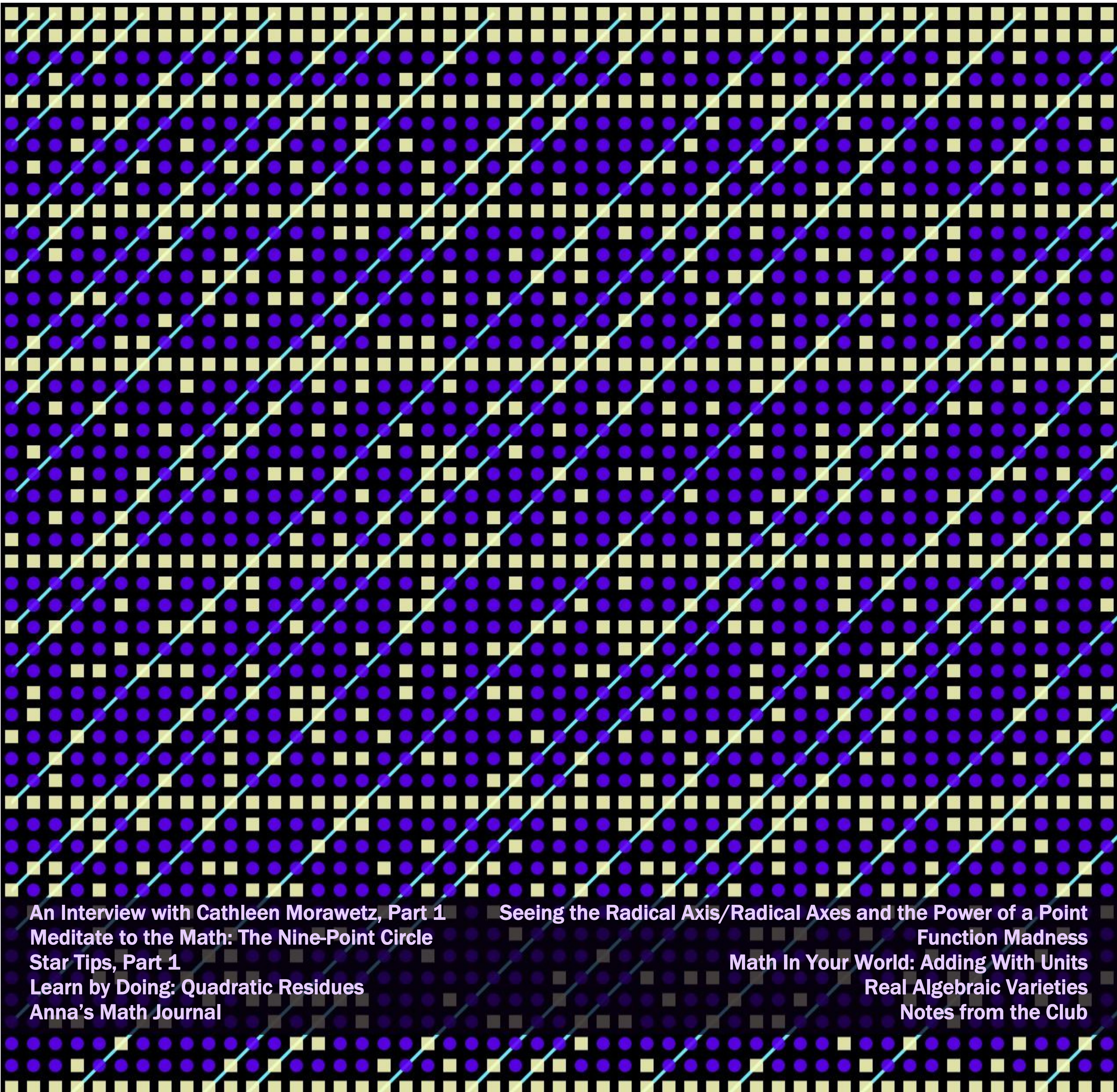
Girls' Angle Bulletin

October 2014 • Volume 8 • Number 1

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Find something that interests you so much that you can't help asking questions about it. Try to answer those questions. If you can't answer them, ask simpler and simpler questions until you can. In the meantime, cultivate an interest in this process of asking questions and finding answers. For inspiration, seek out and talk to others who do the same.

- Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Can you figure out what our cover graphic is about? Send in your thoughts to girlsangle@gmail.com!

An Interview with Cathleen Morawetz, Part 1

Cathleen Morawetz is Professor Emerita at the Courant Institute of Mathematical Sciences at New York University, where she was director from 1984-1988. Last summer, Girls' Angle Program Assistant Margo Dawes interviewed Cathleen, traveling to her home in New York City. The following is a transcript of that interview.

Margo: What is mathematics to you – both as a profession and a field – and how has it affected your life?

Cathleen: We have to begin with the fact that my father was a mathematician. He was born in Ireland, educated in Ireland, and then he went to Canada. I was born in Canada. So I grew up in a household where it was a very natural thing to be a mathematician. In Canada at that time the university education was very specialized, so I got a scholarship to go into mathematics or physics. I went into the so-called "Math, Physics, or Chemistry" option and then narrowed it down to applied mathematics by the time I graduated. So that's how I got into mathematics – it was quite natural.

Margo: Was your mother also a mathematician?

Cathleen: No, she studied mathematics as an undergraduate at Trinity College Dublin, which is where she met my father.

Margo: But she didn't pursue it professionally?

Cathleen: No.

Margo: Do you remember any specific point when you were beginning your study that you realized mathematics was for you, or was it just a gradual introduction?

Cathleen: [laughs] No. I thought of getting out of it more often than I thought of getting into it. The educational system in Canada was very narrow. You couldn't go from math and physics into English literature. You could go into a pass course, but that was a very weak degree to get. I quit after three years – this was during the war – and I went to do war work for a year, and then I came back to finish my last year.

Margo: What was it that made you stick with your course of study and ultimately pursue mathematics as a career?

Cathleen: My mother found very much that she wished she'd had a profession. So I think that influenced me a lot, that one should have a profession. Then I married Herbert [sitting in the room with us] and he had no objections to having a working wife and encouraged me very much. Then he was working in New Jersey, so we came to New Jersey – I had just gotten a Master's Degree at MIT – and my father met Richard Courant at a math society meeting shortly after I was married. They discussed the problems of their daughters and their professions. Courant said, 'I don't think you can help me with my daughter, but perhaps I can help you with yours.' So I went to see him, and he gave me a job editing – for English, really – what became a very

Dear Reader,

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For this issue, those who do not subscribe to the print version will be missing out on the complete interview with Prof. Zamaere, and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
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Meditate$^{\text{Math}}$

The Nine-Point Circle

Look at the geometric diagram on the next page. It shows triangle *ABC* with its sides extended. The midpoints of the sides of triangle *ABC* are connected to form the **medial triangle**, colored purple. The dark blue circle circumscribes the medial triangle and is known as the **9-point circle**.

Inside triangle *ABC* and tangent to all three of its sides is its incircle or inscribed circle, colored light purple. The three large green circles are each mutually tangent to all three lines determined by the sides of triangle *ABC*. These green circles are called the **excircles** of triangle *ABC*.

The altitudes of triangle *ABC* are colored blue.

Find a quiet place to meditate upon this diagram. Ignore time. Just observe. What do you see? What can you explain? Record your observations. Jot down any explanations.

If you'd like a little help to get going, pick a fact listed below and try to see that it is true. Try to explain it. The facts are listed in no particular order.

1. The centroid of the medial triangle coincides with the centroid of triangle *ABC*. (The centroid of a triangle is the intersection of its medians.)

2. If you rotate the medial triangle 180° about its centroid and then perform a dilation centered at the centroid by a factor of 2, you get triangle *ABC*. When you perform this transformation, the circumcenter of triangle *ABC* is mapped onto its orthocenter.

3. The orthocenter, circumcenter, and centroid of triangle *ABC* are collinear. (The line that passes through these points is called the **Euler** line.)

4. The diameter of the 9-point circle is half the diameter of the circumcircle of triangle *ABC*.

5. The center of the 9-point circle is exactly halfway between the orthocenter and circumcenter of triangle *ABC*.

6. The perpendicular bisectors of the sides of triangle *ABC* are also the altitudes of the medial triangle.

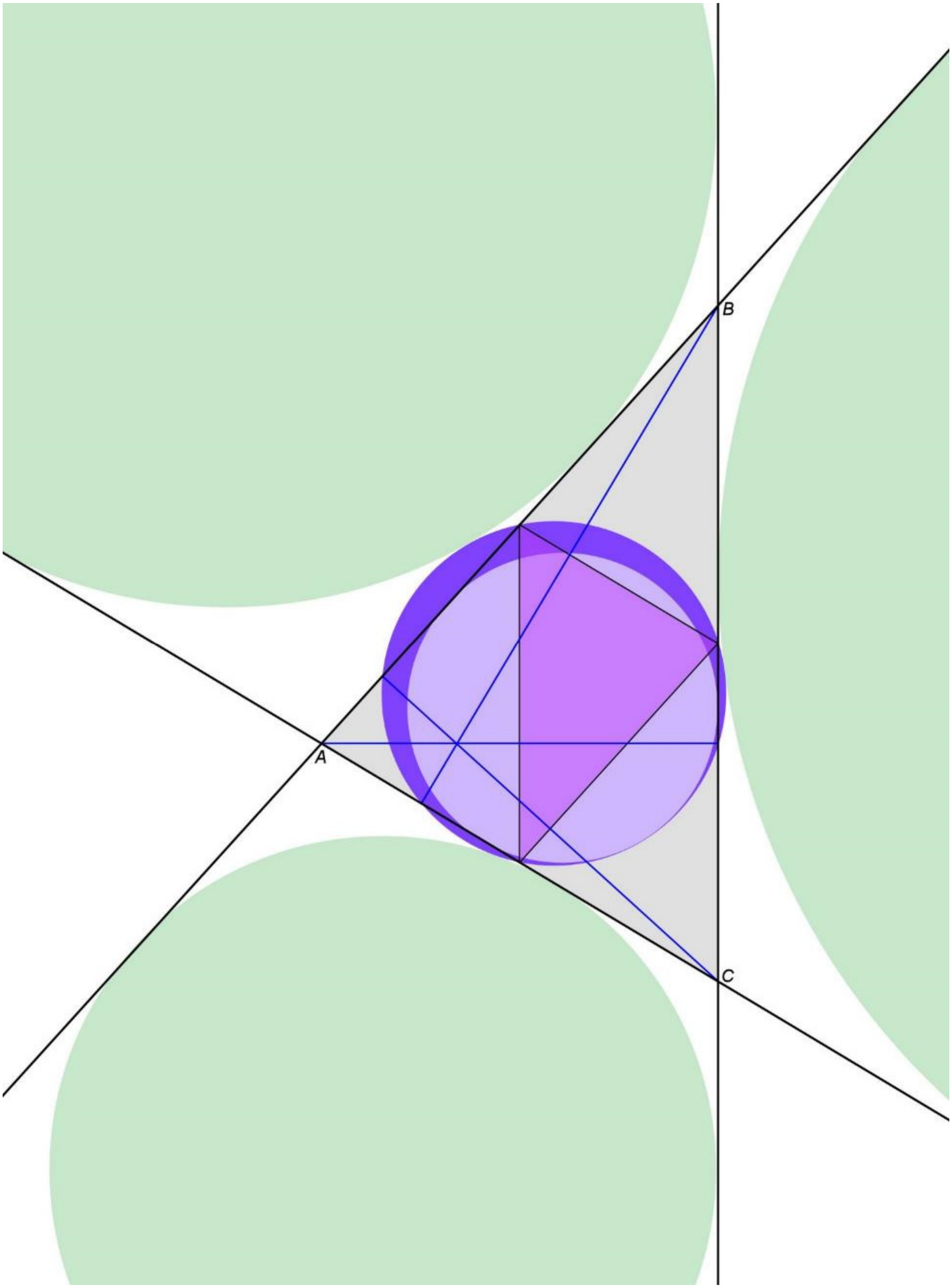
7. The feet of the altitudes of triangle *ABC* are on the boundary of the 9-point circle.

8. The 3 segments that connect the orthocenter (intersection of the altitudes) with the 3 vertices of triangle *ABC* are bisected by the 9-point circle.

9. Any line segment that connects the orthocenter of triangle *ABC* to a point on triangle *ABC*'s circumscribed circle is bisected by the 9-point circle.

10. All 4 circles that are mutually tangent to the sides of triangle *ABC* are also tangent to the 9-point circle.

The Nine-Point Circle



Star Tips, Part 1

by Madison Evans and Ken Fan | edited by Jennifer Silva

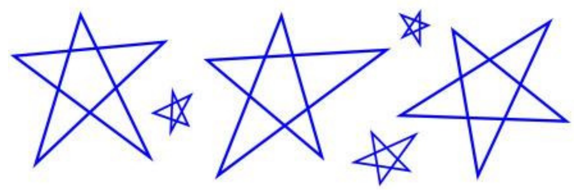
Emily and Jasmine are looking at the stars.

Emily: They're beautiful, aren't they?

Jasmine: Yeah. I used to love stargazing. I'd make up my own constellations, and then forget them a second later.

Emily: Remember when we used to draw stars all over our papers?

Jasmine: Yeah! Those 5-pointed stars, like this …



Emily: I like those more than plain old polygons. Why aren't these discussed in geometry class?

Jasmine: Well, maybe there's not much to say about them. Remember that theorem in geometry about the sum of the interior angles in a polygon? That they always add up to $180(n - 2)$ degrees, where n is the number of sides? If there were theorems like that for 5-pointed stars, I bet they'd discuss them in geometry.

Emily: Hey, maybe we can find a theorem about 5-pointed stars! What's the sum of the angles in a 5-pointed star?

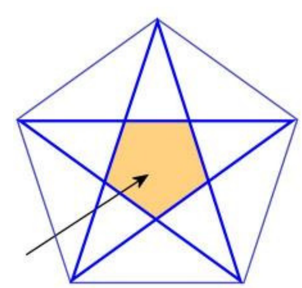
Jasmine: Hmm. A 5-pointed star is really an example of a 10-sided polygon, and the theorem about angle sums in polygons would tell us that the angle sum is $180(10 - 2)$ degrees. So that wouldn't really be a different kind of result.

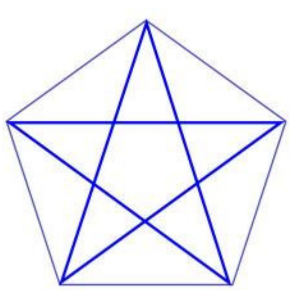
Emily: What if we add up only the angles in the five tips?

Jasmine: That's an idea. Let's try it!

Emily: We might as well start with a regular 5-pointed star.

Jasmine: By "regular," do you mean where the tips would be the vertices of a regular pentagon?





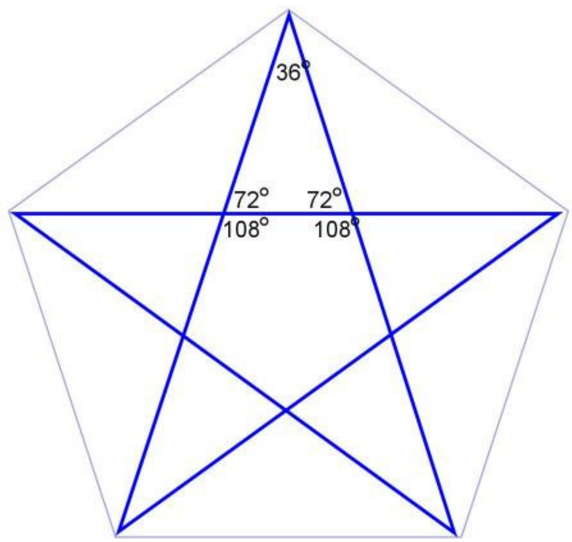
Emily: Yes, like this. (See figure at left.) Is this pentagon inside the 5-pointed star a regular pentagon? (See figure at right.)

Jasmine: I think it has to be, by symmetry. If you make the 5-pointed star by connecting the vertices of a regular pentagon, the resulting shape will retain all of the symmetries of the pentagon.

Emily: That's true. Any symmetry that preserves the pentagon must preserve the entire 5-pointed star. So by symmetry, all of the side lengths and angles of the small pentagon inside our 5-pointed star are equal to each other.

Jasmine: Since the pentagon is regular, we can figure out its internal angles. Then, using that information, we can figure out the base angles in each triangular tip. With that knowledge, we can compute the angles in each of the triangular tips, add those together, and get the total sum of the tip angles.

Emily: Let's do it!

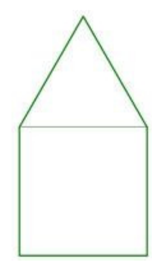


Emily and Jasmine use the formula $180(n - 2)$ with $n = 5$ to find that the sum of the interior angles in a pentagon is 540°. Dividing this by 5, they discover that the interior angle in a regular pentagon is 108°. Next, they use the fact that the base angles in the triangular tips are supplementary to the interior angles of the pentagon to find that the base angles in the triangular tips measure 72°. Knowing that the angles in a triangle add up to 180°, they conclude that each tip angle measures 36°. Hence, the sum of the tip angles in a regular 5-pointed star is 5(36°), or 180°.

Emily: That was fun! Let's work out the sum of the tip angles in another 5-pointed star.

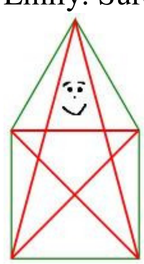
Jasmine: Okay, how about we try a 5-pointed star made by connecting the vertices of a house?

Emily: A house?



Jasmine: You know, a pentagon made by putting an equilateral triangle on top of a square. Like this … (See figure at right.)

Emily: Sure, why not?



Emily draws a 5-pointed star using the vertices of the "house" pentagon.

Emily: Funny how the "house-star" looks like a person.

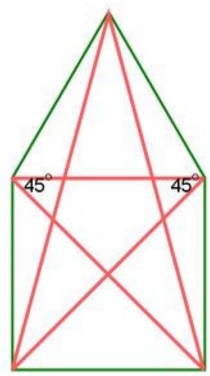
Jasmine: Cute, Emily. Real cute.

Emily: Sorry, but I couldn't resist. Let's see … the bottom two angles of the house would be 90°.

Jasmine: And the top angle would be 60°, because it's part of an equilateral triangle. The remaining two angles are symmetrical and must total 540° – 2(90°) – 60°, which is 300°. Then you divide that number by 2 to get 150°.

Emily: Or you can see that the remaining two angles consist of one angle from the square and one from the triangle, which is 90° + 60°.

Jasmine: True, but we need to find the tip angles of the 5-pointed star, not the house angles.



Emily: Well, the angles that make the "hands" are 45° apiece, because each is the angle between a side of the square and a diagonal of the square, and the diagonal bisects the square.

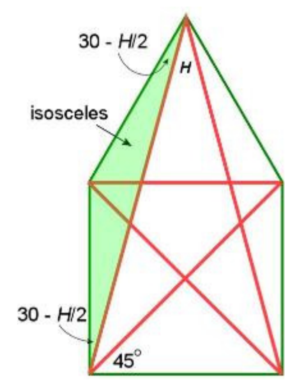
Jasmine: Hmm. The "feet" angles are equal, by symmetry. Do you think the lines that form the "head" angle trisect the top of the house?

Emily: Maybe. I don't know.

Jasmine: I don't see how to compute the head and feet angles.

Emily: I don't, either. Well, let's call the head angle H degrees and see if we can relate other angles to H.

Jasmine: Okay. By symmetry, the two angles on each side of the head angle in the top of the house are equal, and all three angles there add up to 60°; that means the angles that flank the head measure $30 - H/2$ degrees.



Emily: Wait a sec! This triangle is isosceles, (see figure at right) because the house is made from a square and an equilateral triangle, all with the same side lengths. So the angle next to this foot also measures $30 - H/2$ degrees.

Jasmine: Oh, right! And the other angle next to that foot is 45°. So that means that the foot angle must be $90° - (30° - H/2) - 45°$, or $15 + H/2$ degrees.

Emily: Together, the angles of the head and feet add up to $H + 2(15 + H/2)$ degrees, and that simplifies to $30 + 2H$ degrees. So the total sum of all of the tip angles in our 5-pointed house-star is $45 + 45 + 30 + 2H$, or $120 + 2H$ degrees. If we could just figure out what H is …

Jasmine: Emily, I just noticed something.

Emily: What?

Jasmine: That isosceles triangle you observed … we know the angles in it, because its apex angle is one of the house angles which we already computed to be 150°. So its base angle measures 15°, and that's the $30 - H/2$ angle next to the foot!

Emily: Ah! So H is 30°!

Jasmine: Then the sum of the tip angles is $120 + 2(30)$, or 180°. That's what we got for the regular 5-pointed star!

Emily: Do you think that the sum of the angles in *any* 5-pointed star is 180°?

Jasmine: I'm beginning to think so. Let's try to prove it!

Emily: Okay. But how are we going to do this? With an arbitrary 5-pointed star, we don't know the measures of any of the angles.

Jasmine: The only fact I can think of that works in general is the basic angle-sum theorem for polygons. So we know that the five angles in the pentagon contained inside our 5-pointed star add up to 540°.

Emily: Maybe we can try to relate the angles in the tips to the interior angles of the pentagon?

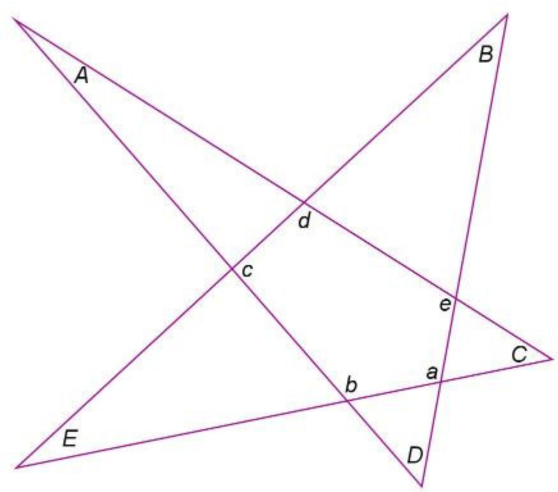
Jasmine: That's an idea! I'll label the angles of the tips A, B, C, D, and E.

Emily: And I'll label the angles of the pentagon a, b, c, d, and e.

Jasmine: But I just labeled the tip angles A, B, C, D, and E!

Emily: Oh, sorry! I meant *lower* case a, b, c, d, and e!

Jasmine: That's fine. It looks like there's a unique angle inside the pentagon opposite each tip angle, so let's label the angle opposite capital A with lower case a, the angle opposite capital B with lower case b, and so on.



Emily: Great. I'll do that.

Emily and Jasmine stare at the figure for some time. Can you do what they are trying to do?

Jasmine: I think I just noticed something useful! Tip angles A and C and pentagon angle b are the 3 angles of a single triangle, so they add up to 180°.

Emily: Nice observation! By symmetry, we get analogous equations for five different triangles:

$$\begin{aligned} A + C + b &= 180^\circ \\ B + D + c &= 180^\circ \\ C + E + d &= 180^\circ \\ D + A + e &= 180^\circ \\ E + B + a &= 180^\circ \end{aligned}$$

Jasmine: If we add up these equations, we'll get an equation involving the sum of the tip angles and the sum of the interior angles. It looks like each tip angle will be counted twice and each interior angle just once, so we get

$$2(A + B + C + D + E) + a + b + c + d + e = 5(180^\circ).$$

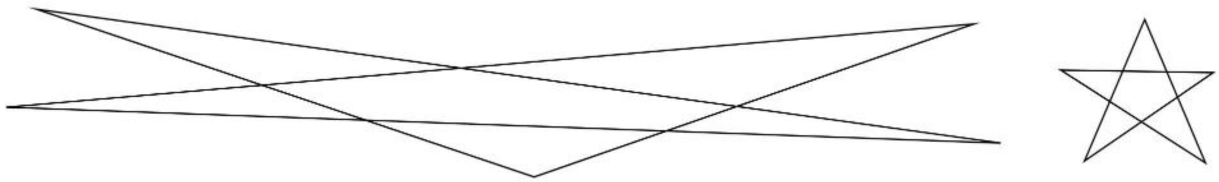
Emily: And since the sum of the interior angles of a pentagon is 3(180°), we get

$$2(A + B + C + D + E) = 2(180°),$$

which shows that the sum of the tip angles really is 180° in general!

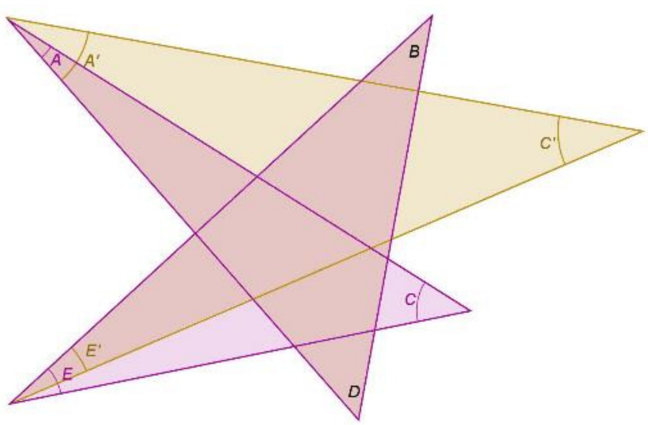
Jasmine: Cool! We did it!

Emily: That's amazing! No matter how askew or not our 5-pointed star is, the sum of its tip angles will always add up to a straight angle.



Jasmine: Wait a sec. As we move one of the tip vertices, we're affecting the sizes of three different angles. The result implies that the sum of the three angles must be an invariant of the motion.

Emily: You're saying that in the figure below, $A + C + E = A' + C' + E'$.



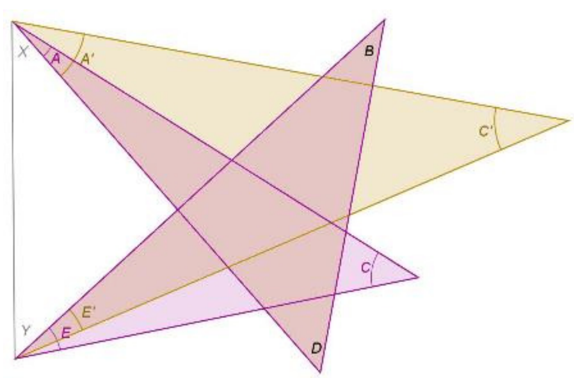
Jasmine: Right. Doesn't that seem weird?

Emily: Hmm. I think I can explain that. If we connect the two tips that the moved vertex is connected to, we close off a triangle. I'll label two of this triangle's angles by X and Y. (See figure below.) The sum of the angles in a triangle is 180°. That means $C + A + E + X + Y = 180°$ and $C' + A' + E' + X + Y = 180°$. That is,

$$C + A + E + X + Y = C' + A' + E' + X + Y.$$

If we subtract $X + Y$ from both sides of this equation, we get the invariance.

Jasmine: Nice. What you're saying is that the invariance of the sum of the three angles that change as you move one of the tip vertices around is essentially the same as the invariance of the sum of the three angles in a triangle.



Emily: Right.

Jasmine: Hey, now I'm wondering: does all this generalize to stars with even more tips?

Tune in next time as Emily and Jasmine explore 6, 7, 8, and higher-pointed stars.

Learn by Doing

Quadratic Residues
by Girls' Angle Staff

For this installment of Learn by Doing, we assume familiarity with modular arithmetic. For an introduction to modular arithmetic, take a look at any book on number theory, such as Harold Davenport's *The Higher Arithmetic* or Oystein Ore's *Invitation to Number Theory*. Also see Hana Kitasei's Bulletin's series *Prueba del 9* in Volume 2, Numbers 1-5 or Robert Donley's series *Fermat's Little Theorem* in Volume 6, Numbers 1-4.

Problem 1. Find a perfect square that is 6 greater than a multiple of 10. (A perfect square is the square of an integer.)

Problem 2. For what digits d (from 0 to 9, inclusive) is there a perfect square that has units digit d?

Problem 3. For what numbers d between 0 and 12, inclusive, is there a perfect square that is d more than a multiple of 13?

In general, a number d is a **quadratic residue** modulo n, if and only if there is a perfect square that is d more than a multiple of n. In other words, d is a quadratic residue modulo n, if there is a solution to the modular arithmetic equation $x^2 = d \pmod{n}$.

Problem 4. If d is a quadratic residue, modulo n, show that $d + n$ is also a quadratic residue modulo n.

Because of problem 4, once we know which of the numbers between 0 and $n - 1$ (inclusive) are quadratic residues modulo n, we know all the quadratic residues modulo n.

Problem 5. Show that $x^2 = (x + n)^2 \pmod{n}$ and $x^2 = (n - x)^2 \pmod{n}$.

Problem 5 shows that the list of perfect squares $0^2, 1^2, 2^2, \ldots, (n/2)^2$ (if n is even) or $0^2, 1^2, 2^2, \ldots, ((n - 1)/2)^2$ (if n is odd) will contain, modulo n, all the quadratic residues.

Problem 6. Show that the quadratic residues modulo 8 are 0, 1, and 4, up to multiples of 8.

Problem 6 shows that the squares from 0^2 to $\lfloor n/2 \rfloor^2$ may repeat values, modulo n. *Henceforth, we will work modulo p, where p is an odd prime number.*

Problem 7. Let $0 \le a < b \le (p - 1)/2$. Show that $a^2 - b^2$ is not divisible by p. Conclude that there will be exactly $(p - 1)/2$ nonzero quadratic residues modulo p.

Thus, modulo an odd prime p, all quadratic residues are congruent to one of the $(p + 1)/2$ distinct squares $0^2, 1^2, 2^2, \ldots, ((p - 1)/2)^2$.

Problem 8. List all the quadratic residues between 0 and p for $p = 17$, $p = 19$, and 3 other odd prime moduli. What patterns do you see? Make some conjectures and try to prove them.

In the following problems, we'll describe some patterns that may overlap with your answer to problem 8, so don't proceed until you are satisfied with your own answer to that problem.

Problem 9. Show that quadratic residues (QR) and non-residues (NR), modulo p, satisfy the following multiplication table:

$\times$	NR	QR
NR	QR	NR
QR	NR	QR

Problem 10. Let S be the sum of all the quadratic residues modulo p between 0 and p. Let x be any non-residue modulo p. Show that xS is congruent to the sum of the non-residues modulo p. Show that $S + xS = 1 + 2 + 3 + \ldots + (p - 1) = 0 \pmod p$. Conclude that $S = xS = 0 \pmod p$.

Problem 11. Let $x \neq 0 \pmod p$. Show that x and x^{-1} are either both quadratic residues modulo p or both non-residues modulo p.

Problem 12. Let P be the product of all the quadratic residues between 0 and p modulo p. Show that $P = 1$ if $p = 3 \pmod 4$ and $P = -1$ if $p = 1 \pmod 4$.

Problem 13. Use problem 12 to deduce that $((p - 1)/2)!)^2 = -1 \pmod p$ if $p = 1 \pmod 4$. Notice, that this shows that -1 is a quadratic residue modulo p if $p = 1 \pmod 4$.

Fermat's little theorem says that $x^{p-1} = 1 \pmod p$ for all $x \neq 0 \pmod p$.

Problem 14. Use Fermat's little theorem to show that -1 is not a quadratic residue modulo p when $p = 3 \pmod 4$. Hint: Suppose $-1 = x^2 \pmod p$. Raise both sides to the $(p - 1)/2$ power.

Problems 13 and 14 show that for odd primes p, -1 is a quadratic residue iff $p = 1 \pmod 4$.

In the table below, the quadratic residues between 0 and p for $p = 17$ and $p = 19$ are bold red.

p																		
17:	**1**	**2**	3	**4**	5	6	7	**8**	**9**	10	11	12	**13**	14	**15**	**16**		
19:	**1**	2	3	**4**	**5**	**6**	**7**	8	**9**	10	**11**	12	13	14	15	**16**	**17**	18

Notice that in one case, the quadratic residues show mirror symmetry, and in the other, they show anti-mirror symmetry. Problems 15 explains this pattern.

Problem 15. If -1 is a quadratic residue modulo p, show that x and $-x$ are either both quadratic residues modulo p or both non-residues modulo p. If -1 is not a quadratic residue modulo p, show that x is quadratic residue modulo p if and only if $-x$ is a non-residue.

A **triangular residue** modulo p is a number t that is congruent to a triangular number modulo p, that is, for which there exists an integer x such that $t = x(x + 1)/2 \pmod p$. Let Q be the set of quadratic residues modulo p. Let T be the set of triangular residues modulo p.

Problem 16. Show that $T = \{ (x - 1)/8 \mid x \in Q \}$. (Note: Here, division by 8 takes place modulo p.)

Going back to Problem 7, we know that exactly half of the numbers from 1 to $p - 1$ are quadratic residues modulo p. Next, we'll explore the frequency that consecutive numbers are both quadratic residues, both quadratic non-residues, or a mix of the two.

Let $Q(p)$ be the number of numbers a, with $0 \le a < p$, such that both a and $a + 1$ are quadratic residues modulo p.

Let $N(p)$ be the number of numbers a, with $0 \le a < p$, such that both a and $a + 1$ are quadratic non-residues modulo p.

Let $L(p)$ be the number of numbers a, with $0 \le a < p$, such that a is a quadratic residue modulo p, but $a + 1$ is a quadratic non-residue modulo p.

Let $R(p)$ be the number of numbers a, with $0 \le a < p$, such that a is a quadratic non-residue modulo p, but $a + 1$ is a quadratic residue modulo p.

Also, recall the Legendre symbol:

$$\left(\frac{a}{p}\right) \equiv \begin{cases} 1 \text{ if } a \text{ is a square modulo } p \text{ and } a \neq 0 \pmod{p}, \\ 0 \text{ if } a = 0 \pmod{p}, \\ -1 \text{ otherwise.} \end{cases}$$

Problem 17. For the first 7 odd prime numbers p, make a table of all the quadratic residues modulo p, and compute $Q(p)$, $N(p)$, $L(p)$, and $R(p)$.

Problem 18. A. Explain why $Q(p) + L(p) = (p + 1)/2$. Similarly, show that
B. $Q(p) + R(p) = (p + 1)/2$, and
C. $N(p) + R(p) = N(p) + L(p) = (p - 1)/2$.

Problem 19. Show that $Q(p) + N(p) - L(p) - R(p) = 1 + \left(\frac{-1}{p}\right) + \sum_{a=0}^{p-1} \left(\frac{a(a+1)}{p}\right)$.

Problem 20. Note that $a(a + 1) = a^2(1 + 1/a)$. Use this fact to prove that $\sum_{a=0}^{p-1} \left(\frac{a(a+1)}{p}\right) = -1$.

Problem 21. Find expressions for $Q(p)$, $N(p)$, $L(p)$, and $R(p)$.

Problem 22. Let $C_p = \{ (x, y) \mid x^2 + y^2 = 1 \pmod{p}, 0 \le x < p, \text{ and } 0 \le y < p \}$. Determine $\#C_p$ for all prime numbers p.

Problem 23. Prove that $\sum_{a=0}^{p-1} \left(\frac{a(a+1)(\mathrm{a}+2)}{p}\right) = 0$, where p is an odd prime number congruent to 3 modulo 4.

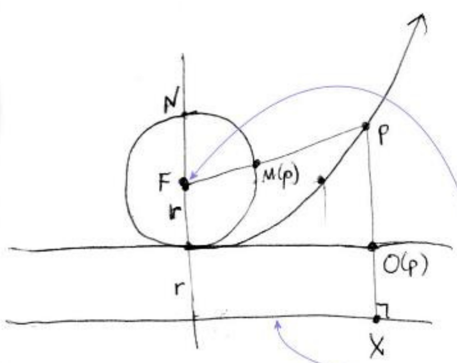
Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna finds a synthetic proof of the projection result she found last time.

Last time, I discovered a neat result. If you center a sphere at the focus of a paraboloid of revolution so that it just touches the paraboloid's vertex, then stereographic projection onto the tangent plane at the vertex, composed with the "light map," is the same as vertical projection.



$O(p)$ is vertical projection down to the tangent plane at the vertex.

$M(p)$ is the "light" map

$S(p)$ is stereographic projection of $M(p)$ onto the tangent plane at the vertex.

The light map sends a point on the paraboloid to the point on the sphere where vertically incoming light bounces off the point and intersects the sphere on its way to the focus.

My proof used analytic geometry, and I can't help but wonder if there is a nicer proof - perhaps one that involves the definition of a parabola as the locus of points equidistant from a point and a line, known as the directrix.

Let's see. Because the parabola is the locus of points equidistant from the focus *F* and the directrix, I know that *FP* = *PX*. And since the vertex of the parabola is halfway between the focus and directrix, both *FM*(*P*) and *O*(*P*)*X* are equal to the radius of the sphere. Therefore, *PM*(*P*) = *PO*(*P*).

$$FP = PX$$

$$PM(p) = PO(p)$$

$$\angle NFP \cong \angle FPO(p)$$

These angles are equal - they're alternate interior angles.

$$\triangle FNM(p) \sim \triangle PM(p)O(p)$$

So these triangles are similar, by side angle side....they're both isosceles triangles.

$$\angle NM(p)F \cong \angle PM(p)O(p)$$

So these angles are congruent...

$\Rightarrow$ $N, M(p), O(p)$ are collinear.

...but that means *N*, M(*P*), and *O*(*P*) are collinear!...

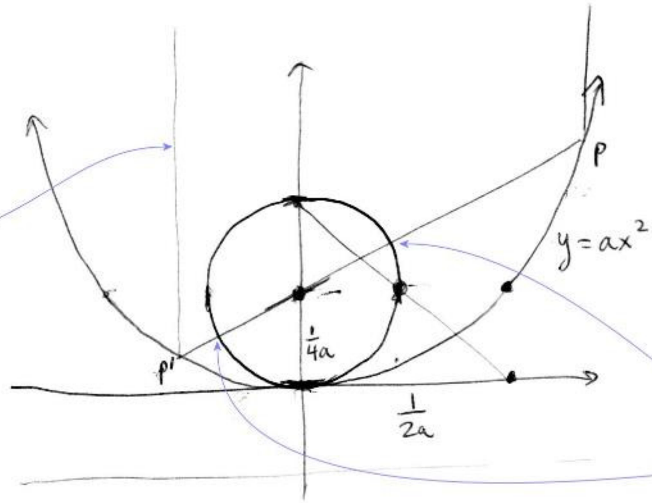
$\Rightarrow$ $O(p) = S(M(p))$.

...and that shows that *O*(*P*) is the stereographic projection of M(*P*)! Wow, that proof practically wrote itself!

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

I wonder if the result can be used to see anything else.

When light comes in vertically, it bounces off the paraboloid, say at point P, goes through the focus, then reflects again off the paraboloid, say at P', and then heads away straight up. I feel that the result should tell me something about how P and P' are related.



I know that $\mathcal{O}(P)$ and $\mathcal{O}(P')$ are the stereographic projections of the points where PP' intersects the sphere. And since PP' passes through the center of the sphere, these intersection points are diametrically opposed.

I'll define I(P) to be P'.

$$I(p) = p'$$

I remember that, through stereographic projection, reflection of the sphere in its equatorial plane corresponds to inversion of the plane. If I take a point q on the sphere, reflect in the equatorial plane, then rotate 180 degrees about the north-south axis, I end up sending q to the point diametrically opposed to it...

$O(p') = -$ Inversion of $O(p)$ in circle of radius $\frac{1}{2a}$ centered at paraboloid vertex.

...that means that $\mathcal{O}(P')$ must be the negative of the inversion of $\mathcal{O}(P)$ in the circle that is the stereographic projection of the equator. The radius of the sphere is $1/(4a)$, so the circle of inversion must have radius $1/(2a)$.

$$p = (x, ax^2)$$

So if P is (x, ax^2), then, using inversion, P' must have horizontal coordinate $-1/(4a^2)$ divided by x, so the vertical coordinate is a times that squared...

$$I\left((x, ax^2)\right) = \left(-\frac{1}{4a^2x}, \frac{1}{16a^3x^2}\right)$$

To make a stereographic projection of a globe, make a large paraboloid of revolution big enough so that it fully contains the globe when the globe is centered at the focus of the paraboloid. Put a light at the focus. Place the plane of the canvas perpendicular to the axis to capture the reflected light.

What else can be said? I guess the fact that $\mathcal{O}(P)$ equals $\mathcal{S}$(M(P)) gives an amusing way to create a stereographic projection using a light placed at the center of the sphere, instead of the north pole. I'll go ahead and record this idea here.

ABB 10.26.14

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Seeing the Radical Axis

by Anna B. | edited by Jennifer Silva

If you are not familiar with the power of a point and the radical axis, please read Lightning Factorial's *Radical Axes and the Power of a Point* on page 20.

Lately, paraboloids of revolution have been swirling around in my mind. A paraboloid of revolution is the surface traced out by a parabola when you spin the parabola about its axis. These bowl-like surfaces are put to good use as the shape of the collecting mirror in reflecting telescopes.

While thinking about them, I stumbled upon a way to understand basic facts about a geometric object known as the **radical axis**. The radical axis is the locus of points that have the same power with respect to two different circles in the plane. If the circles are concentric, then the locus of such points is empty and there is no radical axis. But if the circles are nonconcentric, then the radical axis turns out to be a line that runs perpendicular to the line that passes through the circle centers. Here, I'd like to explain how these facts about the radical axis can be understood just by meditating on paraboloids of revolution. Hardly a computation will be needed!

For the sake of definiteness, let's work in the xyz-coordinate space, oriented so that the z-axis points straight up. Let H be the horizontal plane that contains the x- and y-axes. The plane H is where any circles we encounter will live. By "standard paraboloid," I mean the paraboloid of revolution represented by the equation $z = x^2 + y^2$ or any translate of it. Finally, by "standard parabola," I mean any parabola congruent to the parabola $y = x^2$ in the xy-coordinate plane. (Note the following quirk of these temporary definitions: I am demanding that standard paraboloids be vertically oriented, whereas standard parabolas need not be.)

Observation 1

The graph of the power of points with respect to a fixed circle is a paraboloid of revolution. Specifically, it is the standard paraboloid that contains the circle.

Let C be a circle in H. Let $p(x, y)$ be the power of the point $(x, y, 0)$ with respect to C. Let P_C be the graph of p, i.e., the set of points $(x, y, p(x, y))$. Observation 1 asserts that P_C is the standard paraboloid that contains C.

To see this, we use the formula $d^2 - r^2$ for the power of $(x, y, 0)$ with respect to C, where d is the distance of the point from the center of C and r is the radius of C. Let $(a, b, 0)$ be the coordinates of the center of C. In terms of our coordinates, $d^2 = (x - a)^2 + (y - b)^2$. Hence, the power of the point $(x, y, 0)$ with respect to C is $p(x, y) = (x - a)^2 + (y - b)^2 - r^2$. This formula reveals that the graph of p is the translate of the paraboloid $z = x^2 + y^2$ by a units in the x direction, b units in the y direction, and $-r^2$ units in the z direction. When $(x, y, 0)$ is on the circle, $d = r$ and the power $p(x, y)$ is 0, so the graph contains the circle C.

I like to imagine C as a ring made out of some idealized metal, and that we have a bowl shaped like the standard paraboloid. We obtain P_C by "dropping the bowl into the ring C." The bigger the radius of C, the deeper the bowl will dip below H.

Observation 2

Vertical planar cross sections of standard paraboloids are vertically-oriented standard parabolas. The plane containing the axis of the paraboloid and the vertex of the parabola (if distinct from the vertex of the paraboloid) is perpendicular to the cross-sectional plane.

To see this second observation, it suffices to check that vertical planar cross sections of the standard paraboloid $z = x^2 + y^2$ are vertically-oriented standard parabolas, because translates of vertical planes are also vertical planes. We can see directly from the equation $z = x^2 + y^2$ that planes parallel to the yz-coordinate plane (i.e., planes given by equations of the form $x = constant$) intersect $z = x^2 + y^2$ in vertically-oriented standard parabolas. Furthermore, in these cases, the vertex of the parabolic intersection is situated directly above the x-axis. Hence, if the vertex of the parabolic intersection is not the origin, then the plane containing it and the z-axis is perpendicular to planes parallel to the yz-coordinate plane.

By the rotational symmetry of the paraboloid of revolution, we obtain Observation 2.

Observation 3

Let P_1 and P_2 be distinct standard paraboloids. Then P_1 and P_2 intersect in a vertically-oriented standard parabola, unless they are vertical translates of each other, in which case their intersection is empty.

Standard paraboloids represent the graphs of functions, so if P_1 and P_2 are vertical translates of each other, they have no points of intersection. Let's assume they are not vertical translates of each other and that they have distinct axes. Let A be the unique vertical plane that contains both of their axes. The two paraboloids will intersect A in two vertically-oriented standard parabolas. These two parabolas must intersect. (For example, one can check that the graph of their difference is a non-horizontal straight line which necessarily crosses the horizontal axis. In the xy-coordinate plane, this amounts to checking that $((x - a)^2 + b - ((x - c)^2 + d))$ is equal to a linear function with slope $2(c - a)$, and noting that when $a \neq c$, this slope is not 0.) Let R be the unique vertical plane perpendicular to A that contains the point of intersection of the two parabolas.

From Observation 2, we know that R intersects both P_1 and P_2 in vertically-oriented standard parabolas. Since the two intersections share the same parabolic vertex, they must be one and the same parabola.

Observation 4

Let C and C' be two nonconcentric circles in H. Then P_C and $P_{C'}$ intersect in a parabola contained in a vertical plane perpendicular to the line segment joining the circle centers.

Observation 1 tells us that P_C and $P_{C'}$ are standard paraboloids. Observation 3 tells us that P_C and $P_{C'}$ intersect in a vertically-oriented standard parabola contained in a vertical plane. In our demonstration of Observation 3, we saw that this vertical plane is perpendicular to the line segment joining the centers of the circles.

Observation 5

Let C and C' be two nonconcentric circles in H. The radical axis of C and C' is a straight line perpendicular to the line segment joining their centers.

Let R be the vertical plane that contains the intersection of P_C and $P_{C'}$. The intersection of R and H is the radical axis of C and C', for these are exactly the points whose powers with respect to both circles are equal. Using Observation 4, we conclude that the radical axis is a line that runs perpendicular to the line segment joining the circle centers.

Radical Axes and the Power of a Point

by Lightning Factorial | edited by Jennifer Silva

Anna asked me to provide a basic introduction to the power of a point and the radical axis. So here goes!

As soon as you draw a circle in the plane, every point in the plane gets *power*!

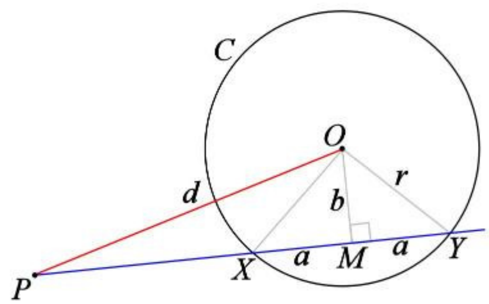
Let me explain.

Fix a circle C in the plane with radius r. Let P be a point in the plane.

The **power of the point P with respect to the circle C** is equal to $d^2 - r^2$, where d is the distance between P and the center of C.

That's actually all there is to the definition of the power of a point (with respect to a circle), so what makes the notion so powerful?

Draw a line through P that intersects circle C at X and Y. If your line is tangent to the circle, take $X = Y$. What's amazing about the power of a point is that no matter which of these lines you use, the power of P with respect to circle C will equal $PX \cdot PY$ or $-PX \cdot PY$, depending on whether P is outside or inside the circle, respectively.



To see why, take a look at the figure to the right. Point M is the midpoint of the chord XY. Notice that ΔXMO and ΔPMO are both right triangles, so $r^2 = a^2 + b^2$ and $d^2 = PM^2 + b^2$.

When P is outside the circle, we compute that $PX \cdot PY = (PM - a)(PM + a) = PM^2 - a^2$. Using $PM^2 = d^2 - b^2$, we find $PM^2 - a^2 = d^2 - b^2 - a^2 = d^2 - r^2$, as desired.

Exercise: Do a similar computation to show that $-PX \cdot PY = d^2 - r^2$ for points P inside the circle.

To illustrate how wonderful it is to be able to compute the power of a point in more than one way, try to prove the following result of Euler's first without using the power of a point:

> Theorem. In triangle T, let C be the circumcenter and let I be the incenter. Let R be the radius of the circumscribed circle and let r be the radius of the inscribed circle. Then $R^2 - CI^2 = 2rR$.

After you've tried to prove this result without using the power of a point, prove the theorem by showing that the equation $R^2 - CI^2 = 2rR$ can be interpreted as an equality between two different expressions for the power of the incenter I with respect to the circumscribed circle. The two different expressions can be obtained by computing this power using two different lines that pass through the incenter.

The Radical Axis

Let C_1 and C_2 be two nonconcentric circles in the plane. The **radical axis** is the locus of points whose powers with respect to circle C_1 and circle C_2 are equal. The reason we must insist that C_1 and C_2 not be concentric is that if they are concentric, then there is no radical axis.

For more on the radical axis, check out *Geometry Revisited* by Coxeter and Greitzer, and read Anna's *Seeing the Radical Axis* on page 18.

Function Madness!

We've got a problem. Lunga Lee loves functions, but is terribly long-winded. How much can you simplify Lunga's lengthy descriptions of the following three functions?

Function 1

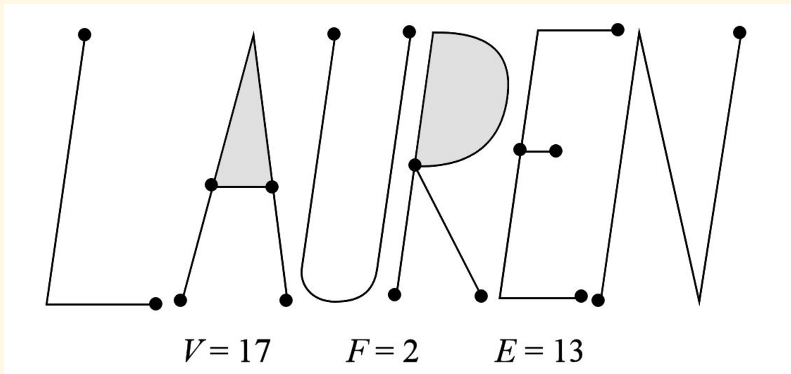
INPUT: Your first name

Carefully write your name in capital letters. Mark with a dot all points that would not have exactly two line segments or curve segments emanating out from the dot. Shade all enclosed regions. (See the example.)

Let V be the number of dots that you marked.

Let F be the number of enclosed regions that you shaded.

Let E be the number of lines or curves that have dots at their ends (a closed loop that begins and ends in a dot counts).



Let $C = V - E + F$.

OUTPUT: C

Function 2

INPUT: A triangle in the coordinate plane whose vertices all have integer coordinates

Let the vertices be (x_1, y_1), (x_2, y_2), and (x_3, y_3). Let $A = x_2 - x_1$. Let $B = y_2 - y_1$. Let $C = x_3 - x_1$. Let $D = y_3 - y_1$. Let $E = AD - BC$. Let $F = |E|/2$. Let $G = F + 1$. Let H be the number of lattice points (i.e. points with integer coordinates) that are inside the triangle and *not* on the triangle's perimeter. Let $I = G - H$. Let $J = 2I$.

OUTPUT: J

Function 3

INPUT: You!

Let h be your height in inches (round to the nearest inch). Let x be equal to four times h. Add 39 to x to get y. Multiply y by 50 to get z. If you already had your birthday this year (2014), let $f = 64$, otherwise, let $f = 63$. Let w equal z plus f. Let v equal w minus the four digit year of your birth. Let u be equal to v minus $200h$.

OUTPUT: u

Adding With Units

by Addie Summer
edited by Jennifer Silva

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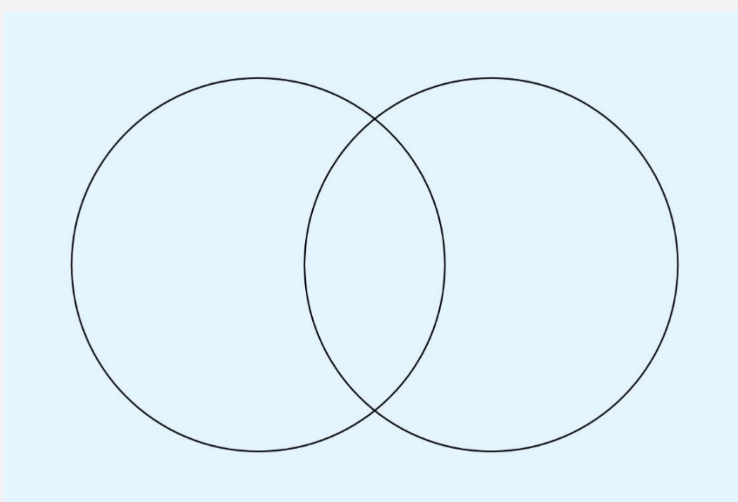
Real Algebraic Varieties

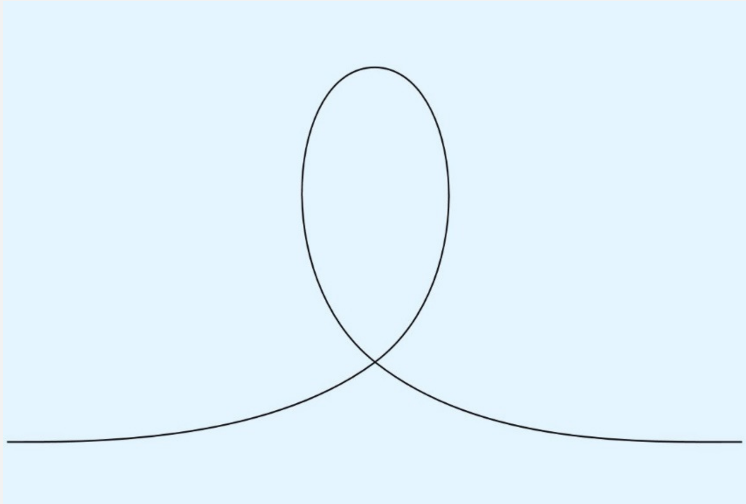
In her interview in the previous issue of this Bulletin, Christine Berkesch Zamaere introduced the concept of an **algebraic variety**. An algebraic variety is the common set of zeroes of a finite collection of polynomials. For example, in the *xy*-coordinate plane, the zeroes of the polynomial $3x - 4y + 12$ correspond to a line, so a line is an example of an algebraic variety.

Each problem gives a set of polynomials in three real variables x, y, and z. The corresponding algebraic variety lives in 3-dimensional (real) space. Can you sketch or describe each variety?

1. $\{ x, y, z \}$
2. $\{ x^2 + y^2 - 1, z - 5 \}$
3. $\{ x - y \}$
4. $\{ y^2 + z^2 \}$
5. $\{y, z\}$
6. $\{ z - xy \}$
7. $\{ x^2 + y^2 + z^2 - 25 \}$
8. $\{ x^2 + y^2 + z^2 - 25, x + y - 5 \}$
9. $\{ x^2 + y^2 + z^2 - 25, x + y - 10 \}$
10. $\{ xyz \}$



Can you specify a finite set of polynomials whose common zeroes look like this figure?



Can you specify a finite set of polynomials whose common zeroes look like this figure?

11. $\{ xyz - 1 \}$

12. $\{ z - x^2 - y^2 \}$

13. $\{ z^2 - x^4 - 2x^2y^2 - y^4 \}$

14. $\{ z^2 - x^2 - y^2 \}$

15. $\{ (x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2) \}$

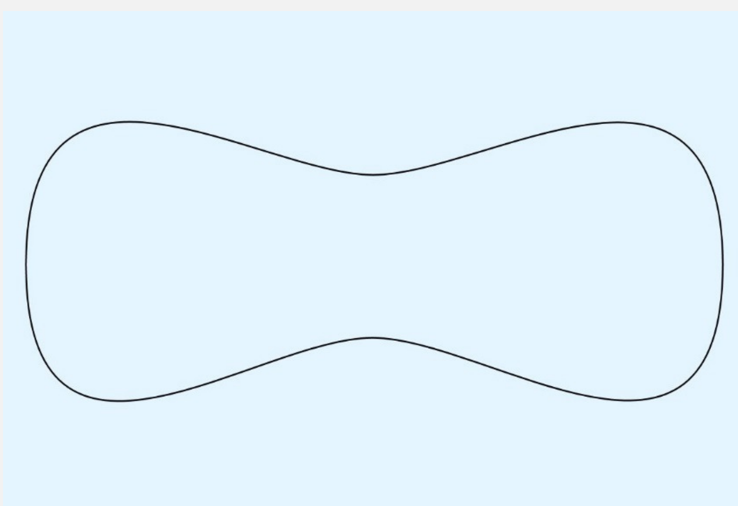
16. $\{ (x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2), x - 1 \}$

18. $\{ x^5 - 10x^3y^2 + 5xy^4 - 1, 5x^4y - 10x^2y^3 + y^5 \}$

17. $\{ x^2 + z^2 - 1, y^2 + z^2 - 1 \}$

19. $\{ y^2 + zx^2 + z(1 - z)^2 - z \}$

20. $\{ x^2 + y^2 - z^2(1 - z^2) \}$



Can you specify a finite set of polynomials whose common zeroes look like this figure?

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 15 - Meet 1 September 11, 2014	Mentors:	Bridget Bassi, Anthea Chung, Tiera Guinn, Hannah Larson, Jennifer Matthews, Wangui Mbuguiro, Cynthia Odu, Rediet Tesfaye, Isabel Vogt, Sibo Wang

For our icebreaker activity, members arranged themselves into a rectangular array. We then had each girl introduce herself, starting with the girl in the front right corner. After a girl introduced herself, she would pass a marker to the girl behind her and to her left. If there was no such girl, we would "wrap around" the rectangle in the sense that girls in the front row were thought of as being just behind the girls in the back row and girls in the right column were thought of as being just to the left of the girls in the left column. The girls would continue until the marker was returned to the girl in the front right corner.

To give everyone a chance to learn everyone else's name, we repeated this activity several times with rectangles of different dimensions. To spice things up, we'd add in a few mentors to the mix for some of the rounds.

Soon, a question emerged: for which rectangles will everybody in the rectangle get to introduce herself? In general, given the dimensions of the rectangle, how many girls will get to introduce themselves?

Session 15 - Meet 2 September 18, 2014	Mentors:	Bridget Bassi, Anthea Chung, Tiera Guinn, Wangui Mbuguiro, Cynthia Odu, Rediet Tesfaye, Isabel Vogt, Sibo Wang

Some members analyzed and supplied the details to Zagier's one-sentence proof that every prime number congruent to 1 modulo 4 is a sum of two perfect squares. This wonderful gem of a proof can be found in *The American Mathematical Monthly*, Volume 97, Number 2 (February, 1990), on page 144. A reprint is also available on Prof. Zagier's website.

Session 15 - Meet 3 October 2, 2014	Mentors:	Bridget Bassi, Anthea Chung, Tiera Guinn, Jennifer Matthews, Wangui Mbuguiro, Rediet Tesfaye, Isabel Vogt, Jane Wang, Sibo Wang
	Visitor:	Emily Pittore, iRobot

Emily Pittore is a robotic vision engineer at iRobot, the company that manufactures the autonomous Roomba vacuum cleaner. At Girls' Angle, Emily discussed fundamental problems in robotic vision and showed a variety of ways in which math is used to tackle them. To begin, it's important to understand that in a robot, images are rectangular arrays of tiny dots. Each dot is called a **pixel**. Typically, each pixel consists of a triple of numbers that give the amount of red, green, and blue light in that pixel. When you look at a computer monitor, you are, in fact, looking at a rectangular array of pixels, and if you look at your monitor with a magnifying glass, you will be able to see that each pixel is composed of a red, green, and blue dot. One of the challenges in the field of robotic vision is to devise algorithms that process these numerical arrays and enable the robot to extract useful information that it can use to accomplish its tasks.

For example, Emily showed an image of a yellow bag and a green bag. What steps can an algorithm take to be able to "see" that there are two objects and that they are both bags? In this case, the program could separate the image into yellow pixels and green pixels and then try to detect characteristics of a bag in each of the two sets of pixels separately. Grouping pixels according to similar characteristics is known as **segmentation** and recognizing objects in images is known as **identification**. In the case of the simple scene with two bags, segmentation and identification is relatively straightforward. But then Emily showed us an image of two overlapping trees. For such an image, even humans have a tough time figuring out what leaves belong to which tree and the problem of segmentation becomes enormously complicated.

Another basic image data processing problem is to identify edges and corners. Emily explained a technique for finding edges that uses **Sobel operators**. To simplify the situation, assume that the image is a grayscale image so that each pixel corresponds to a single number instead of three. The idea is to replace each pixel with a number that depends on the pixel's value and those of its neighbors. The choice of application determines how the replacement number depends on the pixel values. For example, to find vertical edges, each pixel value might be replaced by a weighted sum of neighboring values with weights given by the matrix:

$$\begin{matrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{matrix}$$

The central 0 is the weight of the original pixel and the other 8 entries are the weights for the surrounding 8 pixels where each weight is attached to the pixel in the same relative position to the original pixel as that weight is to the central weight. The idea is that if a pixel is on a vertical boundary, there will be a sharp contrast between pixels to the left and to the right. On the other hand, if the pixel is not on a vertical boundary, then the pixel values to the left and right will not differ by too much. The weights of the above matrix are designed to produce large replacement values when there is a sharp contrast across a vertical boundary and small replacement values when there is a low contrast across a vertical boundary.

Another edge-detection technique is based on the idea that if an image shifts parallel to an edge, the image won't change much, but if it shifts perpendicular to an edge, the image will change noticeably.

Emily concluded her presentation by having the girls try to find the least expensive path for a robot to take to travel back to a power station. This problem is the same as the one that Kate Jenkins addressed last year when she explained Dijkstra's algorithm (see Volume 7, Number 2 of this Bulletin).

Session 15 - Meet 4 October 9, 2014	Mentors:	Bridget Bassi, Anthea Chung, Karia Dibert, Hannah Larson, Jennifer Matthews, Rediet Tesfaye, Anu Vajapeyajula, Isabel Vogt, Jane Wang, Sibo Wang

Some members played a game called "Round and Round." In this game, players sit in a circle together with a mentor, who typically participates. The mentor describes a sequence and gives both a starting and stopping point. For example, the sequence might be: the perfect squares in order from smallest to largest, starting at 1 and stopping at 400. One of the participants then states the starting number. Going around in a circle, the next participant says the next number in the sequence. They continue going round and round until someone reaches the stopping point. If any player errs, the next player restarts from the starting point.

It's a surprisingly fun game and can be adapted to address different educational goals. It can be used to practice computation, either with or without paper and pencil. It can be used to draw attention to interesting sequences. And it can be used to induce players to discover patterns. For example, if you use perfect squares, players will often notice that the differences between successive perfect squares are the odd numbers. Then, rather than computing each perfect square using multiplication, players give themselves the option of adding the appropriate odd number to the last spoken square.

In one set of rounds, we counted up by 7 from 0 to 98, counted down by 11 from 100 to "the smallest multiple of 11 greater than -100," went through perfect squares from 0 to 400, went through perfect cubes from 0 to 1,000, went through Fibonacci numbers from the first 1 to 144, and went through powers of 2 from 1 to 1,048,576.

Session 15 - Meet 5 October 16, 2014	Mentors:	Bridget Bassi, Karia Dibert, Rediet Tesfaye, Anu Vajapeyajula, Isabel Vogt, Jane Wang, Sibo Wang

Some members played "X Box." In this game, we have a small box with an "X" written on it. The mentor places a secret number inside the box. She then gives players a clue about the number. Players try to figure out the number inside the box. When players feel certain that they have the right number, the box is opened to reveal the secret number inside. This game prepares players for the concept of the variable. The game can be adapted to a wide range of abilities by changing the clues that are given about the secret numbers.

Some members began work on a quest to open a treasure chest by cracking a secret mathematical code designed by our high school mentor intern Jennifer Matthews.

Session 15 - Meet 6 October 23, 2014	Mentors:	Bridget Bassi, Clara Chan, Anthea Chung, Karia Dibert, Alexandra Fehnel, Lydia Goldberg, Jenni Matthews, Rediet Tesfaye, Anu Vajapeyajula, Isabel Vogt, Jane Wang, Sibo Wang

Some members have been reexamining rules that they learned by rote to deepen their understanding of them. For example, some knew that $1/(a/b) = b/a$, as a rule, but did not understand why it is so. We worked on a combination of building models of division that provide insight into the rule and deriving the rule from definitions.

Some members helped Lunga Lee simplify her descriptions of functions. If you'd like to try your hand at this, see page 21.

Session 15 - Meet 7 October 30, 2014	Mentors:	Anthea Chung, Karia Dibert, Alexandra Fehnel, Lydia Goldberg, Rediet Tesfaye, Anu Vajapeyajula, Isabel Vogt, Jane Wang, Sibo Wang

Congratulations to the code breakers for cracking the code just in time for Halloween!

Calendar

Session 15: (all dates in 2014)

September	11	Start of the fifteenth session!
	18	
	25	No meet
October	2	Emily Pittore, iRobot
	9	
	16	
	23	
	30	
November	6	
	13	Cornelia A. Van Cott, University of San Francisco
	20	
	27	Thanksgiving - No meet
December	4	
	11	

Session 16: (all dates in 2015)

January	29	Start of the sixteenth session!
February	5	
	12	
	19	No meet
	26	
March	5	
	12	
	19	
	26	No meet
April	2	
	9	
	16	
	30	
May	7	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $36 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.

☐ Enclosed is $216 for one session (12 meets)

☐ I will pay on a per meet basis at $20/meet.

Nonmembers: Please choose one.

☐ I will pay on a per meet basis at $30/meet.

☐ I'm including $36 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________