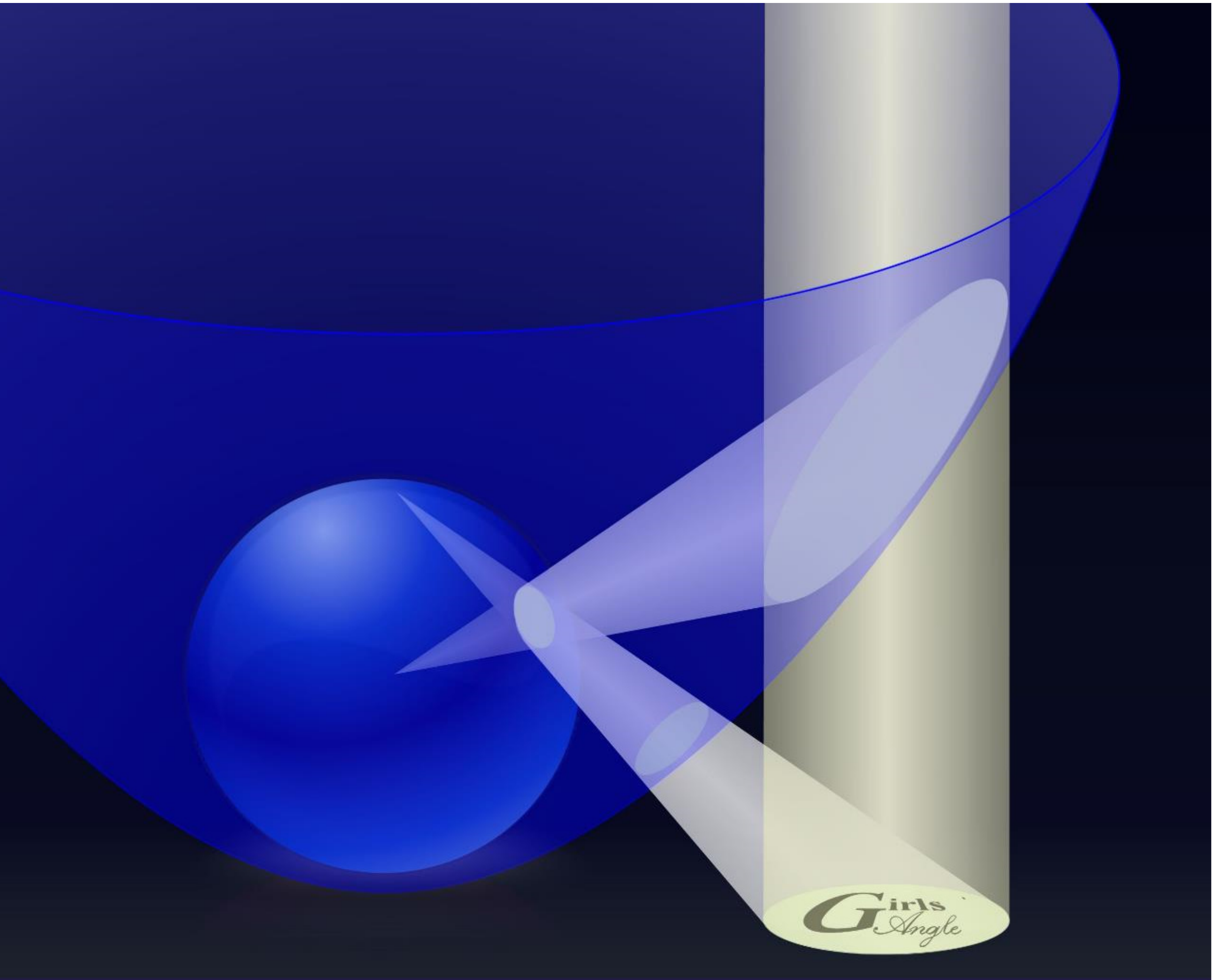
Girls' Angle Bulletin

August 2014 • Volume 7 • Number 6

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Every math problem that challenges you to the point where you get stuck is an opportunity to develop techniques for finding new points of view. That skill is helpful not just in math, but with all of life's problems. The Bulletin contains problems at a variety of difficulty levels, but if you still don't find something challenging enough, let us know!

- Ken Fan, President and Founder

Girls' Angle Bulletin

The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Shining light on the paraboloid and sphere. See *Anna's Math Journal* for details. Image paraboloid made with MATLAB, a powerful suite of mathematical software produced by MathWorks.

An Interview with Christine Berkesch Zamaere

Christine Berkesch Zamaere is an assistant professor of mathematics at the University of Minnesota. She earned her doctoral degree in mathematics from Purdue University under the supervision of Uli Walther.

...anything worth pursuing takes hard work, so do not be afraid to dive in and get busy.

Ken: To get things started, I'm interested to know what mathematics meant to you at various stages in your life. For example, what did mathematics mean to you in kindergarten? In middle school? In high school? In college? In graduate school? And today?

Christine: For me, mathematics meant pattern recognition, arithmetic, logic, and geometry in elementary school. In middle school, I was introduced to algebra and how to use mathematics to solve problems in everyday life situations, for example, in computing interest, optimization problems, and basic physics. During high school, I began to grasp the power, practicality, and ubiquity of the subject; mathematics is the foundation for economics, science, and engineering. Through graduate school and beyond, I have learned to appreciate how various mathematical disciplines work together to yield new discoveries. For example, in trying to understand geometric objects and their associated equations in my own research, I use tools from algebra, geometry, analysis (the study of continuous structure), and combinatorics (the study of discrete structure).

Ken: When did becoming a mathematician become a goal? What turned you on to mathematics?

Christine: When I began college, I planned to be a violinist, but I also continued to study mathematics because I found it fun. After my sophomore year, I had the opportunity to participate in a Research Experience for Undergraduates (REU) funded by the National Science Foundation. It opened up my eyes to the world of mathematical research, which I discovered I really enjoyed!

I worked with a small group of students from around the country on a research project in combinatorial representation theory. We used computers to run large searches to generate examples much faster than we could have done by hand. We then hunted for patterns and used our observations to make conjectures. Finally, we constructed proofs to turn our conjectures into theorems. While each step in this process was rewarding, the part I found the most exciting was that by the end, we understood mathematics that no one else had ever figured out before us. Because of this great experience, I decided to pursue mathematics as a career and set my sights on graduate school!

Ken: How do you learn mathematics? Did you ever encounter a mathematical subject or concept that proved difficult to grasp? If so, what kinds of things did you do to grasp it eventually?

Christine: Learning new mathematics for me involves several steps. Even if I am fortunate enough to attend lectures on the subject, I still spend time reading and working on problems from books and old exams. Other people are also an important resource to me! I talk with experts on the new topic, as well as with others who are learning along with me.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the complete interview with Prof. Zamaere, and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
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Girls' Angle

Finding the Maximum Subsequence, Part 2[1]

by Kate Jenkins | edited by Jennifer Silva

Did you think of a way to find the maximum subsequence in a sequence of length N using fewer than N addition problems?

Recall that we are trying to come up with an algorithm that, given any sequence of numbers, will find the subsequence with the largest sum by doing as little work as possible. Here is the example sequence we've been looking at:

1	**-2**	**2**	**-1**	**4**
1st	2nd	3rd	4th	5th

Let's make the following 2 definitions:

Let $M(j)$ be the maximum of all the sums of subsequences that end at position j.

Let $s(j)$ be the start position for the maximum subsequence that ends at position j.

If we knew the values of $M(j)$ and $s(j)$ for all values of j between 1 and the length of our sequence, then we would have the answer to our original problem. This is because the maximum subsequence has to end at some value of j, so – by definition – its start position would be $s(j)$ and its sum $M(j)$.

Let's see if we can efficiently compute $M(j)$ and $s(j)$.

In our example, $M(1) = 1$ and $s(1) = 1$. We know this because the only subsequence that ends at position 1 is the one that also begins there, and its sum is the value of the 1st number in the sequence, which happens to be 1.

For $j = 2$, there are only two subsequences to consider – the one that begins and ends at the 2nd term, or the one that begins with the 1st term and ends with the 2nd term. In other words, you can either start over with a new subsequence beginning with the 2nd term, or extend the one beginning at the 1st term by one term. Which has the bigger sum? It depends on whether $M(1)$ is positive or negative. For our purpose, it's better to start over than to extend a subsequence whose sum is negative. In this case, since $M(1) = 1$ is greater than 0, our best bet is to extend the sequence to the one starting at the 1st term and ending with the 2nd term. So

$$s(2) = 1 \text{ and } M(2) = M(1) + (2^{nd}\ \textit{term of sequence}) = 1 + (-2) = -1.$$

For any value of $j > 1$, the choice is to either extend a sequence that ends at $j - 1$, or to begin a new sequence at the jth position. If you are extending an existing sequence, you can't possibly do better than to extend the subsequence with the maximal sum that ends at $j - 1$.

These considerations suggest the following algorithm to find the maximum subsequence by computing $M(j)$ and $s(j)$ for a sequence of length N:

[1] This content supported in part by a grant from MathWorks.

Maximum Subsequence Finder Algorithm

Step 1. Let $j = 1$, let $M(j)$ be the 1st term in the sequence, and let $s(j) = 1$.

Step 2. Let $j = 2$.

Step 3. If $M(j - 1) < 0$, then let $M(j)$ be the jth term of the sequence and let $s(j) = j$. If $M(j - 1) \geq 0$, then let $M(j)$ be the sum of $M(j - 1)$ and the jth term of the sequence and let $s(j) = s(j - 1)$.

Step 4. Increase the value of j by 1.

Step 5. If $j \leq N$, go to Step 3.

Step 6. Stop.

As the steps are worked through, keep track of the value of j that corresponds to the largest value of $M(j)$ so far computed. When the algorithm terminates, we will know the value of j where M is maximal. Let's denote by J this special value of j. Then the maximum subsequence is the sequence from $s(J)$ to J, and the sum of this subsequence is $M(J)$.

Here's how the algorithm works when applied to our sample sequence:

1	**-2**	**2**	**-1**	**4**
1st	2nd	3rd	4th	5th

Step	j	$M(j)$	$s(j)$	Value of j such that $M(j)$ is the largest value of M so far computed
1	1	1	1	1
2	2			
3		-1*	1	1
4	3			
5				
3		2	3	3
4	4			
5				
3		1*	3	3
4	5			
5				
3		5*	3	5
4	6			
5				
6				

*In the table above, an asterisk * indicates a place where we performed an addition operation.*

The algorithm finds that the largest subsequence sum is 5, achieved by the subsequence that runs from the 3rd to 5th positions.

Notice that this algorithm only requires at most one addition operation for each value of j greater than 1 and less than or equal to N. Hence, for a sequence of length N, the algorithm will perform fewer than N addition operations! (In our analysis, we ignored the cost of performing the checks for when something is greater than zero. These checks are also work, but since they are done just twice for each value of j from 2 to N [once to check if $M(j - 1) > 0$, and once to check if a larger value of M has been discovered], these checks do not change the efficiency of the algorithm.)

The work involved in this algorithm is therefore $O(N)$. Algorithms that are $O(N)$ are referred to as **linear time algorithms**. The first brute-force algorithm we considered is a **cubic time algorithm**, and the second algorithm we considered is a **quadratic time algorithm**.

Let's see how our latest algorithm compares to our other algorithms. The chart below shows how many addition operations each algorithm must perform for various sequence sizes.

Sequence Length	Algorithm 1	Algorithm 2	Algorithm 3
5	20	10	4
6	35	15	5
10	165	45	9
100	166,650	4,950	99
1000	166,666,500	499,000	999

(The numbers under "Algorithm 3" are the maximum number of addition operations needed. The algorithm might perform fewer additions, depending on what the input sequence is.)

Wouldn't you much rather do Algorithm 3 instead of one of the others, given a sequence of length 100 or more?

Remember the longer sequence I gave you at the beginning of Part 1? Did you find the maximum subsequence of that sequence? Find it using Algorithm 3 to check if you got the same answer! For your convenience, here is the sequence again:

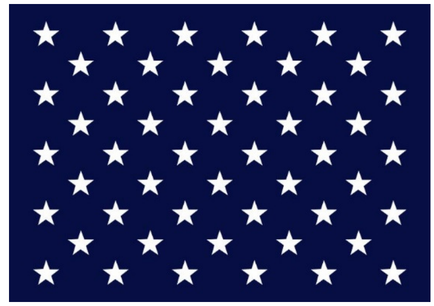
10	**-8**	**4**	**2**	**-9**	**8**	**4**	**-10**	**-1**	**12**
1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th

For the answer, see page 29.

I hope this maximum subsequence problem has given you a sense of what people consider when they think about algorithms and why algorithms can be interesting and useful, particularly when solving large problems. I have shared just one small example. There are many other neat algorithms for important problems that people really care about! And we keep inventing new ones all the time. I hope you find opportunities to learn more about them. For me, they have been the foundation of an interesting and rewarding career.

Star-Spangled Numbers

by Lightning Factorial | edited by Jennifer Silva



Robert Heft designed the US flag's 50-star field.

Emily and Jasmine decide to design a 51-star flag.

Jasmine: I like Robert Heft's 50-star flag design.

Emily: I do too, but I'm curious, what do you like about it?

Jasmine: Well, if I'd made it, I probably would've put the stars in a blocky 5 by 10 rectangular array. The way Heft did it, it feels more dynamic.

Emily: It's actually two rectangular arrays, one embedded within the other.

Jasmine: Hey, you're right! It's a 4 by 5 rectangular array of stars inside a 5 by 6 rectangular array of stars.

Emily: Why don't we try a similar pattern for a 51-star flag?

Jasmine: Okay!

Emily: Hmm. How do we figure out the dimensions of the rectangles?

Jasmine: Well, since we don't know what the dimensions are yet, let's say the inner rectangular array is W stars by L stars.

Emily: Okay. If we want the inner rectangle to fit snugly inside the outer rectangle, as in Heft's design, the outer rectangle would have dimensions of $L + 1$ by $W + 1$.

Jasmine: That means there would be LW stars in the inner rectangle and $(L + 1)(W + 1)$ stars in the outer rectangle, for a total of $LW + (L + 1)(W + 1)$ stars.

Emily: That simplifies to $2LW + L + W + 1$.

Jasmine: So we have to solve the equation $2LW + L + W + 1 = 51$.

Emily: Wait a sec! Doesn't that equation have tons and tons of solutions? I mean, you can pretty much substitute any value of W and then solve for L, right?

Jasmine: Yeah, but we need L and W to be positive *integers*.

Emily: Oh yeah, that's right. Well, we can still solve for, say, L in terms of W, and then try to see which integers W yield integer values of L.

Both girls take a moment to isolate L in the equation $2LW + L + W + 1 = 51$.

Jasmine: I got $L = \frac{50 - W}{2W + 1}$.

Emily: I got that, too. But how are we supposed to figure out when $2W + 1$ divides evenly into $50 - W$?

Jasmine: I don't know. I guess we can just try all values of W from 1 to 51. We know that W can't be bigger than 51 since there aren't more than 51 stars.

Emily: Actually, we just have to go until $2W + 1 > 50 - W$, and that happens when $3W > 49$. So we only need to try numbers from 1 to 16 for W. That's not too bad. We might as well get started.

W	**$2W + 1$**	**$50 - W$**	**$(50 - W)/(2W + 1)$**
1	3	49	16 1/3
2	5	48	9 3/5
3	7	47	6 5/7
4	9	46	5 1/9
5	11	45	4 1/11
6	13		

Jasmine: Emily, maybe all that work isn't necessary. I just had a thought. What if we divide $2W + 1$ into $50 - W$ using polynomial long division? We get $L = \frac{50 - W}{2W + 1} = -\frac{1}{2} + \frac{101}{4W + 2}$. This is an integer if and only if $\frac{101}{4W + 2}$ is a half-integer.

Emily: What's a half-integer?

Jasmine: It's a number halfway between two consecutive integers: an odd number divided by 2.

Emily: Okay, but you know what? For some reason, I remember that 101, 103, 107, and 109 are all prime numbers. Since 101 is prime, we'll only get a half-integer if $4W + 2$ is either 2 or 202. If $4W + 2 = 2$ then $W = 0$, which won't work. If $4W + 2 = 202$ then $W = 50$, which means that $L = 0$, so that doesn't work either. Hey, that was much faster than making the table!

Jasmine: Yes, but that also means that we can't make a 51-star flag in the form of a rectangle embedded in a rectangle. Bummer!

Emily and Jasmine pause while they consider what to do with this newfound knowledge. Then they look at each other.

Emily: Are you thinking what I'm thinking?

Jasmine: You want to figure out for what N can N stars be arranged as a rectangle embedded in a rectangle?

Emily: Exactly! How'd you know?

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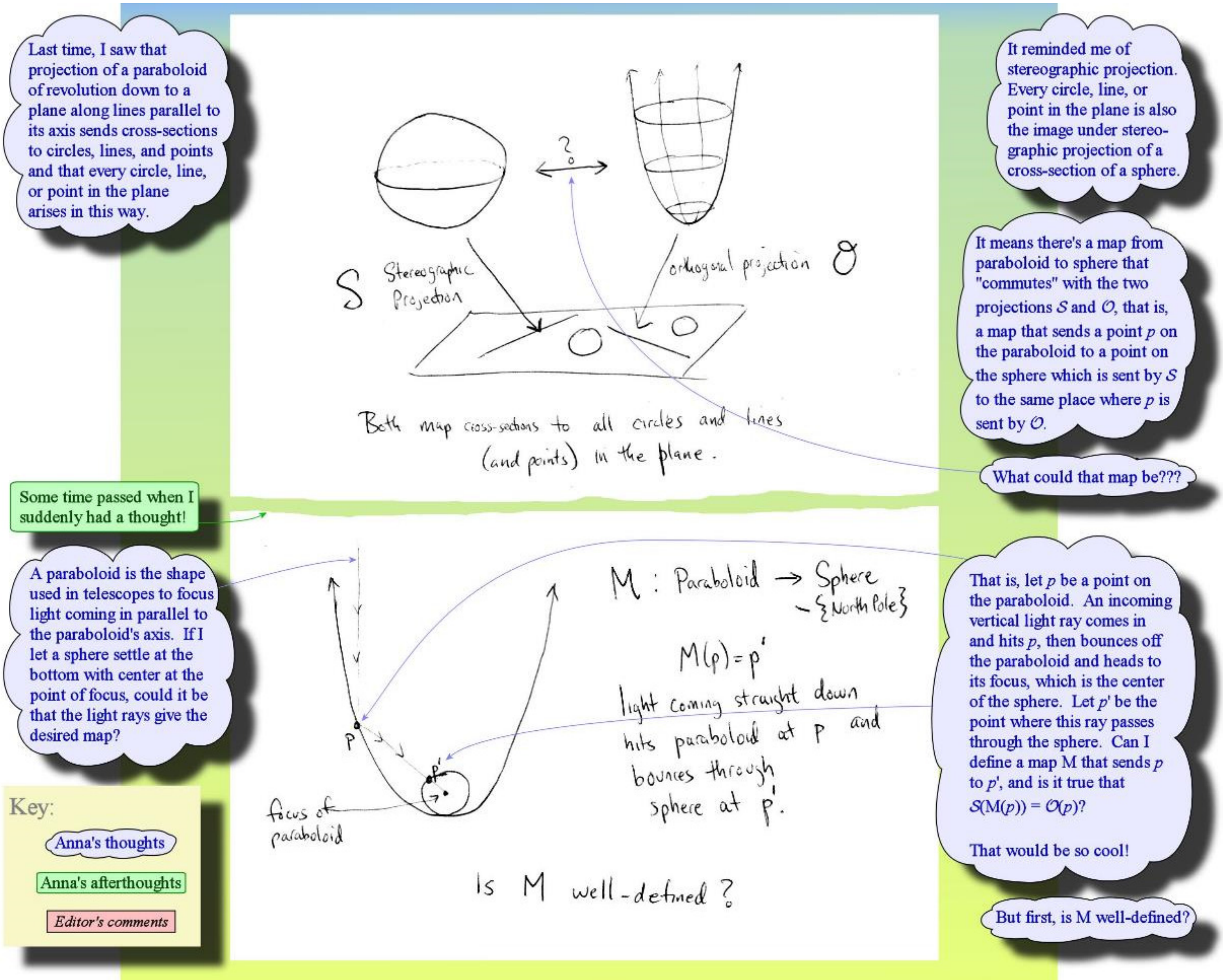
Anna's Math Journal

By Anna B.

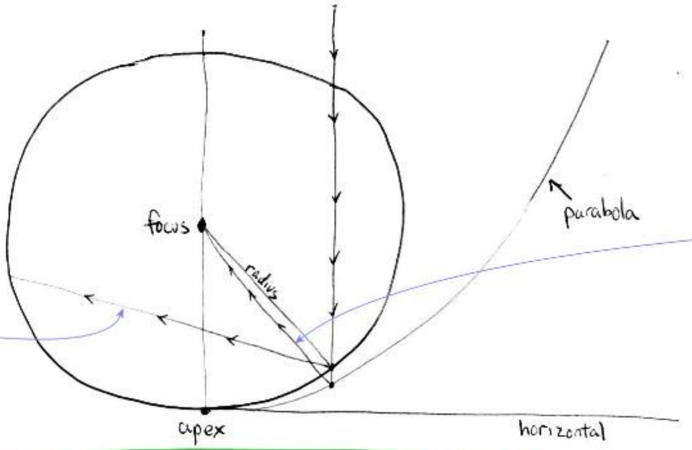
Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna follows a hunch that connects a paraboloid to a sphere and obtains a neat result.



What concerns me is the possibility that the light ray not meet the sphere after it bounces off the parabola.

If the light ray bounces off the sphere, the reflected ray and the incoming ray make equal angles with the radial line.



What if the sphere sits below the paraboloid in places? If that happens, the definition of M would make no sense.

If the ray bounces off the paraboloid, it heads straight for the focus, which is the center of the sphere.

That means the paraboloid must be less sloped than the sphere, because otherwise, light bouncing off the paraboloid would miss the focus from below.

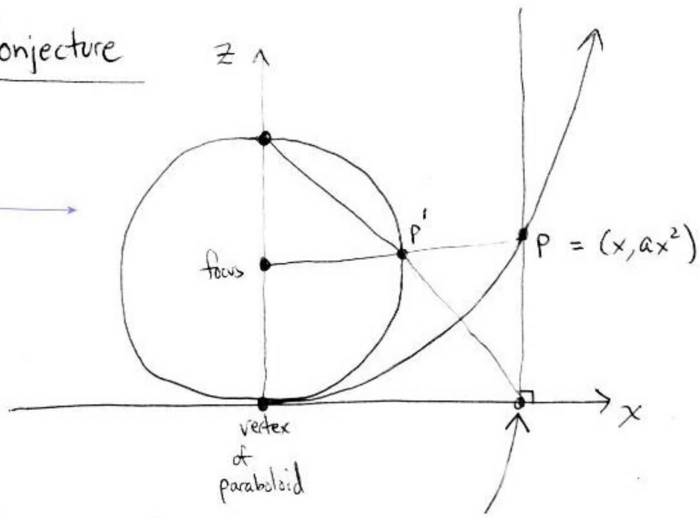
And if the paraboloid is always shallower than the sphere, the sphere must sit above the paraboloid, so the map M is well-defined.

light reflects off the paraboloid
at a sharper angle
so the slope of the tangent
must be shallower than the circle $\Rightarrow$ M is well-defined.

Actually, this reasoning is flawed! It assumes the sphere is above the paraboloid, which is what I wanted to prove! However, it's also not necessary because the proof of the conjecture below also shows that M is well-defined.

Conjecture

What I want to show is that $\mathcal{O}(p) = \mathcal{S}(p')$, where $p' = \mathrm{M}(p)$. Because everything is rotationally symmetric about the z-axis, I can restrict attention to a vertical cross-section. In the cross-section, we have a circle sitting in a parabola.



See this issue's cover for a 3D illustration.

Does $\mathcal{O}(p) = \mathcal{S}(p')$ where $p' = M(p)$?

Strategy: Find focus
Define p' to be intersection of focus $-p$ and North pole $-\ \mathcal{O}(p)$.
See if p' is on circle.

To clarify, the strategy is to define p' as this intersection, instead of as $\mathrm{M}(p)$. If p' is on the circle, then since it is on the line segment connecting focus to p, it would then be equal to $\mathrm{M}(p)$, and also, $\mathcal{S}(p') = \mathcal{O}(p)$, thereby proving the conjecture.

Hm. What strategy can I use to prove this conjecture? Here's a plan: I'll find exactly where the focus is. Then I will define p' to be the intersection of the line segment connecting the focus with p and the line segment connecting the top of the circle to $\mathcal{O}(p)$. If p' is on the circle, the conjecture follows, because then $\mathcal{O}(p)$ would also be $\mathcal{S}(p')$ and, by construction, p' would also be $\mathrm{M}(p)$.

Assume parabola is $z = ax^2$.

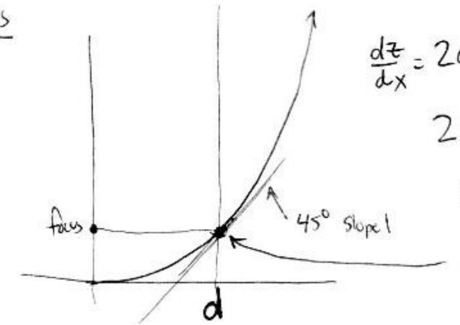
Without loss of generality, I'll say that the parabola is given by $z = ax^2$.

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

To execute the strategy, I'll first find the exact location of the focus.

By symmetry, the x-coordinate of the focus is 0.

Focus



$$\frac{dz}{dx} = 2ax$$

$$2ad = 1$$

$$d = \frac{1}{2a}$$

$$\left(\frac{1}{2a}, a\left(\frac{1}{2a}\right)^2\right) = \left(\frac{1}{2a}, \frac{1}{4a}\right)$$

I know the location of the focus is well-known, but I can find it quickly by using the derivative. The point where the slope of the parabola is 1 has the same z-coordinate as the focus because light coming straight down will bounce off this point horizontally.

So focus is at $\left(0, \frac{1}{4a}\right)$. Radius is $\frac{1}{4a}$.

The bottom of the sphere is the vertex of the parabola, so the radius of the sphere is $1/(4a)$.

Following the strategy, I now define p' to be the intersection of the line segment connecting the focus to p and the line segment connecting the top of the circle to $(x, 0)$, where $p = (x, ax^2)$. Also, let (m, n) be the coordinates of p'.

$p' = (m, n)$ is the intersection of the line segment focus $-$ p and north pole $-$ $(x, 0)$.

I can find m using similarity. The focus, north pole, and p' form a triangle similar to the triangle with vertices p, $(x, 0)$, and p'.

Similarity: $$\frac{m}{x-m} = \frac{1/4a}{ax^2} = \frac{1}{4a^2x^2}$$

$$4a^2x^2m = x - m$$

$$m(4a^2x^2+1) = x$$

$$m = \frac{x}{1+4a^2x^2}$$

I'll use similarity to find n as well.

$$n = \frac{1}{2a}\left(\frac{x-m}{x}\right) = \frac{1}{2a}\left(1 - \frac{m}{x}\right) = \frac{1}{2a}\left(1 - \frac{1}{1+4a^2x^2}\right)$$

$$= \frac{1}{2a}\left(\frac{4a^2x^2}{1+4a^2x^2}\right) = \frac{2ax^2}{1+4a^2x^2}$$

Gosh, m and n are kind of messy, when expressed in terms of a and x. Could this point really be a distance of $1/(4a)$ from the focus?

Check if p' is on circle.

Does $$m^2 + \left(n - \frac{1}{4a}\right)^2 = \frac{1}{16a^2} \ ?$$

I guess I'll go ahead and plug away...

$$m^2 + \left(n - \frac{1}{4a}\right)^2 = \frac{x^2}{(1+4a^2x^2)^2} + \left(\frac{2ax^2}{1+4a^2x^2} - \frac{1}{4a}\right)^2$$

$$= \frac{x^2}{(1+4a^2x^2)^2} + \left(\frac{2ax^2 \cdot 4a - (1+4a^2x^2)}{4a(1+4a^2x^2)}\right)^2$$

...this doesn't look very promising... oh dear...

The fact that p' is on the circle also shows that M is well-defined.

$$= \frac{1}{(1+4a^2x^2)^2}\left(x^2 + \left(\frac{4a^2x^2-1}{4a}\right)^2\right)$$

$$= \frac{1}{(1+4a^2x^2)^2}\left(\frac{(4a)^2x^2 + (4a^2x^2-1)^2}{(4a)^2}\right)$$

$$= \frac{1}{(4a)^2}\frac{1}{(1+4a^2x^2)^2}\left(16a^4x^4 + 8a^2x^2 + 1\right) = \frac{1}{16a^2} \checkmark !$$

HEY! It works! $\mathcal{O}(p) = \mathcal{S}(\mathrm{M}(p))$!

Anna proved a nifty result which I've never heard before. Have you seen it?

ABB 7.12.14

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Summer Fun!

In the last issue, we invited members to submit solutions to a batch of Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil! Move your mind! You might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Please refer to the previous issue for the problems.

Members and Subscribers: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Summer Fun!

Magic Squares

by Lightning Factorial

1. Let $\begin{matrix} a & b \\ c & d \end{matrix}$ be a 2 by 2 square of numbers. To be a magic square, we must have

$$a + b = c + d = a + c = b + d = a + d = b + c.$$

From $a + b = a + c$, we see that $b = c$. From $a + b = a + d$, we see that $b = d$. From $a + d = b + d$, we see that $a = b$. Hence, $a = b = c = d$ and distinct entries are impossible.

2. Here's a 3 by 3 magic square: $\begin{matrix} 1 & 8 & 12 \\ 18 & 7 & -4 \\ 2 & 6 & 13 \end{matrix}$. There are infinitely many.

3. Let $\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$ be a 3 by 3 magic square. If we add both diagonals to the middle row and column we get:

$$\underbrace{(a + e + i)}_{\text{NW-SE diagonal}} + \underbrace{(g + e + c)}_{\text{SW-NE diagonal}} + \underbrace{(d + e + f)}_{\text{middle row}} + \underbrace{(b + e + h)}_{\text{middle column}}.$$

This sum simplifies to $a + b + c + d + 4e + f + g + h + i$. Using the fact that

$$S = a + b + c = d + e + f = g + h + i,$$

our sum can be written as $3S + 3e$. Thus, $3S + 3e = 4S$. Isolating e, we find that $e = S/3$.

4. We get $\begin{matrix} a & b & c \\ \frac{-2a+b+4c}{3} & \frac{a+b+c}{3} & \frac{4a+b-2c}{3} \\ \frac{2a+2b-c}{3} & \frac{2a-b+2c}{3} & \frac{-a+2b+2c}{3} \end{matrix}$.

Notice that setting $a = 1$, $b = 8$, and $c = 12$ produces the magic square in our answer to #2.

5. We get $\begin{matrix} x+z & z-x-y & y+z \\ -x+y+z & z & x-y+z \\ -y+z & x+y+z & -x+z \end{matrix}$.

Notice that if x, y, and z are integers, this matrix will consist entirely of integers. However, the expressions in #4 may not result in integers even if a, b, and c are integers.

Summer Fun!

6. Suppose $0 < x < y < z - x$ and $y \neq 2x$.

First, we'll show that all the entries in the square are positive.

Since $z - x > y$, we know that $z > x + y$. Since x and y are both positive, we can conclude that z is also positive. Therefore $x + z$, $y + z$, $x + y + z$, and z will all be positive.

Also, from $z - x > y$, we see that $z - x - y > 0$.

To see that $-x + y + z > 0$, observe that $-x + y + z = (z - x - y) + 2y$, and both $z - x - y$ and $2y$ are positive. Similarly, we can see that $x - y + z > 0$ because $x - y + z = (z - x - y) + 2x$.

To see that $-y + z > 0$, note that $-y + z = (z - x - y) + x$.

Finally, we're given that $-x + z > 0$.

We conclude that the square will consist of positive numbers.

Next, we'll check that all entries are distinct so that we may conclude that the square is, in fact, a magic square.

First note that $x + y + z > y + z > x + z > z > -x + z > -y + z > z - x - y$. This shows that the 7 entries in the top row, bottom row, and center are distinct.

Now observe that $y + z > -x + y + z > z$. This tells us that of the 7 entries from the previous paragraph, the only one that $-x + y + z$ might equal is $x + z$. However, if $-x + y + z = x + z$, then $2x = y$, which is not the case by assumption.

Similarly, observe that $z > x - y + z > -y + z$, which tells us that of the other 8 entries, the only one that $x - y + z$ might equal is $-x + z$. However, if $x - y + z = -x + z$, then again, $2x = y$, which is not true by assumption.

Thus, if $0 < x < y < z - x$ and $y \neq 2x$, then the square will be a magic one.

7. All rows, columns and diagonals add up to $n(n^2 + 1)/2$. Hint: Add up the entries in a normal magic square in 2 different ways.

8. The central number in a normal 3 by 3 magic square must be 5.

Solutions for 9 and 10 are omitted.

11. This is closely related to the representation of the numbers 0 through 15 in base 4. If n is an integer between 1 and 16, inclusive, let x be 4 if n is divisible by 4 and let x be the remainder left when n is divided by 4 otherwise. Let $X = n - x$. Note that X must be in the set $\{0, 4, 8, 12\}$. Then $n = X + x$. If $n = Y + y$ is another representation with Y in $\{0, 4, 8, 12\}$ and y in $\{1, 2, 3, 4\}$, then $X - Y = y - x$, which shows that $y - x$ is divisible by 4. This is only possible if $x = y$, and if $x = y$, then also $X = Y$, so the representation is unique.

12. This follows from #11.

13. There are $4! = 24$ ways to assign the numbers 0, 4, 8, and 12 to the variables a, b, c, and d. There are $4! = 24$ ways to assign the numbers 1, 2, 3, and 4 to the variables α, β, γ, and δ. Since the assignment of numbers to the Latin variables is independent of the assignment of number to the Greek variables, there are a total of $24^2 = 576$ ways to make the number assignments.

The assignments are not all different in the sense that some can be obtained from others by rotations and reflections of the array.

Summer Fun!

Center of Mass and Mass Points

by Girls' Angle Staff

1. The ratio of the mass at A to that at B is 3 : 1.

2. A. A unit mass should be placed at vertex B so that the center of mass of the two masses at vertices A and B is located at F. A unit mass should be placed at vertex C so that the center of mass of the two masses at A and C is located at E.

B. The center of mass of the masses at B and C is located at point D.

C. Use the "piecemeal" property of the center of mass to see that it must be located on all 3 medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, and hence the medians are concurrent.

To find out what we mean by "piecemeal" property of the center of mass, see Volume 7, Number 3 of this Bulletin, particularly pages 20-21.

D. The center of mass of 3 unit masses, one placed at each vertex of the triangle, is located at the intersection of the medians. Consider the median $\overline{AD}$. We compute the center of mass of the 3 unit masses by first replacing the 2 unit masses at points B and C with a 2 unit mass located at their center of mass, which is point D. We now compute the center of mass of the original 3 unit masses by computing the center of mass of the unit mass at A and the 2 unit mass at D. By the law of the lever, we know that this center of mass will split the median into 2 pieces that are in the ratio 1 : 2. The same argument can be applied to all 3 medians.

3. Label the vertices of the quadrilateral A, B, C, and D, in clockwise order. Place unit masses at the 4 vertices of the quadrilateral. Now compute the center of mass of these 4 point masses in two different ways, using the "piecemeal" property of the center of mass.

4. Hint: Think of each unit mass at each vertex of P as 2 point masses each of mass 1/2.

5. The ratio $DX : DY = 2 : 1$.

6. The angle bisector theorem tells us that $AY : YC = 7 : 3$ and $BX : XC = 7 : 6$. Therefore, if we place a 7 unit point mass at C, a 3 unit point mass at A, and a 6 unit point mass at B, the center of mass of the masses at A and C will be at Y and the center of mass of the masses at B and C will be at X. Using the "piecemeal" property of the center of mass, we conclude that the center of mass of all 3 masses will be at M. We can then readily compute that $BM : MY = 10 : 6$.

7. If we can assign point masses to A, B, and C so that their center of mass is located at P, we would then be able to compute the ratio of the heights of triangles ABP and ABC. Assign a point mass of mass a to A, b to B, and c to C. We desire that $a : b + c = 5 : 8$ and $b : a + c = 4 : 9$. That is, $8a = 5(b + c)$ and $9b = 4(a + c)$. We solve this linear system for a and b in terms of c and find that $a = 5c/4$ and $b = c$. Hence, if we let $a = 5$, $b = 4$, and $c = 4$, the center of mass of the 3 point masses will be at P. Let F be where the cevian from A through P meets $\overline{AB}$. Using the "piecemeal" property of the center of mass, we find that $FP : PC = 4 : 9$. Therefore, $FP : FC = 4 : 13$. We conclude that the area of triangle ABP is $13(4/13) = 4$ square units.

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8. Without loss of generality, we position a coordinate plane so that the circumcircle has unit radius and center at the origin. By rotating if necessary, we can ensure that one of the vertices is at $A = (1, 0)$ and the other 2 vertices are located at $B = (\cos a, \sin a)$ and $C = (\cos b, \sin b)$, where $0 < a < b < 360°$. The center of mass of the 3 equal unit masses is located at the average of their coordinates: $P = ((1 + \cos a + \cos b)/3, (\sin a + \sin b)/3)$. In order for the center of mass of the 3 unit masses and the mass m to be at the circumcenter, we need m to be on the circumcircle opposite P, and, by the law of the lever, we must have $m = 3\,|P|$, where $|P|$ is the distance of P from the origin. Therefore, $m^2 = (1 + \cos a + \cos b)^2 + (\sin a + \sin b)^2$. This can be rearranged to $m^2 = 3 + 2(\cos a + \cos b + \cos(a - b))$.

Now observe that

$$A = (-a + b)/2,$$
$$B = 180 - b/2,$$
$$\text{and } C = a/2.$$

Therefore, $m^2 = 3 + 2(\cos 2C + \cos 2B + \cos 2A)$.

B. The centroid of the triangle is contained in the circumcircle. Therefore, $0 \le |P| < 1$. Hence,

$$0 \le m^2 = 3 + 2(\cos 2C + \cos 2B + \cos 2A) < 9.$$

Equality holds if and only if $|P| = 0$, and this can only happen when the triangle is equilateral.

9. At A, place a point mass of mass $wxx'yy'$. At B, place a point mass of mass $vxx'yy'$. At C, place two point masses, one of mass $xx'wxy'$ and one of mass $xx'vx'y$. These masses are specifically chosen so that the following 3 facts are true:

1. The center of mass of the point masses at A and B is F.
2. The center of mass of the point mass at A and the point mass at C of mass $xx'wxy'$ is E.
3. The center of mass of the point mass at B and the point mass at C of mass $xx'vx'y$ is D.

From fact 1, we know that the center of mass of all 4 masses is located along $\overline{CF}$. Facts 2 and 3 combined imply that the center of mass of all 4 masses is also located along $\overline{ED}$. Therefore, the center of mass of all 4 point masses is located at P. We can then apply fact 1 to deduce that

$$PF : PC = (xx'wxy' + xx'vx'y) : (wxx'yy' + vxx'yy').$$

This can be simplified to $PF : PC = (wxy' + vx'y) : yy'(w + v)$.

If $x/y = x'/y'$, then

$$\frac{PF}{PC} = \frac{wxy' + vx'y}{yy'(w+v)} = \frac{w\frac{x}{y} + v\frac{x'}{y'}}{w+v} = \frac{w\frac{x}{y} + v\frac{x}{y}}{w+v} = \frac{x}{y}.$$

10. *A*. The center of mass of unit point masses located at each of the integers from 0 to n, inclusive, is located at $(0 + 1 + 2 + 3 + ... + (n - 1) + n)/(n + 1) = n/2$.

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B. We claim that $m_k = k + 1$. To see this, we show that the center of masses of the point masses at 0, 1, 2, 3, ..., n is located at $2n/3$. First, the weighted sum of the positions of each point mass from the origin, weighted by their masses, is

$$\begin{aligned} & 1(0) + 2(1) + 3(2) + \dots + n(n-1) + (n+1)(n) \\ = & \sum_{k=1}^{n}(k+1)k \\ = & \sum_{k=1}^{n}(k^2+k) \\ = & \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k \\ = & \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \end{aligned}$$

The total mass is $1 + 2 + 3 + \dots + (n-1) + n + (n+1) = (n+1)(n+2)/2$. Therefore, the center of mass is located at $\frac{2}{(n+1)(n+2)}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$, which simplifies to $2n/3$.

C. When $R = 1 - 1/p$, where p is a positive integer, we claim that $m_k = {}_{p-2+k}C_{p-2}$, where ${}_nC_k$ is the binomial coefficient n choose k. This can be proven by induction. We omit the details.

11. We think of the plane as the complex plane. Let $w = e^{-\frac{2\pi i}{n}}$. The vertices of the polygon are located at $1, w, w^2, w^3, \dots, w^{n-1}$ and the point mass at w^k has mass $k + 1$. The center of mass of these point masses is located at

$$\frac{1}{M}\sum_{k=0}^{n-1}(k+1)w^k,$$

where M is the total mass $n(n+1)/2$. To compute this sum, we use the algebraic identity

$$1 + 2x + 3x^2 + \dots + nx^{n-1} = \frac{(n+1)x^n(x-1) + (1-x^{n+1})}{(1-x)^2}.$$

(This identity can be derived in the following way. First, note that $1/(1-x)$ is the sum of the infinite geometric series $1 + x + x^2 + \dots$. If we square this, we find that $1/(1-x)^2$ is the infinite series $1 + 2x + 3x^2 + \dots + (k+1)x^k + \dots$. If we subtract from this the quantity $x^{n+1}/(1-x)^2$, the first $n + 1$ terms will be unaffected but from the term $(k+1)x^k$, with $k > n$, we will subtract the like term $(k-n)x^k$, resulting in $(n+1)x^k$ for all terms $k > n$. Therefore, if we subtract $(n+1)x^n/(1-x)$ from $(1-x^{n+1})/(1-x)^2$, we will obtain the above identity.)

We substitute w for x in the identity and divide by the total mass. After simplification, we find that the center of mass is located at $\frac{1}{n+1}(-1, \cot\frac{\pi}{n})$. As n tends to infinity,

$\frac{1}{n+1}\cot\frac{\pi}{n} = \frac{\cos \pi/n}{(n+1)\sin \pi/n}$ tends to $1/\pi$.

Summer Fun!

Quadratic Reciprocity

by Cailan Li

(Recall that in this Summer Fun problem set, p and q always denote distinct odd prime numbers.)

1. Starting from $ax^2 + bx + c = 0 \pmod{p}$, we proceed by "completing the square" to transform this equation to $(2ax + b)^2 = b^2 - 4ac \pmod{p}$. This shows that $b^2 - 4ac$ is a square modulo p if and only if there exists x such that $ax^2 + bx + c = 0 \pmod{p}$. (Since p is an odd prime, $4a$ is invertible modulo p.)

2. (We should have asked to show that the *nonzero* squares modulo p are given by g^x where $1 < x < p$ is even, since 0 is a square modulo p but isn't a power of a primitive root.) The nonzero residues modulo p are given by $g, g^2, g^3, ..., g^{p-1}$. If we square these, we will find all nonzero squares modulo p. Thus, the set of squares are $\{g^2, g^4, g^6, ..., g^{2(p-1)}\}$. Since $g^{p-1} = 1 \pmod{p}$, we know that $g^{2k} = g^{2k+p-1}$. This implies that

$$\{g^2, g^4, g^6, ..., g^{2(p-1)}\} = \{ g^2, g^4, g^6, ..., g^{p-1}\},$$

as desired.

3. Problem 2 informs us that there are $(p - 1)/2$ nonzero squares modulo p. We determine all the squares to find the answers. Modulo 3, the squares are 0 and 1, so $\left(\frac{2}{3}\right) = -1$. Modulo 7, the squares are 0, 1, 4, and 2, hence $\left(\frac{2}{7}\right) = 1$. Modulo 13, the squares are 0, 1, 4, 9, 3, 12, and 10, hence $\left(\frac{5}{13}\right) = -1$. Modulo 19, the squares are 0, 1, 4, 9, 16, 6, 17, 11, 7, and 5, hence $\left(\frac{3}{19}\right) = -1$. To compute $\left(\frac{1041}{101}\right)$, we could systematically compute the 51 squares modulo 101 and check if 1041 is among them, though doing so would be tedious. The development of the theory of quadratic residues leads to much more efficient ways of computing Legendre symbols.

4. From Fermat's little theorem, $(x^{(p-1)/2})^2 = 1 \pmod{p}$. Hence $x^{(p-1)/2} = \pm 1 \pmod{p}$. If $x = y^2 \pmod{p}$, then $x^{(p-1)/2} = y^{(p-1)} = 1 \pmod{p}$ by Fermat's little theorem. On the other hand, $x^{(p-1)/2} - 1 = 0 \pmod{p}$ cannot have more than $(p - 1)/2$ roots. Since there are $(p - 1)/2$ nonzero squares modulo p, all the non-squares must satisfy $x^{(p-1)/2} = -1 \pmod{p}$.

5. Did you figure out that, contrary to what we asked, $\left(\frac{-1}{p}\right) = 1$ if and only if $p = \mathbf{1} \pmod{4}$? To see this, we use Euler's criterion: $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$. Thus, $\left(\frac{-1}{p}\right) = 1$ if and only if $(p - 1)/2$ is even, say equal to $2k$ for some integer k.
In that case, $p = 4k + 1$.

6. If $k = j$, then $| r_k | = | r_j |$. Assume $| r_k | = | r_j |$. Then $kb = \pm jb \pmod p$. Dividing by b, we see that $k = \pm j \pmod p$, i.e. either p divides $k - j$ or p divides $k + j$. Because both k and j are between 1 and $(p - 1)/2$, inclusive, it cannot be the case that $k + j$ is divisible by p. So $k - j$ must be divisible by p, and since both k and j are between 1 and $(p - 1)/2$, inclusive, we must have $k = j$.

7. From #6, the numbers $r_1, r_2, r_3, \ldots, r_{(p-1)/2}$ consist of the numbers $\pm 1, \pm 2, \pm 3, \ldots, \pm(p - 1)/2$, with a definite sign for each entry in this list. By definition of N, there are exactly N negative numbers in the list. Thus $b(2b)(3b) \cdots ((p - 1)b/2) = (-1)^N \cdot 1 \cdot 2 \cdot 3 \cdots ((p - 1)/2) \pmod p$.

8. Using Gauss's lemma, $\left(\frac{-1}{p}\right) = (-1)^N$, where N is the number of k such that $1 \le k \le (p - 1)/2$ and $r_k = (-1)k < 0$. This is true for all such k, so $N = (p - 1)/2$. Therefore, -1 is or is not a square modulo p according to whether $(p - 1)/2$ is even or odd, i.e., $p = 1 \pmod 4$ or $p = 3 \pmod 4$.

9. By Gauss's lemma, $\left(\frac{2}{p}\right) = (-1)^N$, where N is the number of k such that $1 \le k \le (p - 1)/2$ and $r_k < 0$, where r_k is the unique number between $-p/2$ and $p/2$ such that $r_k = 2k \pmod p$. Of the numbers $2, 4, 6, 8, \ldots, p - 3, p - 1$, the numbers up to $(p - 1)/2$ are congruent to positive numbers in the range $-p/2$ to $p/2$ and the remaining numbers are congruent to negative numbers in the range $-p/2$ to $p/2$. Therefore, N is the least integer greater than or equal to $(p - 1)/4$. If $p = 1$ or $7 \pmod 8$, then N is even and if $p = 3$ or $5 \pmod 8$, then N is odd. (Verify this!)

10. Using Euler's criterion, we see that $\left(\frac{xy}{p}\right) = (xy)^{(p-1)/2} = x^{(p-1)/2} y^{(p-1)/2} = \left(\frac{x}{p}\right)\left(\frac{y}{p}\right)$.

11. From #9, $\left(\frac{2}{3}\right) = -1$ and $\left(\frac{2}{7}\right) = 1$. We compute $\left(\frac{5}{13}\right) = \left(\frac{13}{5}\right) = \left(\frac{3}{5}\right) = \left(\frac{5}{3}\right) = \left(\frac{2}{3}\right) = -1$. We compute $\left(\frac{3}{19}\right) = -\left(\frac{19}{3}\right) = -\left(\frac{1}{3}\right) = -1$. Finally, $\left(\frac{1041}{101}\right) = \left(\frac{31}{101}\right) = \left(\frac{101}{31}\right) = \left(\frac{8}{31}\right) = \left(\frac{2}{31}\right)^3 = 1$.

12. By Gauss's lemma, $\left(\frac{q}{p}\right) = (-1)^{N_1}$ and $\left(\frac{p}{q}\right) = (-1)^{N_2}$. Therefore $\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{N_1+N_2}$. It follows that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$ if and only if $(-1)^{N_1+N_2} = 1$.

13. The diagonal AD is on the line $py = qx$. Since p and q are relatively prime, any integer solutions to this equation must satisfy $p \mid x$ and $q \mid y$. However, for points (x, y) on the interior of the diagonal, $0 < x < p/2$ and $0 < y < q/2$.

14. Let (a, b) be a lattice point in the interior of H above diagonal AD. By examining the boundary of this region, we see that $0 < a < p/2$, $qa/p < b < qa/p + 1/2$, and $b < q/2$. The second of these inequalities is equivalent to $-p/2 < qa - pb < 0$. Therefore aq is congruent, module p, to a negative number greater than $-p/2$ and contributes to the value of N_1. Conversely, for any a between 0 and $p/2$ where aq is congruent, modulo p, to a number between $-p/2$ and 0, there is an integer b such that $-p/2 < aq - bp < 0$. The left inequality implies that $b < aq/p + 1/2 < (q + 1)/2$.

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Since q is an odd prime and b is an integer, this implies that $b < q/2$. Therefore, (a, b) corresponds to a lattice point in the interior of H and above the diagonal AD.

A similar argument shows that N_2 is the number of lattice points in the interior of H below the diagonal AD.

Combining this result with #13, we conclude that there are $N_1 + N_2$ lattice points in the interior of H.

15. The interior of H is defined by the following inequalities:

$$0 < x < p/2$$
$$0 < y < q/2$$
$$qx - q/2 < py < qx + p/2$$

Let (x, y) be a lattice point in H. Since p and q are both odd, $((p + 1)/2 - x, (q + 1)/2 - y)$ is also a lattice point. We now verify that $((p + 1)/2 - x, (q + 1)/2 - y)$ is in the interior of H by checking that its coordinates satisfy each of the inequalities above. Since $0 < x < p/2$, we know that $(p + 1)/2 - p/2 < (p + 1)/2 - x < (p + 1)/2$, which simplifies to $1/2 < (p + 1)/2 - x < (p + 1)/2$. Since $(p + 1)/2 - x$ and $(p + 1)/2$ are integers, if $(p + 1)/2 - x < (p + 1)/2$, then in fact $(p + 1)/2 - x < p/2$. And since $0 < 1/2$, we conclude that $0 < (p + 1)/2 - x < p/2$.

Similar reasoning shows that $0 < (q + 1)/2 - y < q/2$.

Next, we compute that

$$\begin{aligned} q((p + 1)/2 - x) - q/2 &= q(p + 1)/2 - qx - q/2 \\ &= qp/2 - qx \\ &= qp/2 + p/2 - p/2 - qx \\ &< qp/2 + p/2 - py \\ &= p((q + 1)/2 - y). \end{aligned}$$

A similar computation shows that $p((q + 1)/2 - y) < q((p + 1)/2 - x) + p/2$.

We conclude that $((p + 1)/2 - x, (q + 1)/2 - y)$ is in the interior of H.

Now observe that $(x, y) = ((p + 1)/2 - x, (q + 1)/2 - y)$ if and only if $x = (p + 1)/4$ and $y = (q + 1)/4$. But $((p + 1)/4, (q + 1)/4)$ is a lattice point if and only if $p = q = 3 \pmod 4$. So only when $p = q = 3 \pmod 4$ will the number of lattice points in the interior of H be odd since all lattice points in the interior of H aside from $((p + 1)/4, (q + 1)/4)$ can be paired with their image under the involutory transformation $(x, y) \to ((p + 1)/2 - x, (q + 1)/2 - y)$.

16. From #14, the number of lattice points in the interior of H is $N_1 + N_2$. From #15, $N_1 + N_2$ is odd if and only if $p = q = 3 \pmod 4$. Thus $\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{N_1+N_2} = -1$ if and only if $p = q = 3 \pmod 4$. Thus $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$ unless p and q are both congruent to 3 modulo 4, in which case $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.

Summer Fun!

Signs of Permutations

by Ken Fan

1. In one-line notation, the permutation that leaves each object in place is given by:

$$1\ 2\ 3\ 4\ 5\ .\ .\ .\ (n - 1)\ n$$

2. Let's build up the permutation step by step by assigning a place for the object in the 1^{st} box, followed by the 2^{nd} box, followed by the 3^{rd} box, etc. First take out all the objects.

The object from the 1^{st} box can be put into any of the n boxes. The object from the 2^{nd} box can be put into any box other than the one where the object from the 1^{st} box was placed. So there are $n - 1$ choices for the object from the 2^{nd} box. After the objects from the 1^{st} and 2^{nd} boxes are placed, there are $n - 2$ choices for where we can place the object from the 3^{rd} box. By the time we get to the object from the $(k + 1)$st box, there are only $n - k$ unoccupied boxes remaining. Since each choice is independent, the total number of ways to rearrange the objects is $n(n - 1)(n - 2) \cdot \cdot \cdot 3 \cdot 2 \cdot 1 = n!$. Here's a table of the first few values of $n!$:

n	1	2	3	4	5	6	7	8
***n*!**	1	2	6	24	120	720	5040	40320

3. With $s = 3\ 1\ 5\ 2\ 4$ and $t = 5\ 2\ 4\ 1\ 3$, if we apply t followed by s, the object in box 1 will end up in box 4, the object in box 2 will end up in box 1, the object in box 3 will end up in box 2, the object in box 4 will end up in box 3, and the object in box 5 will end up in box 5.

4. In one-line notation, the permutation obtained by applying s first, followed by t, is:

$$4\ 5\ 3\ 2\ 1$$

This is different from what you get when you apply t first, followed by s, which, in one-line notation, is the permutation 4 1 2 3 5.

5. We'll prove this by induction on n. When $n = 1$, we adopt the convention that the empty product corresponds to the permutation that leaves every object in place. When $n = 2$, there are only 2 permutations: 1 2 and 2 1. The first can be represented by the empty product and the second is a transposition.

Now assume that every permutations on the numbers 1 through N can be written as a product of transpositions. We shall show that any permutation on $N + 1$ numbers can be written as a product of transpositions.

Let s be a permutation on the numbers 1 through $N + 1$. Suppose that $s(J) = N + 1$. If $J = N + 1$, then the restriction of s to the numbers 1 through N is a permutation of the numbers 1 through N. By induction, this restriction can be written as a product of transpositions. If we extend all these transpositions to transpositions on the numbers 1 through $N + 1$ by leaving the object in box $N + 1$ alone, we see that s is a product of transpositions. So assume that $J < N + 1$.

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Let t be the transposition which swaps the objects in boxes J and $N + 1$ and leaves all the other objects in place. Notice that $st\,(N + 1) = N + 1$. By the same reasoning of the previous paragraph, we use induction to see that st can be written as a product of transpositions $t_1 t_2 t_3 \cdots t_m$. Then $t_1 t_2 t_3 \cdots t_m \cdot t$ is a product of transpositions equal to s.

By induction, all permutations can be expressed as products of transpositions.

6. A transposition swaps the positions of 2 objects and leaves all the other objects in place. Once we have selected which 2 objects we are going to swap, the transposition is determined. So the number of transpositions is equal to the number of ways we can pick 2 objects from n objects. We have n choices for the first object and $n - 1$ choices for the second. However, if we simply take the product $n(n - 1)$, we would be counting every possibility twice since each possibility can be selected in 2 ways, depending on which of the 2 elements is selected first. So the number of transpositions is $n(n - 1)/2$.

7. Since $\mathbf{1}$ leaves all objects in place, there are no (x, y), with $1 \le x < y \le n$, such that $p(x) > p(y)$. Hence $N(\mathbf{1}) = 0$. Conversely, suppose $N(p) = 0$. If $p(1) > 1$, there must be some $k > 1$ such that $p(k) = 1$. But then $(1, k)$ would be a pair of box labels such that $1 < k$ but $p(1) > p(k)$, in contradiction to the assumption that $N(p) = 0$. Therefore $p(1) = 1$. By similar reasoning, $p(2) = 2$, and so on for all the boxes. We conclude that if $N(p) = 0$, then $p = \mathbf{1}$.

8. If $N(p) = n(n - 1)/2$, *every* pair of box labels (x, y), with $1 \le x < y \le n$ satisfies $p(x) > p(y)$. If we let f be the permutation represented by $n\ (n - 1)\ (n - 2) \cdots 3\ 2\ 1$ in one-line notation, then pf must satisfy $pf(x) < pf(y)$ for every pair of box labels (x, y), with $1 \le x < y \le n$ (because $f(x) > f(y)$ and so $pf(y) > pf(x)$). From #7, it follows that $pf = \mathbf{1}$. Therefore $pff = \mathbf{1}f$. But $pff = p\mathbf{1} = p$ and $\mathbf{1}f = f$. Hence $p = f$.

9. Let t be a transposition that swaps the contents of boxes i and j with $1 \le i < j \le n$. Throughout, assume that $1 \le x < y \le n$. We consider cases according to how $\{x, y\}$ and $\{i, j\}$ intersect. If $\{x, y\}$ and $\{i, j\}$ do not intersect, then $t(x) = x$ and $t(y) = y$. If $\{x, y\} \cap \{i, j\} = \{i\}$, then either $x = i$ or $y = i$. If $x = i$, then $t(x) > t(y)$ if and only if $j > y$. Since $y > x$, there are exactly $j - i$ values of y such that $t(x) > t(y)$. If $x = j$, then $t(x) < t(y)$. By similar reasoning, we count $j - i$ ordered pairs (x, y) where $t(x) > t(y)$ and $\{x, y\} \cap \{i, j\} = \{j\}$. If $\{x, y\} \cap \{i, j\} = \{i, j\}$, then $x = i$ and $y = j$ and $t(x) > t(y)$. We conclude that there are $2(j - i) + 1$ ordered pairs (x, y) such that $t(x) > t(y)$. The $\sigma(t) = (-1)^{2(j - i) + 1} = -1$.

10. Suppose t swaps the contents of boxes i and j with $1 \le i < j \le n$. Then

$$pt(x) = \begin{cases} p(x), & \text{if } x \neq i \text{ and } x \neq j \\ p(j), & \text{if } x = i \\ p(i), & \text{if } x = j \end{cases}$$

Let $S = \{\ (x, y) \mid 1 \le x < y \le n \text{ and } p(x) > p(y)\ \}$ and $S' = \{\ (x, y) \mid 1 \le x < y \le n \text{ and } pt(x) > pt(y)\ \}$. By definition $\sigma(p) = (-1)^{\#S}$ and $\sigma(pt) = (-1)^{\#S'}$. We can also assume, without loss of generality, that $p(i) < p(j)$, for if $p(j) < p(i)$, we can simply switch the roles of p and pt.

Summer Fun!

10 continued. Throughout, assume $1 \le x < y \le n$. We again consider cases according to how $\{x, y\}$ and $\{i, j\}$ intersect. If the intersection is empty, then (x, y) is in S if and only if (x, y) is in S'. If the intersection is $\{i, j\}$, then $x = i$ and $y = j$ and since $p(i) = pt(j)$ and $p(j) = pt(i)$, we know that (x, y) is in S' but not in S. (Recall that we are assuming that $p(i) < p(j)$.)

Now consider the cases where $\{x, y\}$ and $\{i, j\}$ intersect in a single element. There are 4 such cases: either $x = i$, $x = j$, $y = i$, or $y = j$. If $x = i$, then $p(x) = p(i)$, $pt(x) = p(j)$, and $p(y) = pt(y)$. If $p(y) < p(i)$, then (x, y) is in both S and S'. If $p(i) < p(y) < p(j)$, then (x, y) is in S' but not S. If $p(y) > p(j)$, then (x, y) is in neither S nor S'. A similar argument shows that if $x = j$, then (x, y) is in both S and S' if $p(y) < p(i)$, in S but not S' if $p(i) < p(y) < p(j)$ (note that unlike the case $x = i$, here, more pairs end up in S than in S'), and in neither S nor S' if $p(y) > p(j)$. Therefore, the cases $x = i$ and $x = j$ account for a net difference of $\#\{y \mid i < y < j \text{ and } p(i) < p(y) < p(j)\}$ more pairs in S' than in S.

Similar reasoning reveals that the cases $y = i$ and $y = j$ account for a net difference of $\#\{x \mid i < x < j \text{ and } p(i) < p(x) < p(j)\}$ more pairs in S' than in S.

Since the net difference from the cases $x = i$ and $x = j$ equals the net difference from the cases $y = i$ and $y = j$, all 4 cases where $\{x, y\}$ and $\{i, j\}$ intersect in a single element account for a net difference of an even number more pairs in S' than in S.

The only case that affects the relative parity of $\#S$ and $\#S'$ is where $x = i$ and $y = j$.

Therefore, $\#S$ and $\#S'$ are of opposite parity and $\sigma(pt) = -\sigma(p)$.

To show that $\sigma(tp) = -\sigma(p)$, we can adapt the above argument or use inverses, noting that $\sigma(p) = \sigma(p^{-1})$, where p^{-1} is the unique permutation that satisfies $p^{-1}p = \mathbf{1}$.

11. For #11, combine the results of #5 and #10.

12. Actually, the statement isn't true if $p = 2$. Did any of you catch that? So let's assume that p is an odd prime number. In this case, see Cailan's solution to problem 2 of his Summer Fun problem set on quadratic reciprocity on page 23.

13. Again, we must assume that p is an odd prime number. If $p = 2$, then $\sigma(s) = 1$ regardless of the parity of k since, in this case, $g = 1$ and $g^k = 1$ for all k. So assume that p is an odd prime number.

If we start at any box x and repeatedly apply s, the contents of box x will go from x to ax to a^2x to a^3x, etc. The contents of box x will return to box x whenever we apply s a total of m times where $a^m = 1 \pmod{p}$. The first time this happens is when $m = (p - 1)/(k, p - 1)$. From this we see that the $p - 1$ boxes can be organized into $(p - 1)/m$ sets of m boxes where s cyclically rotates the contents of the boxes within each set in such a way that each object visits each box in the set. Such a cycle can be written as a product of $m - 1$ transpositions (check this!). Therefore, s is a product of $(m - 1)(p - 1)/m$ transpositions and $\sigma(s) = (-1)^{(m-1)(p-1)/m}$. This shows that $\sigma(s) = -1$ if and only if $(m - 1)(p - 1)/m$ is odd, and this can only happen if m is even and $(p - 1)/m$ is odd. If $(p - 1)/m$ is odd, then the highest power of 2 that divides $p - 1$ must also divide m, and this can only happen if k is odd (since otherwise, $(k, p - 1)$ would be even and its factors of 2 would cancel with factors of 2 in $p - 1$. Since p is an odd prime, $p - 1$ is even, so the highest power of 2 that divides $p - 1$ is at least 2. Therefore if $(p - 1)/m$ is odd, then m will also be even.

Summer Fun!

Calendar

Session 15: (all dates in 2014)

September	11	Start of the fifteenth session!
	18	
	25	No meet
October	2	Emily Pittore, iRobot
	9	
	16	
	23	
	30	
November	6	
	13	Cornelia A. Van Cott, University of San Francisco
	20	
	27	Thanksgiving - No meet
December	4	
	11	

Finding the Maximum Subsequence Answer (see page 8): The maximum subsequence runs from the 6^{th} term to the 10^{th} term (8, 4, -10, -1, 12) and the sum of this subsequence is 13.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____

The $36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $36 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton Univeresity
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, The Dartmouth Institute
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, assistant professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, associate professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $36 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ______________________________________