2	-1	4
1st	2nd	3rd	4th	5th

End \ Start	1st	2nd	3rd	4th	5th
1st	1 = **1**				
2nd	… + (-2) = **-1**	-2 = **-2**			
3rd	… + 2 = **1**	… + 2 = **0**	2 = **2**		
4th	… + (-1) = **0**	… + (-1) = **-1**	… + (-1) = **1**	-1 = **-1**	
5th	… + 4 = **4**	… + 4 = **3**	… + 4 = **5**	… + 4 = **3**	4 = **4**

Our modified algorithm still comes up with the same answer, but it requires substantially fewer addition operations to get there. Instead of the 20 we had to do with the original version, the new way only needs 10.

For a sequence of length N, how many addition operations would we need to do? One addition operation is performed for each subsequence that has an end position greater than its start position. There are $N - 1$ such subsequences with start position 1, $N - 2$ with start position 2, $N - 3$ with start position 3, etc., so the number of such subsequences is

$$(N - 1) + (N - 2) + (N - 3) + \ldots + 3 + 2 + 1 = N(N - 1)/2 = N^2/2 - N/2.$$

Since the dominant term, up to a constant factor, is N^2, computer scientists would call it $O(N^2)$.

The table that follows shows how many operations are needed by the two algorithms for a few different sequence lengths.

Sequence Length	Algorithm 1	Algorithm 2
5	20	10
6	35	15
10	165	45
100	166,650	4,950
1000	166,666,500	499,500

Algorithm 2 is much nicer, but it's still a lot of work for long sequences. Can we do better?

The answer is yes. It turns out that there's a way to find the answer with fewer than N addition operations. To do this, you don't compute the sum for every possible subsequence, only for some of them. But you still can prove that you get the right answer.

Play with the problem a bit and see if you can figure it out.

Continued at the bottom of page 29...

A Brilliant Mind

by Valeria Golosov[1] | edited by Jennifer Silva

Brahmagupta was a real person who lived in India in the 7th century. This is a fictionalized imagining of how he discovered his formula for the area of a cyclic quadrilateral.

Over a thousand years ago, on a hot, sunny day in India, Brahmagupta was walking along the Shipra River. The river went on for what seemed like miles. Around a bend, there appeared in the middle of the river a perfectly circular island covered in trees. Brahmagupta was gripped with curiosity and decided to investigate more closely. After a long and refreshing swim, he reached the distant shore. The island was rich with beautiful flowers, delicious fruit and tall palm trees. It was so tranquil and peaceful.

He walked deeper into the island and discovered a wall. It was tall and made of dark stones. It seemed to stretch to the edge of the island. Brahmagupta thought to himself that the wall was like the chord of a circle. Then he heard a voice.

Wealthy Merchant: Who dares disturb me on my island?

Brahmagupta looked up and saw, sitting atop the wall, a short, stubby man draped in the softest-looking and most colourful silks he had ever seen.

Brahmagupta: I'm sorry kind sir, but I was only curious to see this island because it appeared to be a perfect circle; I didn't know it belonged to someone.

The man looked inquiringly at the stranger. All of a sudden, he recognized him.

Wealthy Merchant: I know you – you are a famous mathematician. You are Brahmagupta. I will forget about your trespassing on my island if you can help me with a problem I have that requires your brilliant mind.

Brahmagupta was flattered and decided to help the man.

Brahmagupta: What is the problem?

Wealthy Merchant: I am building a grand monument dedicated to my father. We shared a love of mathematics, so I decided that I should build a monument in the shape of a beautiful geometric figure in his honour. My engineer is working on a flying machine so that people will be able to see the monument's perfect geometric shape from up high. I desire that you find the area of my monument so that I know the right amount of material for its roof.

Brahmagupta looked puzzled; how would he find the area if he could only see a wall?

Brahmagupta: Merchant, I can only see one side of the monument. I don't know what shape it is. For all I know, this wall *is* the monument.

Wealthy Merchant: Very well, I tell you that the shape is a cyclic quadrilateral. My engineer knows only its edge lengths.

Brahmagupta: Fine sir, what are the edge lengths?

Wealthy Merchant: That is a secret! Deliver me a formula so that my engineer can substitute whatever edge lengths he sees fit.

Brahmagupta: Kind sir, even knowing the edge lengths, there are still different cyclic quadrilaterals...

Wealthy Merchant: Enough already! I've business to attend to.

The Merchant disappeared and left Brahmagupta to think alone.

Brahmagupta took a stick and started to think. He decided to call the edge lengths a, b, c, and d. He reasoned that there are $4 \times 3 \times 2 \times 1 / 4 = 6$ different ways to order the edges of the quadrilateral, but the 6 orderings can be organized into 3 pairs where each pair consists of an

[1] Valeria Golosov is entering her last year at the Wimbledon High School in London, England.

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Anna's Math Journal

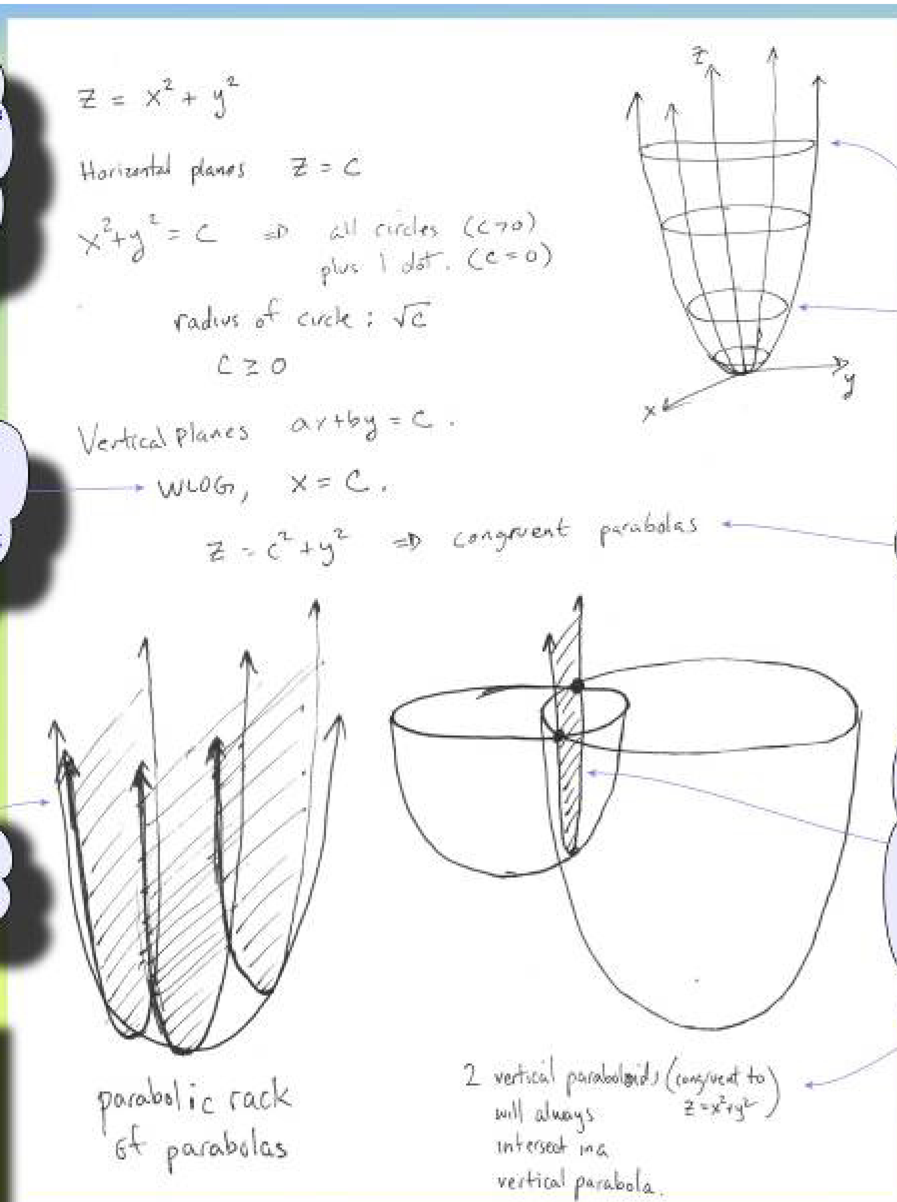
By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna explores paraboloidal cross sections (see the cover).

I was thinking about conic sections and how they can be obtained as cross sections of a double napped cone. It seemed to me that cross sections of a paraboloid of revolution should also be conics. I decided to investigate.

I'll begin by looking at cross sections of a basic paraboloid of revolution: $z = x^2 + y^2$, starting with hoizontal cross sections.



Horizontal cross sections are circles with centers on the z-axis.

I'll try vertical cross sections next. This paraboloid has rotational symmetry, so without loss of generality, I can assume the intersecting plane is $x = c$.

So, vertical cross sections are all parabolas... and they're all congruent to $y = x^2$.

Hmm. As c varies, it's like a parabola tracing out a surface. The paraboloid of revolution is a parabolic "rack" of parabolas!

Wait a sec! All vertical cross section are congruent to each other. That means if I take two paraboloids and slice them anywhere with a vertical plane, I can glue the two together along the cut. Or, another way of putting it, any two such vertically oriented paraboloids must intersect in a vertical parabola... absolutely vertical!

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Now I'll look at intersections with generic planes that are not vertical or horizontal.

Generic plane $ax+by+cz=d$.

Rotate so intersection of plane with xy-plane is a line parallel to y-axis.

By symmetry, I can rotate so that the plane intersects the *xy*-plane in a line parallel to the *y*-axis.

WLOG, $z = mx+b$.

$$mx+b = x^2+y^2$$
$$b = x^2 - mx + y^2$$
$$= \left(x-\tfrac{m}{2}\right)^2 + y^2 - \tfrac{m^2}{4}$$
$$b + \tfrac{m^2}{4} = \left(x-\tfrac{m}{2}\right)^2 + y^2$$

I'll combine the equations for the paraboloid and the plane... I get the equation of a circle.

Oh, but the circle isn't the cross section...it's the projection of the cross section onto the *xy*-plane.

The projection of the cross-section onto the xy-plane is a circle.

If $b+\frac{m^2}{4}=0$, the projected circle is a point ...

Hey...if the cross section projects to a point, it must mean that the plane is tangent to the paraboloid...

so $z = mx - \frac{m^2}{4}$ must be tangent to $z=x^2+y^2$ at $\left(\frac{m}{2}, 0, \frac{m^2}{4}\right)$.

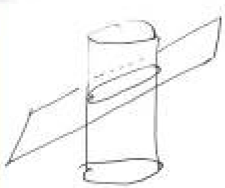
Since the projection to the *xy*-plane is a circle, the cross section is also the intersection of the paraboloid with a vertical cylinder.

Cross-section is the intersection of the plane $z=mx+b$ with the cylinder $b+\frac{m^2}{4} = \left(x-\frac{m}{2}\right)^2+y^2$.

$\Rightarrow$ an ellipse with semiminor axis $\sqrt{b+m^2/4}$

semimajor axis: $\sqrt{1+m^2}\,\sqrt{b+m^2/4}$

...but it is also the intersection of the cylinder with the intersecting plane, and a plane and cylinder intersect in an ellipse.



Summary Cross sections of $z=x^2+y^2$ are

1. parabolas congruent to $y=x^2$ obtained by vertical planes
2. ellipses which become circles when the intersecting plane is horizontal.
3. points, when the intersecting plane is tangent.

I think I'll summarize what I've found so far.

So, paraboloidal cross sections, at least for this particular paraboloid, don't yield all conic sections. Hyperbolas are left out.

Projection to the xy-plane sends cross sections that are parabolas to lines
ellipses to circles
and all lines and circles are possible.

The projections of cross sections yield all the lines and circles in the plane... that's curious and intriguing... it makes me think of stereographic projection, because that sends cross sections of a sphere also onto all the lines and circles in a plane. Hmm...

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Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Chocolate Chip Cookies Count

by Addie Summer
edited by Jennifer Silva

Logo Design by Hana Kitasei

"17 of your choice chocolate chip cookies, please!" I requested of Mr. ChemCake.

"17?" responded Mr. ChemCake. "That's an unusual number! Ah, this must be for Yellow Pig Day?"

"Yep! I've got a friend at HCSSiM this summer, and I want to surprise her."

"Well, let's see. I sell them in packs of 4 and 7, so how many of each size would you like?"

I began thinking. I didn't want to order more than 4 packs of 4, because that would give me more than 17 cookies. If I got 4 packs of 4, I'd have 16 cookies and I'd need 1 more, but I couldn't purchase single cookies. If I got 3 packs of 4, that would be 12 cookies, and I'd still need 5 more – not possible. If I got 2 packs of 4, I'd have 8 cookies and I'd need 9 more, but 9 is also not a multiple of 7. With 1 pack of 4, I'd need 13 more cookies, which is also not a multiple of 7. And 17 isn't a multiple of 7, so 17 can't be had with only packs of 7. Argh!

"It's impossible!" I declared, wondering how I always seem to end up doing math at Cake Country. "Oh well. I'll go ahead and get 18 cookies: 1 pack of 4 and 2 packs of 7, please."

"How about this, Addie," said Mr. ChemCake as he slipped on some gloves. "I'll sneak one out of the pack of 4. Nobody will know the difference! Besides, I'm the one who packages these. I'll charge you for 17, and you can have this extra one compliments of the house!"

"That'd be perfect! Thanks, Mr. ChemCake!

On the way home, I wondered this: How many chocolate chip cookies can one buy if they only come in packs of 4 and 7? If I purchase x packs of 4 and y packs of 7, then I would have a total of $4x + 7y$ chocolate chip cookies. So if I want to be able to purchase N cookies, I would need to solve the equation $4x + 7y = N$, and the solutions x and y would have to be nonnegative integers. I figured out that there are no such solutions when $N = 17$, but $x = 1$ and $y = 2$ is a solution for $N = 18$ cookies.

In the xy-coordinate plane, the solutions to the equation $4x + 7y = N$ correspond to a straight line. Different values of N correspond to different lines that are all mutually parallel to each other; they all have slope -4/7. The points in the xy-coordinate plane that have integral x and y coordinates form a square **lattice**. So the question of whether N cookies can be purchased is equivalent to whether there is a lattice point with nonnegative coordinates on the straight line $4x + 7y = N$.

I decided to ignore the nonnegative condition for the moment and focus on understanding what lattice points, if any, are on the line $4x + 7y = N$. By writing the equation as $7y = N - 4x$, I could see that the equation has an integral solution if there is an integer x for which $N - 4x$ is a multiple of 7. Here, I made use of a fundamental fact about numbers: because 4 and 7 are

Suppose A and B are relatively prime. Consider the first B multiples of A:

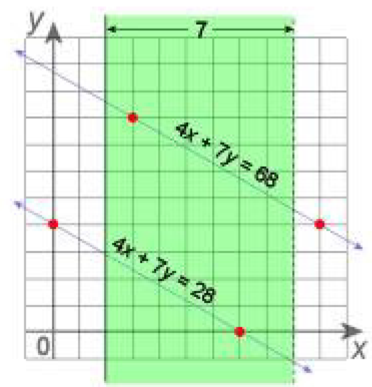
$$A, 2A, 3A, 4A, ..., BA.$$

If any two of these multiples, say jA and kA with $j < k$, leave the same remainder upon division by B, then their difference, $(k - j)A$, is divisible by B. Since A and B are relatively prime, this implies that B must divide $k - j$, but $0 < k - j < B$, so this is impossible.

Therefore, if we divide these first B multiples of A by B, we will get all B possible remainders.

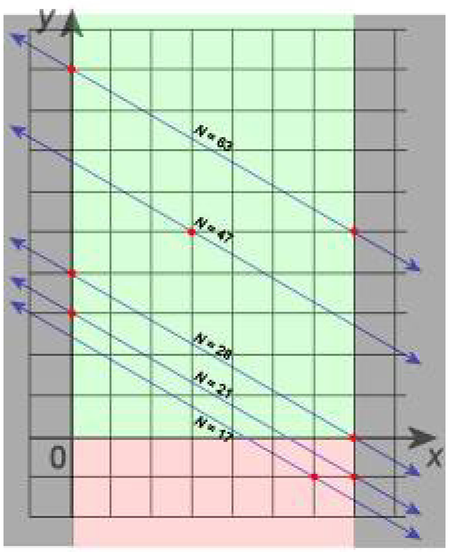
Since kA and $(k + B)A$ leave the same remainder upon division by B, the sequence of remainders left by multiples of A is periodic with period B.

relatively prime, the multiples of 4, when divided by 7, leave remainders that cycle through all possible values 0, 1, 2, 3, 4, 5, and 6 in some fixed order (for proof, see the blue box at left). So not only do I know that there are values of x such that $N - 4x$ is divisible by 7, but also that such values of x form an infinitely long arithmetic sequence with common difference 7. For example, when $N = 18$, we have the solution $x = 1$, $y = 2$. Since $x = 1$ is part of a solution, so will be $x = 8$, 15, 22, 29, etc., as well as $x = -6$, -13, -20, etc.



This regularity means that every 7-unit wide vertical strip that includes its left boundary but not its right will contain exactly 1 lattice point on the line $4x + 7y = N$. We could also use strips that include their right boundary but not their left. By including one boundary but not both, we avoid the situation where the strip just manages to contain 2 lattice points on the line, one on each boundary.

Armed with this knowledge, let's reinstate the nonnegative coordinate condition. Because our lines have negative slope, the smaller x is, the larger y is. Therefore, if a line has a lattice point with nonnegative coordinates, it must have one in the vertical strip of points whose x-coordinates satisfy $0 \le x < 7$. For this reason, we confine our attention to this strip as indicated in the graph below. I've colored the vertical strip green and red. Red points correspond to points with $y < 0$. I'll use the word "segment" to refer to the part of the line $4x + 7y = N$ inside this vertical strip defined by the inequality $0 \le x < 7$. (Note that the vertical strip does not include its right boundary.)



When N is large, the segment will be high up on the graph and all of its points will have nonnegative coordinates. Let's imagine gradually decreasing N. As N decreases, the segment drops. When N reaches 4×7, the lower tip of the segment reaches the x-axis. As N drops below 4×7, the segment pokes into the fourth quadrant. Because the segment has negative slope, the first lattice point it will "brush by" with negative y-coordinate will be the point (7, -1), which sits on the line $4x + 7y = 21$. (Note that (7, -1) is the lower tip of the segment, but is not technically part of the segment since its x-coordinate is not strictly less than 7.) Can we purchase 21 cookies? Sure we can: because (7, -1) is on the right boundary of the strip, the other end of the segment, which is (0, 3), will give us a way to get 21 cookies: buy 3 packs of 7. To find the largest unobtainable number, we must touch a lattice point inside the red region, not brush by one on its excluded right border (where (7, -1) is located).

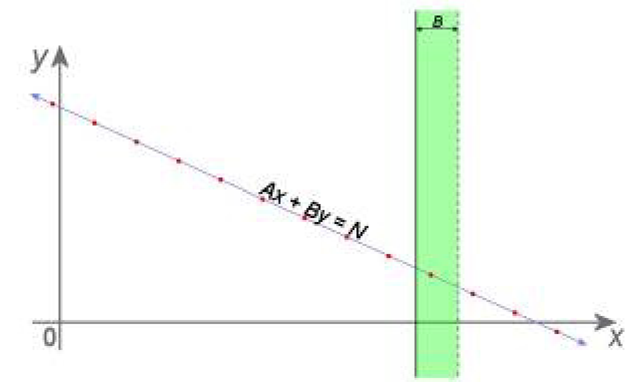
As we slide our segment further down, gradually decreasing N, the line segment will slip between the lattice points (6, -1) and (7, -1). All the values of N we pass through on this leg of our descent represent numbers of chocolate chip cookies that we can obtain because the lattice point on such segments can't be in the red region. The first lattice point in the red region that the segment will actually touch will be (6, -1). Here, for the first time, we reach a value of N that cannot be obtained. The lattice point (6, -1) is on the line $4x + 7y = 17$.

It turns out that 17 is the largest number of chocolate chip cookies that cannot be purchased at Cake Country. What bad luck!

We can now completely answer our question. We just have to check which of the finite number of numbers between 0 and 18 cannot be obtained, and this will constitute a complete list of unobtainable numbers of cookies. They are: 1, 2, 3, 5, 6, 9, 10, 13, and 17. All numbers of cookies above 17 are obtainable, as the argument above shows.

Generalizing As soon as I arrived home, I felt compelled to work through the entire argument from the top using packs of A and B cookies, instead of 4 and 7. If A and B share a common factor, then any number of cookies that I purchase would have to be a multiple of that common factor, and I would never be able to buy a quantity of cookies that isn't a multiple of the common factor. For this reason, I assumed that A and B are relatively prime positive integers. Also, without loss of generality, I assumed that $A < B$. (If $B > A$, then I would just switch the packs.)

Everything in the argument above goes through without complication in this more general setting. In the general setting, the equation to solve is the line $Ax + By = N$. We again seek solutions that are lattice points with nonnegative coordinates. For fixed N, because A and B are relatively prime, there will be lattice points on the line and they will be equally spaced every B horizontal units.



As before, we focus attention on the vertical strip in the xy-coordinate plane given by $0 \le x < B$. For all integers N, the segment of the line $Ax + By = N$ inside the strip will contain exactly 1 lattice point. Critically, if the line $Ax + By = N$ has a lattice point with nonnegative coordinates, it must have one in the strip defined by $0 \le x < B$.

When N is large, the segment consists entirely of points with nonnegative coordinates. As N decreases, the segment drops. When $N = AB$, the lower tip of the segment reaches the x-axis at the point $(B, 0)$ (which is the endpoint of the segment, but not technically contained in the vertical strip; where is the lattice point on $Ax + By = AB$ with $0 \le x < B$?). As the segment continues to drop, it will brush by the lattice point $(B, -1)$ when $N = AB - B$, a lattice point at one endpoint of the segment on the excluded right border of the vertical strip. The segment's other endpoint, $(0, A - 1)$, is a lattice point with nonnegative coordinates. As N further decreases from $AB - B$, the first lattice point with negative coordinates that our segment will actually touch is $(B - 1, -1)$, and this lattice point is on the line $Ax + By = A(B - 1) + B(-1) = AB - A - B$. Thus, $N = AB - A - B$ is the largest number of cookies that cannot be obtained by purchasing packs of A and B cookies. Using this number as an upper limit, we can then check the finite number of smaller cases to determine exactly which numbers of cookies are unobtainable.

The number of unobtainable cookie quantities We can calculate how many quantities are unobtainable by carefully counting lattice points in the region of the plane defined by the inequalities $0 \le x < B$, $y < 0$, and $Ax + By \ge 0$. (When finding the unobtainable quantities of cookies, it would be helpful to know exactly how many we should be looking for.)

Challenge. Try this on your own before reading further!

Because every line of the form $Ax + By = N$, where N is an integer, has precisely 1 lattice point inside the vertical strip $0 \le x < B$, the number of lattice points inside this vertical strip that are above the line $Ax + By = 0$ and below the line $Ax + By = AB$ is exactly equal to $AB - 1$. (That is, each integer value of N strictly between 0 and AB corresponds to a unique lattice point in the described region, and vice versa.) I'll refer to the region inside the vertical strip $0 \le x < B$ and (strictly) between the lines $Ax + By = 0$ and $Ax + By = AB$ as R.

The region R, together with its boundary, is a parallelogram. This parallelogram, like all parallelograms, enjoys 180-degree rotational symmetry about its center – in this case, the point $(B/2, 0)$. Under this symmetry, lattice points are mapped to lattice points. Lattice points below the x-axis are swapped with those above the x-axis.

Let U be the number of quantities that are unobtainable. These unobtainable quantities correspond precisely to the lattice points in R that have negative y-coordinates. By symmetry, there are U lattice points in R in the first quadrant (which excludes the axes). In addition to these points, there are also $B - 1$ lattice points in R on the positive x-axis (namely, the points (1, 0), (2, 0), (3, 0), ..., $(B - 1, 0)$). We have so far accounted for all lattice points in R except for those on the y-axis, of which there are $A - 1$ (specifically (0, 1), (0, 2), (0, 3), ..., $(0, A - 1)$). We conclude that $AB - 1 = 2U + (B - 1) + (A - 1)$. Solving for U, we find

$$U = \frac{1}{2}(A-1)(B-1).$$

(Note that we do not want to count lattice points on the right border of the parallelogram, i.e., points whose x-coordinate is equal to B, because then we would make the mistake of double counting the numbers N that are multiples of B.)

Take It To Your World In the setup of the above discussion, show that the second highest unobtainable quantity of cookies is $AB - 2A - B$. Can you find a formula for the third highest unobtainable quantity of cookies? (Hint: split into cases depending on whether or not $A < B/2$.)

Using only 39-cent and 20-cent stamps, what is the largest postage value you would not be able to create exactly? How many postage values would you not be able to (exactly) create?

In the past, McDonald's sold its Chicken McNuggets in boxes of 6, 9, and 20. If you only purchased boxes with 9 or 20 nuggets, what is the largest number of nuggets you would not be able to (exactly) purchase? What would the answer be if you used only the boxes with 6 or 20 nuggets? What if you used only the boxes with 6 or 9 nuggets? What is the largest number of nuggets you would not be able to buy (exactly) using all 3 box types?

Following W. J. C. Sharp, rederive the formula $U = (A - 1)(B - 1)/2$ for the number of unobtainable quantities by studying the polynomial

$$p(x) = (1 + x^A + x^{2A} + \ldots + x^{AB})(1 + x^B + x^{2B} + \ldots + x^{AB}).$$

(Hint: compute $p(1)$ in two different ways.)

The problem Addie discusses is a special case of a more general problem known as the **Frobenius coin problem**, after the mathematician F. G. Frobenius. The Frobenius coin problem asks for the largest amount of money that cannot be produced exactly using a finite set of coin denominations. The formulas for the largest unobtainable amount and the number of unobtainable amounts for coins of 2 relatively prime denominations were known to James Joseph Sylvester in the 19th century. To see another lattice argument similar to the one used to find U, see Cailan Li's Summer Fun problem set on page 24.

Summer Fun!

The best way to learn math is to do math, so here are the 2014 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems before seeing solutions.

Some problems are quite a challenge and could take several weeks to solve, so please don't approach these problem sets with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can feel frustrating to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Magic Squares

by Lightning Factorial

A magic square is a square array of distinct numbers in which the numbers along each row, each column, and both diagonals add up to the same number. In this Summer Fun problem set we'll explore some of the mathematics behind magic squares.

Any number can serve as a 1 by 1 magic square, so let's move on to 2 by 2 squares.

1. Let $\begin{matrix} a & b \\ c & d \end{matrix}$ be a 2 by 2 square of numbers. To be a magic square, we must have

$$a + b = c + d = a + c = b + d = a + d = b + c.$$

Show that these equations force all 4 numbers to be equal to each other. Since magic squares are supposed to consist of distinct numbers, we conclude that there are no 2 by 2 magic squares.

Let $\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$ be a 3 by 3 magic square.

2. Before we analyze 3 by 3 magic squares systematically, try to construct one.

3. Let S be the sum of any row, column, or diagonal. If we add the numbers along both diagonals and the middle row and column, we will get $4S$. On the other hand, show that this sum is also equal to $a + b + c + d + 4e + f + g + h + i$. From these two facts deduce that

$$e = S/3 = (a + b + c)/3.$$

4. Problem 3 tells us that once we specify the top row, we have also determined e, the entry in the center of the square:

$$\begin{matrix} a & b & c \\ ? & \dfrac{a+b+c}{3} & ? \\ ? & ? & ? \end{matrix}$$

Express the unknown entries in the square in terms of a, b, and c by applying the fact that the numbers in each row, each column, or either diagonal add up to the same number $a + b + c$.

Using Problem 4, we can construct infinitely many squares whose rows, columns, and diagonals add up to the same number. However, if we just substitute any numbers for a, b, and c, the resulting square may have duplicate numbers. Next, we'll explore criteria that guarantee that all 9 numbers in the square will be different from each other.

Summer Fun!

5. Following the 19th century mathematician Edouard Lucas, replace a, b, and c in the square you derived in Problem 4 with the following substitutions:

$$\begin{aligned} a &= x+z \\ b &= z-x-y \\ c &= y+z \end{aligned}$$

6. (Lucas) Show that the square will be a magic square of positive numbers when:

$$0 < x < y < z - x \text{ and } y \neq 2x.$$

7. An n by n magic square is **normal** if it contains the integers from 1 to n^2, inclusive. Find a function of n that gives the sum of any row, column, or diagonal in a normal n by n magic square.

8. In a 3 by 3 normal magic square, what must the central number be?

9. Construct a normal 3 by 3 magic square.

10. Construct a normal 4 by 4 magic square.

According to the Online Encyclopedia of Integer Sequences, there are 880 different 4 by 4 normal magic squares and 275,305,224 different 5 by 5 normal magic squares (see sequence A006052). Here, magic squares are considered "different" if one cannot be obtained from the other by rotating or reflecting the square. It is unknown how many 6 by 6 (or larger) normal magic squares there are.

Because there are so many normal magic squares, a variety of ad hoc approaches for devising them have been invented. The remaining problems give an example devised by Euler which he published in Commentationes arithmeticae **2**, in 1849, under the title, "De quadratis magicis."

11. First, show that the numbers from 1 to 16 can uniquely be written as $X + x$, where X is taken from the set {0, 4, 8, 12} and x is taken from the set {1, 2, 3, 4}.

12. Examine the following square array of sums: $\begin{array}{cccc} a+\alpha & b+\delta & c+\beta & d+\gamma \\ d+\beta & c+\gamma & b+\alpha & a+\delta \\ b+\gamma & a+\beta & d+\delta & c+\alpha \\ c+\delta & d+\alpha & a+\gamma & b+\beta \end{array}$. Suppose we assign the numbers 0, 4, 8, and 12 to the 4 Latin letters a, b, c, and d in some order and assign the numbers 1, 2, 3, and 4 to the 4 Greek letters α, β, γ, and δ in some order. Show that the resulting array of numbers will be a 4 by 4 normal magic square.

13. Show that Euler's method in Problem 12 can be used to produce 576 normal 4 by 4 magic squares. Are they all "different"?

For further reading, check out *Magic Squares and Cubes*, by W. S. Andrews.

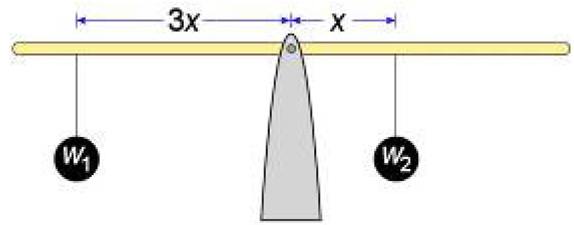
Summer Fun!

Center of Mass and Mass Points

by Johnny Tang and Girls' Angle Staff, edited by Margo Dawes

The technique known as "Mass Points" is based on the fact that the center of mass can be computed piecemeal. The following problems can be solved using the technique.

If these problems cause you difficulty, please review Volume 7, Number 3 of this Bulletin.



1. Let X be the center of mass of two points masses located at points A and B. If $AX : XB = 1 : 3$, what is the ratio of the mass at A to that at B?

2. In triangle ABC, let D, E, and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively.

A. Place a unit mass point at vertex A. What mass should be placed at vertex B so that the center of mass of the two masses is located at F? What mass should be placed at vertex C so that the center of mass of the two masses at A and C is located at E?

B. With masses assigned to the vertices of triangle ABC as in Part A, where is the center of mass of the point masses at vertices B and C?

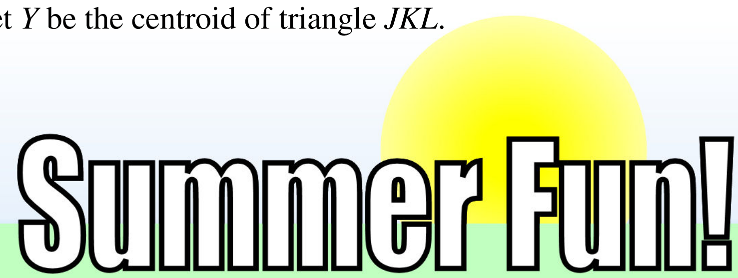
C. Using properties of the center of mass, deduce that the center of mass of all 3 point masses must be on $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Conclude that the medians of a triangle are concurrent. The fact that the medians intersect where the center of mass of 3 equal point masses placed at each vertex is located is the reason why the common point of intersection is called the **centroid** of the triangle.

D. Using mass points, conclude that the centroid splits each median in the ratio 1 : 2.

3. Varignon's theorem states that the midpoints of any quadrilateral form the vertices of a parallelogram. Prove this theorem using mass points.

4. Let P be a polygon. Let P' be a polygon whose vertices are the midpoints of the edges of P. Place unit point masses at the vertices of P and P'. Show that the center of mass of P is located in the same place as the center of mass of P'.

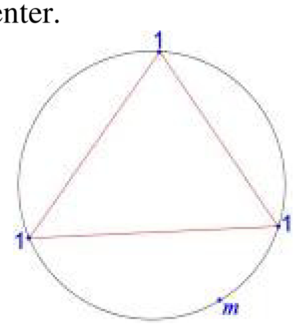
5. Let D be a point outside triangle ABC. Let X be the centroid of ABC. Let J, K, and L be the midpoints of $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$, respectively. Let Y be the centroid of triangle JKL. What is $DX : DY$?



6. In triangle ABC, the angle bisector at vertex A meets $\overline{BC}$ at X and the angle bisector at vertex B meets $\overline{CA}$ at Y. Let M be the intersection of $\overline{AX}$ and $\overline{BY}$. Given that $AB = 7$, $BC = 3$, and $CA = 6$, determine $BM : MY$.

7. In triangle ABC, cevians $\overline{AD}$ and $\overline{BE}$ intersect at P. Suppose that $AP = 8$, $BP = 9$, $DP = 5$, $PE = 4$, and the area of triangle ABC is 13 square units. What is the area of triangle ABP?

8. Place unit point masses at the vertices of triangle ABC. A point mass of mass m is placed on the circumcircle of triangle ABC. The four masses balance at the circumcenter.

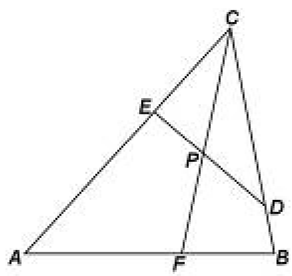


A. Show that $m^2 = 3 + 2(\cos 2A + \cos 2B + \cos 2C)$, where A, B, and C are the angles of the triangle.

B. Using Part A, prove that

$$9 > 3 + 2(\cos 2A + \cos 2B + \cos 2C) \geq 0$$

with equality if and only if triangle ABC is equilateral.



9. In triangle ABC, points D, E, and F are placed on the sides as shown in the figure at left. Point P is the intersection of $\overline{DE}$ and $\overline{CF}$. Given that $AE : EC = x : y$, $BD : DC = x' : y'$, and $AF : FB = v : w$, express $FP : PC$ in terms of x, x', y, y', v, and w. When $\overline{DE}$ is parallel to $\overline{AB}$, i.e., when $x/y = x'/y'$, show that your formula equals x/y.

10. Fix a positive number R. On the number line, we place a point mass of mass m_n at n, for each nonnegative integer n. We start with $m_0 = 1$. The other masses are chosen so that the center of mass of the point masses at 0, 1, 2, 3, …, n is located at Rn.

A. When $R = 1/2$, show that $m_k = 1$ for all k.

B. When $R = 2/3$, what is m_k?

C. When $R = 1 - 1/p$, where p is a positive integer greater than 1, what is m_k?

11. Here's a way to approximate π using mass points. Inscribe a regular n-gon in the unit circle centered at the origin of the xy-coordinate plane so that (1, 0) is one of the vertices. Label the vertices 1 through n starting at (1, 0) and going around in the clockwise direction. Place a point mass at each vertex of mass equal to the label of that vertex. Show that the center of mass of the polygon is located at

$$\frac{1}{n+1}\left(-1, \cot\frac{\pi}{n}\right).$$

Show that as n tends to infinity, the center of mass tends to $(0, 1/\pi)$.

Summer Fun!

Quadratic Reciprocity

by Cailan Li

In this problem set, p and q always denote distinct odd prime numbers.

In algebra, quadratic equations play a fundamental role. Quadratic reciprocity arises when one studies the modular arithmetic version of quadratic equations. Recall that in modular arithmetic, we write $a = b \pmod{n}$ if and only if $a - b$ is a multiple of n.

1. Let a, b, and c be constants. Assume that $a \neq 0 \pmod{p}$. Show that $ax^2 + bx + c = 0 \pmod{p}$ can be solved for x if and only if $b^2 - 4ac$ is a square modulo p, i.e. if and only if there exists y such that $y^2 = b^2 - 4ac \pmod{p}$.

Problem 1 leads us to examine which a (this is a different a from the one in Problem 1) admit a solution to the modular equation $x^2 = a \pmod{p}$. In other words, what a are squares modulo p? The Legendre symbol was introduced by Legendre to conveniently encapsulate this question:

$$\left(\frac{a}{p}\right) \equiv \begin{cases} 1 \text{ if } a \text{ is a square modulo } p \text{ and } a \neq 0 \pmod{p}, \\ 0 \text{ if } a = 0 \pmod{p}, \\ -1 \text{ otherwise.} \end{cases}$$

2. Let g be a primitive root modulo p. (A primitive root modulo p is a number g such that the exponentials $g, g^2, g^3, \ldots, g^{p-1}$ contain a complete set of residues modulo p. For details, see page 14 of Volume 6, Number 5 of this Bulletin.) Show that the squares modulo p are given by g^x where $1 < x < p$ is even.

3. Determine $\left(\frac{2}{3}\right)$, $\left(\frac{2}{7}\right)$, $\left(\frac{5}{13}\right)$, and $\left(\frac{3}{19}\right)$. Think about how you might determine $\left(\frac{1041}{101}\right)$.

4. Assume $x \neq 0 \pmod{p}$. Fermat's little theorem says that $x^{p-1} = 1 \pmod{p}$. Use this to show that $x^{(p-1)/2} = \pm 1 \pmod{p}$. Deduce Euler's criterion, which says that $\left(\frac{x}{p}\right) = x^{(p-1)/2} \pmod{p}$.

5. Use Euler's criterion to show that $\left(\frac{-1}{p}\right) = 1$ if and only if $p = 3 \pmod{4}$.

For the remaining problems, fix $b \neq 0 \pmod{p}$. For each $1 \leq k \leq (p-1)/2$, let r_k be the unique integer such that $-p/2 < r_k < p/2$ and $r_k = kb \pmod{p}$.

6. Show that $|r_k| = |r_j|$ if and only if $k = j$. Therefore, the set of absolute values of r_k is equal to $\{1, 2, 3, \ldots, (p-1)/2\}$.

7. Let N be the number of k such that $1 \leq k \leq (p-1)/2$ and $r_k < 0$. Using Problem 6, observe that $b(2b)(3b) \cdots ((p-1)b/2) = (-1)^N \cdot 1 \cdot 2 \cdot 3 \cdots ((p-1)/2) \pmod{p}$. By cancelling common factors, we deduce Gauss's lemma: $b^{(p-1)/2} = (-1)^N \pmod{p}$.

Combining Euler's criterion and Gauss' lemma, we see that b is a square modulo p if and only if N is even.

Summer Fun!

8. Repeat Problem 5 using Gauss's lemma by counting how many r_k are negative for b = -1.

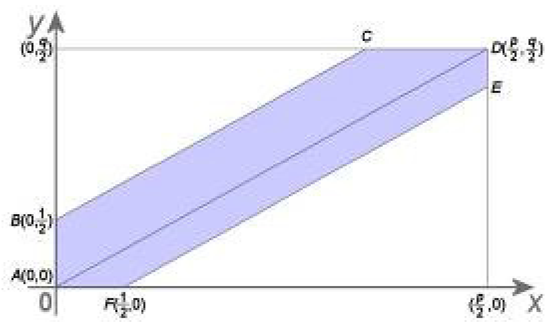
9. Again, by counting the number of negative r_k, use Gauss's lemma to show that $\left(\frac{2}{p}\right)$ = 1 if and only if $p = \pm 1 \pmod 8$.

10. Show that $\left(\frac{xy}{p}\right) = \left(\frac{x}{p}\right)\left(\frac{y}{p}\right)$. For this reason, we will focus on determining $\left(\frac{x}{p}\right)$ for prime numbers x.

Gauss's Law of Quadratic Reciprocity reveals a beautiful pattern concerning squares in modular arithmetic. The law states that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$ unless p and q are both congruent to 3 modulo 4, in which case $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.

11. Before we embark on a proof, use the law to recompute the Legendre symbols in Problem 3.

12. Let N_1 be the number of negative numbers obtained when the first $(p - 1)/2$ multiples of q, namely, q, $2q$, $3q$, ..., $(p - 1)q/2$, are reduced modulo p to integers between $-p/2$ and $p/2$. Similarly, let N_2 be the number of negative numbers obtained when the first $(q - 1)/2$ multiples of p, namely, p, $2p$, $3p$, ..., $(q - 1)p/2$, are reduced modulo q to integers between $-q/2$ and $q/2$. Show that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$ if and only if $(-1)^{N_1+N_2} = 1$.



Consider the figure at right. Lines BC, AD, and FE are parallel. Let H be the hexagon $ABCDEF$.

13. Show that there are no lattice points in the interior of H on the diagonal AD.

14. Show that there are N_1 lattice points in the interior of H that are above the diagonal AD. Similarly, show that there are N_2 lattice points in the interior of H that are below the diagonal AD. Thus, there are $N_1 + N_2$ lattice points in the interior of H.

15. Let (x, y) be a lattice point in the interior of H. Show that $((p + 1)/2 - x, (q + 1)/2 - y)$ is also a lattice point in the interior of H. Use this to show that the number of lattice points in the interior of H is odd if and only if $p = q = 3 \pmod 4$.

16. Combine Problems 14 and 15 to deduce Gauss's Law of Quadratic Reciprocity.

The proof of quadratic reciprocity presented in these problems follows the proof by D. H. Lehmer in *A Low Energy Proof of the Reciprocity Law*, which appeared in the American Mathematical Monthly, Volume 64, Number 2.

Signs of Permutations

by Ken Fan

A permutation is a rearrangement of objects. Place five empty boxes in a row and label them 1 through 5. Place a different object in each box. We can describe a rearrangement of these objects by describing which box the object in box x is moved to. For example, we can say that the object in box 1 moves to box 3, the object in box 2 moves to box 5, the object in box 3 moves to box 2, the object in box 4 stays in box 4, and the object in box 5 moves to box 1. To reduce the amount of writing we have to do, we can conveniently describe all this by writing a list of numbers: 3 5 2 4 1. The kth number in this list tells us which box the object in box k is moved to. For example, 5 4 3 2 1 represents the permutation where the order of the items is flipped. This notation for writing down a permutation is called **one-line notation**.

To be a permutation, each box number must be listed exactly once. Also, there can be any number of boxes, not just 5. If p represents a permutation, we will also write $p(x)$ to denote which box the contents of box x are sent. For example, if p = 3 5 2 4 1, then $p(3) = 2$.

1. Suppose there are n boxes. In one-line notation, what permutation leaves each object in place.

2. With n boxes, show that there are $n!$ permutations. Compute the values of $n!$ for n = 1, 2, 3, 4, and 5. The extreme growth of these numbers is referred to as **combinatorial explosion**.

3. Consider the permutations s = 3 1 5 2 4 and t = 5 2 4 1 3. We can get another permutation by applying t and then s. If we do this, the object that started in box 1 will end up in box 4. That's because the permutation t will move the object in box 1 to box 5, and the permutation s will then move that object from box 5 to box 4. If you apply t then s, where will the objects in boxes 2, 3, 4, and 5 end up?

4. Using the notation in Problem 3, write down the one-line notation for the permutation that is obtained by applying s first, and then t. Notice that this permutation is different from the one where you apply t first, followed by s.

We will write st for the permutation obtained by applying ***t* first and then *s***. It may seem strange to do that because in st, the s is written first. Sometimes people do use the opposite convention, but we'll let st mean "t first, then s" because this is consistent with thinking of permutations as functions and st as function composition. The permutation st is also called the product of s and t. To underscore our convention, when you multiply s by t to get st, it means the permutation obtained by applying t first, then s.

Let's fix a positive integer n and use n boxes. All permutations on these n boxes can be built from special permutations called **transpositions**. A transposition is a permutation that exchanges the contents of two boxes while leaving all other objects in place.

5. Convince yourself that all permutations can be built by multiplying together transpositions. For example, if $n = 3$, the permutation 3 1 2 is equal to the product of the transposition 1 3 2 with the transposition 2 1 3.

Summer Fun!

6. Show that there are $n(n - 1)/2$ transpositions.

Some permutations don't mess up the order of our objects very much. Others reorder things quite a bit. One way we can measure just how much a permutation reorders things is to count how many pairs of box labels (x, y), with $1 \le x < y \le n$, have their contents moved to boxes with the reverse order. In other words, if p is a permutation, we count how many pairs (x, y), with $1 \le x < y \le n$, satisfy $p(x) > p(y)$. We will denote by $N(p)$ this count. Since there are a total of $n(n - 1)/2$ such pairs we know that $0 \le N(p) \le n(n - 1)/2$.

7. Let **1** be the permutation that leaves every object in place. Show that $N(\mathbf{1}) = 0$. If $N(p) = 0$, show that $p = \mathbf{1}$.

8. What permutation p has $N(p) = n(n - 1)/2$?

By definition, the **sign** of a permutation p, denoted $\sigma(p)$, is equal to $(-1)^{N(p)}$. For example, since $N(\mathbf{1}) = 0$, we have $\sigma(\mathbf{1}) = 1$.

The following problems are somewhat more challenging than the first 8 problems.

9. Show that the sign of any transposition is equal to -1.

10. Let p be a permutation and t a transposition. Show that $\sigma(pt) = -\sigma(p)$ and $\sigma(tp) = -\sigma(p)$.

Problems 9 and 10 imply that if you write a permutation as a product of transpositions in 2 ways, the number of transpositions in each product will differ by an even number.

11. Let p and q be permutations. Show that $\sigma(pq) = \sigma(p)\sigma(q)$.

We'll conclude with a lemma of Y. I. Zolotarev that connects the sign of a permutation to modular arithmetic. In Robert Donley's series on Fermat's little theorem, he has shown us how to solve any linear equation, modulo a prime number p. That is, he explained how to solve equations of the form $ax + b = 0 \pmod{p}$, with $a \ne 0 \pmod{p}$. Here, we ask, when is there a solution to the equation $z^2 = a \pmod{p}$ where a is constant and z is unknown? If $a = 0 \pmod{p}$, then $z = 0 \pmod{p}$ is a solution. So let's assume that $a \ne 0 \pmod{p}$.

Assume (or see pages 14-15 of Volume 6, Number 5 of this Bulletin) that there exists g such that the $p - 1$ numbers $g, g^2, g^3, g^4, \ldots, g^{p-1}$ constitute a complete set of nonzero remainders modulo p. Then $a = g^k \pmod{p}$ for some k.

12. Show that $z^2 = a \pmod{p}$ has a solution if and only if k is even.

13. Consider $p - 1$ boxes. Consider the permutation s that sends the contents of box x to box ax (modulo p). For example, if $p = 7$ and $a = 3$, then $s = 3\ 6\ 2\ 5\ 1\ 4$. Show that $\sigma(s) = -1$ if and only if k is odd.

In other words, $z^2 = a \pmod{p}$ has a solution if and only if $\sigma(s) = 1$.

Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 14 - Meet 11 Mentors: Jennifer Matthews, Wangui Mbuguiro, Liz Simon
May 1, 2014

Field trip to MIT's Dept. of Aeronautics and Astronautics

A special Thank You to Prof. Karen Willcox who organized our tour of MIT's Department of Aeronautics and Astronautics. The tour began with a visit to MIT's Wright Brothers Wind Tunnel, where Richard Perdichizzi warned us to remove our glasses before entering. The Wright Brothers Tunnel is the largest wind tunnel at MIT and can blow at speeds up to about 170 mph. Even at 35 mph, it was challenging to stand. It was... a blast.

Next, Patrick Blonigan demonstrated wind flow over various airfoils using a small wind tunnel and some dry ice. With no model inside, the flow was straight and smooth. With a block placed inside, you could see eddies form indicating drag. When he placed a wing inside, the air flowed over without the slightest hitch. Patrick concluded by showing us what happens when a wing stalls.

Karen then led a mathematical activity designed to illustrate how computers are used to model airflow. Each girl became a node in a 1-dimensional mesh. Each girl had to sit, kneel, or stand. With each round, girls used an explicit mathematical formula to compute whether they should be sitting, kneeling, or standing at the beginning of the next round. Each girl's computation depended on her current state and the state of her nearest neighbors. After several rounds, the girls could see a global behavior emerging. Simulation of airflow is achieved in a similar manner using a 3-dimensional mesh and a computer. Each node in the mesh represents a point in space. The state of each node is the speed and direction of wind at that point, and possibly, additional information, such as pressure and temperature. Assuming the physical assumption that the wind speed and direction depend only on local conditions, each node computes and recomputes its wind speed and direction using a mathematical rule that depends only on the states of nearby nodes. The finer the mesh, the more accurate the simulation.

Next, Gwen Gettliffe took us to the Space Systems Lab where she works. For one of her projects, she helped to design "microsatellites," satellites about the size of a shoebox, that search for exoplanets. Because of their size, microsatellites are unable to see deep into space. But, they are inexpensive. The idea is to trade depth of vision for the ability to deploy several for wide coverage of nearby star systems. Gwen pointed out that one of the challenges with Hurricane Katrina was that at the time, there weren't enough satellites to give a detailed sense of the hurricane's movements making it impossible to accurately predict where Katrina would make landfall. With microsatellites, one can string out several in orbit so that frequent updates can be made. To move, they sometimes use carbon dioxide canisters like those found in paint ball guns.

Session 14 - Meet 12 Mentors: Jennifer Matthews, Liz Simon
May 8, 2014

We hosted our traditional end-of-session Math Collaboration!

Calendar

Session 14: (all dates in 2014)

January	30	Start of the fourteenth session!
February	6	
	13	
	20	No meet
	27	
March	6	
	13	Sarah Spence Adams, Olin College
	20	
	27	No meet
April	3	
	10	Anna Frebel, Department of Physics, MIT
	17	
	24	No meet
May	1	Karen Willcox, Dept. of Aeronautics and Astronautics, MIT
	8	

Session 15: (all dates in 2014)

September	11	Start of the fifteenth session!
	18	
	25	No meet
October	2	Emily Pittore, iRobot
	9	
	16	
	23	
	30	
November	6	
	13	
	20	
	27	Thanksgiving - No meet
December	4	
	11	

Finding the Maximum Subsequence, Part 1 (continued from page 8)

Still thinking about how to find the maximum subsequence using fewer than N addition operations? If you'd like a hint, read on.

Are you sure?

Here's the hint. Think about the following: If you know the sum for the biggest subsequence that ends at position j, how does that help you figure out the sum for the biggest subsequence that ends at position $j + 1$? (Here, "biggest subsequence" means the subsequence with the biggest sum – i.e., the one you are looking for.)

Check back next issue for the answer!

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $36 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, assistant professor, UCSF Medical School
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, Tamarkin assistant professor, Brown University
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$36 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ___________________________________