Girls' Angle Bulletin

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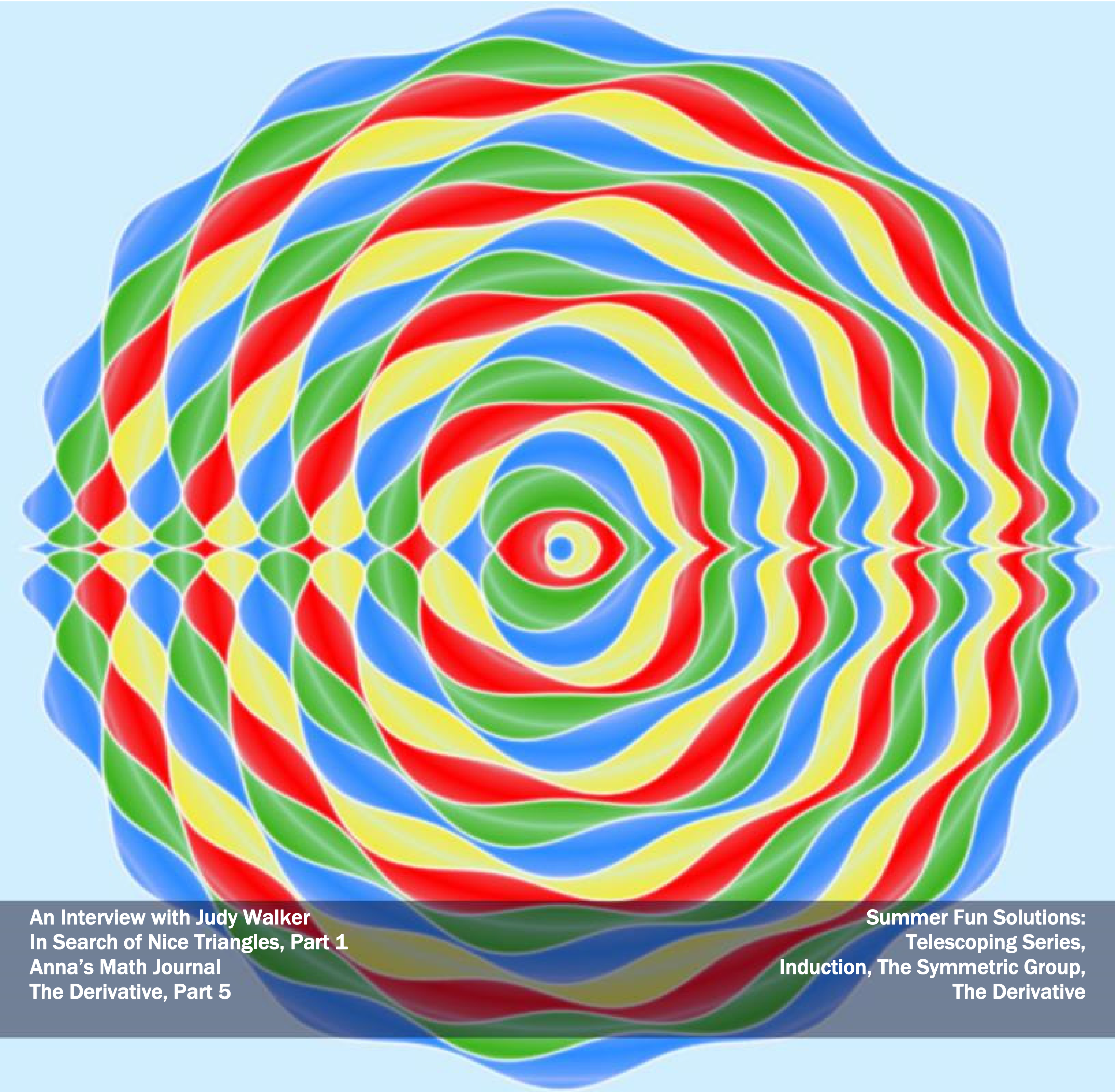
To Foster and Nurture Girls' Interest in Mathematics



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Summer Fun Solutions:
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From the Founder

There's a wide range of mathematical difficulty levels in the Bulletin. When you can work through most material in most issues, it means you're ready to handle the most advanced undergraduate math major programs in the world. If you're struggling to understand something in these pages, don't fret. You can return to it in the future or, if you're a member, you can always contact us. Don't be shy! You might even find that the mere act of writing to us helps you to resolve your own question.

- Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: This "Chebyshev lollipop" is directly based upon an idea of Michael Trott. The mathematical content differs slightly from his creation on the MathWorld website. See: http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html.

An Interview with Judy Walker

Judy Walker is Aaron Douglas Professor of Mathematics and the Chair of the Department of Mathematics at the University of Nebraska–Lincoln. Her main area of interest is Algebraic Coding Theory.

Ken: If it's okay with you, I'd like to start by jumping right into mathematics. What is coding theory? Would you please tell us a problem that illustrates some of the essential features of coding theory?

Judy: Whenever information is transmitted or stored, errors are bound to occur. Think, for example, about music being stored on an iPod, or satellite pictures being sent to earth from outer space. In many situations, the information is stored as a string of 0's and 1's, and the errors would be that some of the 0's come out as 1's and conversely. The goal of coding theory is to find efficient ways of adding redundancy to data so that these errors can be detected and even corrected. For example, you could add an "overall check bit" to the information, so that every string you transmit contains an even number of 1's. If an error occurred, you'd know, because you'd see an odd number of 1's. This scheme is efficient – we're only adding one additional bit no matter how long our initial information string is – but you don't know where the error occurred, and so you cannot fix it. At the other extreme, you could repeat your entire information string twice, so that you transmit it a total of three times. This time you can fix errors by doing a "majority vote", but the system is very inefficient since you're transmitting 3 bits for every 1 bit of information. Coding theory got its mathematical start with a 1948 landmark paper by Claude Shannon, in which he proved that reliable transmission is possible. More precisely, Shannon proved that given any channel (think: a system for transmission or storage of information), there are systems of adding redundancy that are efficient (up to a bound determined by the channel) and that correct lots of errors. His proof wasn't constructive, though, so the big problem in channel coding is to actually find good codes, i.e., codes that are efficient, correct lots of errors, and have efficient decoding algorithms; this is often referred to as Shannon's Challenge. Other related and timely problems include network coding (ensuring efficient and reliable transmission of information through a network, such as the internet), coding for flash memory (USB drives, for example), neural coding (how does the brain encode a stimulus?), and others.

Stay strong. Keep doing math. And be sure to treat mathematics as a team sport.

Ken: What do you find fascinating about coding theory?

Judy: My favorite thing about coding theory is that it centers on problems with immediate applications – problems that have real-world implications – but often very advanced mathematical techniques are needed to solve these problems. What hooked me into the subject originally was the idea of algebraic geometry codes. One way to approach Shannon's challenge is to look for collections of strings of 0's and 1's of a fixed length that don't involve a lot of redundancy and are pairwise different in a lot of coordinates. If we let n be the length of the strings, k the number of coordinates that correspond to information (rather than redundancy), and d the minimum number of coordinates in which any two distinct strings must differ, then there is a bound called the Gilbert-Varshamov bound that gives a lower bound on how large the best possible k/n

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For this issue, those who do not subscribe to the print version will be missing out on this expository article on induction. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
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President and Founder
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Girls' Angle

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In Search of Nice Triangles, Part 1

by Ken Fan | edited by Jennifer Silva

Emily and Jasmine are walking home from school.

Emily: What's up? You haven't said a word since school let out.

Jasmine: Oh, I'm just thinking.

Emily: About what?

Jasmine: Triangles.

Emily: Triangles?

Jasmine: I walked by a geometry class this morning. The teacher had a strange triangle on the board. It had side lengths of 10, 17, and 20 …

Emily: What's strange about that?

Jasmine: … and angle measurements of 30, 60, and 90 degrees.

Emily: Oh! Does such a triangle actually exist?

Jasmine: Nope! A 30-60-90 right triangle with hypotenuse of length 20 should have legs of length 20 sin 30° and 20 cos 30°. It's true that sin 30° is 1/2, so 20 sin 30° is 10. But cos 30° is $\sqrt{3}/2$, so 20 cos 30° is $10\sqrt{3}$, and that can't be equal to 17 since $\sqrt{3}$ is irrational.

Emily: Yes, I agree. If 20 cos 30° = 17, then $\sqrt{3}$ would be 17/10, which is incorrect. Did you tell the teacher?

Jasmine: I went back, but the class was out and the board had been erased. But I began to wonder about which triangles *do* have integer side lengths and nice angle measures.

Emily: What do you consider to be a "nice" angle measure?

Jasmine: That's what I've been thinking about. I was thinking an integer number of degrees, but that doesn't feel right to me because I don't think the degree is a particularly special angle measurement.

Emily: I agree. Degrees are like the choice of 10 as the base of our number system. Arbitrary.

Jasmine: Exactly.

Emily: Some people use grads, which are 1/400 of a full circle, and mathematicians often use radians.

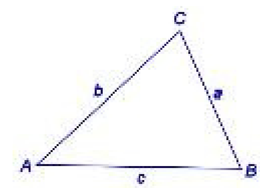
Jasmine: I did think of declaring angles with integer radian measure to be nice, but then the 90° angle wouldn't be considered nice since 90° is $\pi/2$ radians, and I know π is irrational. It doesn't seem reasonable to me to have a definition of "nice angle" that excludes the right angle!

Emily: Hmm. I think the right angle and other angles I like, such as the 30°, 45°, and 60° angles, feel nice because a whole number of each will fill out a full circle. They're the angles you get when you cut a pie into equal slices. Since there are 2π radians in a full circle, perhaps a good definition for nice angle is one whose radian measure is $2\pi/n$ for some whole number n.

Jasmine: I think we should also include angles that are built up from nice angles: an angle that is formed by putting two nice angles together would also be nice.

Emily: Okay. In that case, a nice angle is an angle whose measure, in radians, is a rational multiple of π.

Jasmine: Nice! Let's go with that definition.



Emily: I feel like drawing a figure. Let's sit over there.

Emily and Jasmine find a park bench and Emily draws the figure at right.

Emily: We seek triangles whose side lengths are whole numbers and whose angles are nice, in the sense that they have angle measures that are rational multiples of π in radians.

Jasmine: Because of side-side-side congruence, as soon as side lengths are specified, the triangle is determined up to congruence. Maybe we can express the angle measures in terms of the side lengths and see what happens if we insist that those expressions are rational multiples of π.

Emily: Good idea. This feels like a job for the law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C$.

Jasmine: That means that $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$. Neat! The cosine of C is a rational expression of the side lengths. If all of the sides of a triangle are whole numbers, then the cosines of its angles must be rational.

Emily: But we're interested in knowing if the angle itself has a measure that is a rational multiple of π radians, not whether its cosine is rational.

Jasmine: You're right. Well, cos 90° is 0, so that's an example of a nice angle that has a rational cosine. And 60° works, too, because its cosine is 1/2.

Emily: The cosine of 45° is $\sqrt{2}/2$, which is irrational, so there aren't any triangles with whole number side lengths and an angle measuring 45°.

Jasmine: And the cosine of 30° is $\sqrt{3}/2$, which is also irrational, so no such triangle can have a 30° angle, either.

Emily: In general, we must answer this question: for what rational numbers r is $\cos r\pi$ rational?

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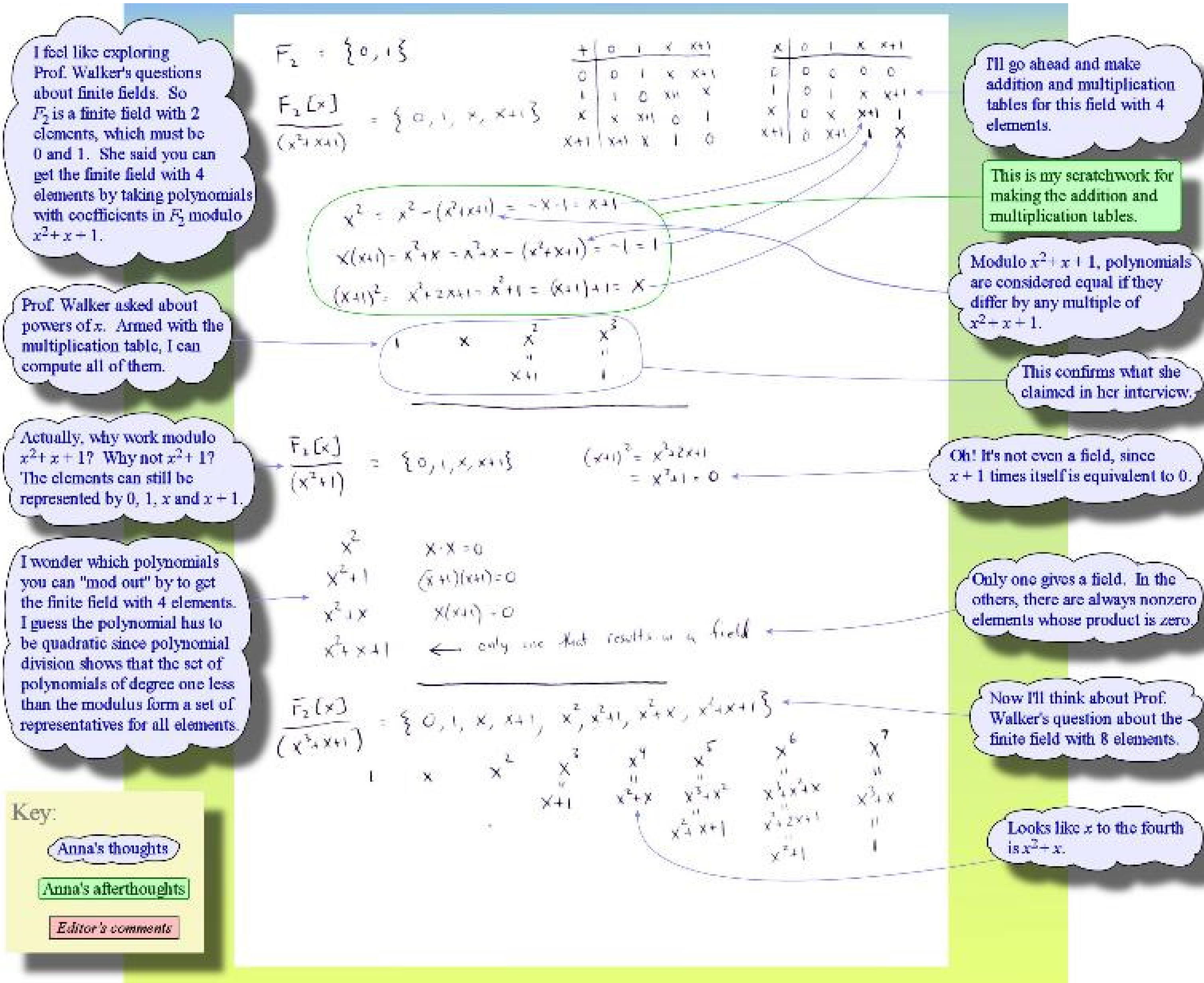
Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna plays with Prof. Walker's finite field questions from her interview (see page 5).



Prof. Walker also suggested defining a field with 8 elements by working modulo y^3+y^2+1.

$$\frac{F_2[y]}{(y^3+y^2+1)} = \{0, 1, y, y+1, y^2, y^2+1, y^2+y, y^2+y+1\}$$

Here are the powers of y. This time, y to the sixth power is equivalent to y^2+y.

$$1 \quad y \quad y^2 \quad y^3 = y^2+1 \quad y^4 = y^3+y = y^2+y+1 \quad y^5 = y^3+y^2+y = y+1 \quad y^6 = y^2+y \quad y^7 = y^3+y^2 = 1$$

Again, I'm curious to know what other polynomials I can "mod out" by to get the field with 8 elements. Having nonzero elements that multiply to something equivalent to zero means that the modulus factors nontrivially. For a degree 3 polynomial that factors, one of the factors must be linear. So that means I just have to find cubics that do not have 0 or 1 as roots. So the constant term must be 1, and the coefficients must add to 1 in F_2. It seems there's only 1 other possibility: y^3+y+1.

$$y^3+1+y, \quad y^3+1+y^2$$

If these two constructions both give rise to a finite field with 8 elements, there must be a way to match up the elements so that the addition and multiplication are compatible. Maybe I can just pair x with y.

A bijective map from one field to another that is compatible with addition and multiplication is called an isomorphism.

$$\frac{F_2[x]}{(x^3+x+1)} \qquad \frac{F_2[y]}{(y^3+y^2+1)}$$

$0 \leftrightarrow 0$
$1 \leftrightarrow 1$
$x \leftrightarrow y$
$x+1 \leftrightarrow y+1$
$x^2 \leftrightarrow y^2$
$x^2+1 \leftrightarrow y^2+1$
$\vdots$

$x^3 = x+1$
$\downarrow \quad \downarrow x$
$y^3 = y^2+1$ ← Not compatible!

Oh no...that doesn't work since x^3 would have to go to y^3, but x^3 is $x+1$ and y^3 is y^2+1, but $x+1$ does not go to y^2+1. It would have to go to $y+1$.

Well, 0 must be paired with 0 and 1 must be paired with 1. Since the other elements are powers of y, x would have to be paired with some power of y.

$0 \leftrightarrow 0$
$1 \leftrightarrow 1$

$x \to y^k$

$x^3 = x+1 \Rightarrow y^{3k} = y^k+1$

And since $x^3 = x+1$ and x^3 must be paired with y^3, we must have $y^{3k} = y^k+1$. I'll make a table to find which k satisfy this.

k	y^{3k}	$y^{k}+1$
2	$y^6 = y^2+y$	y^2+1
3	$y^9 = y^2$	$y^3+1 = y^2$
4	$y^{12} = y^5 = y+1$	$y^4+1 = y^2+y$
5	$y^{15} = y$	$y^5+1 = y$
6	$y^{18} = y^4 = y^2+y+1$	$y^6+1 = y^2+y+1$

Only $k = 3$, 5, and 6 work.

I'll go ahead and make 3 tables showing each of the different pairings when $k = 3$, 5, or 6.

$\frac{F_2(x)}{(x^3+x+1)}$	$\frac{F_2[y]}{(y^3+y^2+1)}$
0	0
1	1
x	$y^3 = y^2+1$
$x+1$	y^2
x^2	y^2+y
x^2+1	y^2+y+1
x^2+x	$y+1$
x^2+x+1	y

$\frac{F_2(x)}{(x^3+x+1)}$	$\frac{F_2[y]}{(y^3+y^2+1)}$
0	0
1	1
x	$y^5 = y+1$
$x+1$	y
x^2	y^2+1
x^2+1	y^2
x^2+x	y^2+y
x^2+x+1	y^2+y+1

$\frac{F_2(x)}{(x^3+x+1)}$	$\frac{F_2[y]}{(y^3+y^2+1)}$
0	0
1	1
x	$y^6 = y^2+y$
$x+1$	y^2+y+1
x^2	$y+1$
x^2+1	y
x^2+x	y^2+1
x^2+x+1	y^2

$(y^2+1)^2 = y^4+2y^2+1 = y^2+y$

$(y^2+y)^2 = y^4+y^2 = y+1$

This is scratchwork.

I'm not sure what to make of all these facts. I'll have to think on it.

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

ABB 8.26.15

The Derivative, Part 5

by Ken Fan | edited by Jennifer Silva

Calculus is a subject well covered in textbooks. Instead of giving another textbook treatment of the subject, we aim to provide an illuminating, though non-rigorous, explanation.

In this installment, we find the derivative of the exponential function.

The exponential function

Let $f(x) = a^x$, where a is a positive real number. I'll assume that you are familiar with the definition of this function and are comfortable with its basic properties, such as $f(0) = 1$ and $f(x + y) = f(x)f(y)$ for all real numbers x and y.

Fix a real number t.

We know that $f(x + t) = f(x)f(t) = a^t f(x)$. I want to highlight this fact, so here it is again:

$$f(x + t) = a^t f(x).$$

We can interpret this identity as saying that the graph of the exponential near $x = t$ looks exactly like a vertically-scaled version of the graph of the exponential near $x = 0$. For instance, suppose you want to know the shape of the graph of the exponential function over values of x between $t - 1$ and $t + 1$. What you can do is take the graph of the exponential function between $x = -1$ and $x = 1$ and stretch it vertically by a factor of a^t.

This observation tells us a lot about the derivative of the exponential function. Vertically stretching a graph changes its derivative by the same factor (that is, the derivative of $cg(x)$ is $cg'(x)$). So we must have $f'(t) = a^t f'(0)$.

In other words, as soon as we know the derivative of the exponential function a^x at $x = 0$, we know the derivative of the function everywhere!

(Note that the equation $f'(t) = a^t f'(0)$ can also be deduced blindly by differentiating both sides of the highlighted fact above with respect to x.)

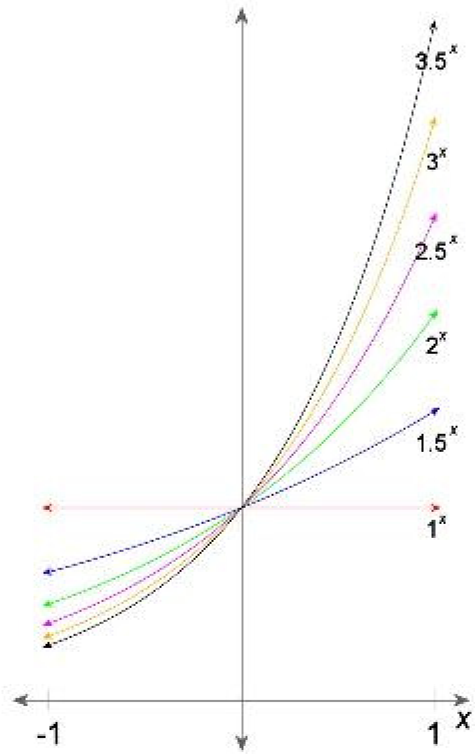
The derivative of a^x at $x = 0$

What is $f'(0)$?

The figure on the next page shows graphs of the exponential function a^x for various values of a and for values of x between -1 and 1.

When $a = 1$, the function a^x is simply the constant function 1 whose derivative is 0. Evidently, as a increases, the slope of the tangent line at (0, 1) increases. Although we haven't proven it, as a tends to infinity, the slope of the tangent line at (0, 1) also tends to infinity.

Apparently, there's a special value of a where the derivative of a^x at $x = 0$ is *precisely* 1. This special value of a is denoted e and is called **Euler's number**.



Thus, by definition, if $f(x) = e^x$, then $f'(x) = e^x f'(0) = e^x$. The exponential function e^x is equal to its own derivative.

The natural logarithm

Because e^x is strictly increasing, it has an inverse function. We denote this inverse function by $\ln x$. The function ln is called the **natural logarithm**.

By definition, $e^{\ln x} = x$ and $\ln(e^x) = x$.

Let $f(x) = a^x$. Then $f(x) = a^x = e^{x \ln a}$. Using the chain rule, we compute that $f'(x) = (\ln a)e^{x \ln a} = (\ln a)a^x$. So now we know that the derivative of a^x at $x = 0$ is $\ln a$.

The derivative of ln *x*

In the last installment of this series, we saw how to express the derivative of an inverse function in terms of the derivative of the function by using the chain rule. Let's apply this technique to find the derivative of the natural logarithm.

We know that $e^{\ln x} = x$. If we differentiate both sides of this equation with respect to x, we find

$$\ln'(x)\, e^{\ln x} = 1.$$

Rearranging terms, we see that the derivative of $\ln x$, with respect to x, is $1/x$.

Properties of ln *x*

The basic properties of the natural logarithm can be derived from its definition as the inverse function to e^x. For example, since $e^0 = 1$, we must have $\ln 1 = 0$. Also, the property

$$(*) \qquad \ln(xy) = \ln x + \ln y$$

is equivalent to the exponential property $e^{x+y} = e^x e^y$.

Just for fun, let's find (*) with differentiation. Let's differentiate $(ab)^x = a^x b^x$ in two different ways, directly and using the product rule. We get $\ln(ab)(ab)^x = \ln(a)a^x b^x + \ln(b)a^x b^x$. Since $(ab)^x = a^x b^x$ is never equal to 0 we can divide by it and find that $\ln(ab) = \ln a + \ln b$.

What is the numerical value of *e*?

We'll leave you with this challenge: without looking it up, can you think of a good way to find a numerical approximation to the value of e?

Summer Fun!

In the last issue, we presented the 2015 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil! Move your mind! You might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Please refer to the previous issue for the problems.

Members and Subscribers*: Don't forget that you are more than welcome to email us with your questions and solutions!*

Summer Fun!

Telescoping Series

by Girls' Angle Staff

1. If we substitute $b_k - b_{k+1}$ for a_k, we find

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots + (b_n - b_{n+1}) \\ &= b_1 - b_2 + b_2 - b_3 + b_3 - b_4 + b_4 - \ldots - b_n + b_n - b_{n+1} \\ &= b_1 - b_{n+1}. \end{aligned}$$

Problems 2-5 are about the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots,$$

i.e. the series associated to the sequence defined by $a_k = \dfrac{1}{1+2+3+\ldots+k} = \dfrac{2}{k(k+1)}$.

2. Let $b_k = 2/k$. We compute $b_k - b_{k+1} = \dfrac{2}{k} - \dfrac{2}{k+1} = \dfrac{2(k+1)-2k}{k(k+1)} = \dfrac{2}{k(k+1)} = a_k$, as desired.

3. By problem 2, $a_k = b_k - b_{k+1}$, so the series telescopes.

4. Using problem 1, the nth partial sum is $b_1 - b_{n+1} = 2 - \dfrac{2}{n+1} = \dfrac{2n}{n+1}$.

5. As n tends to infinity, $b_1 - b_{n+1} = \dfrac{2n}{n+1}$ tends to 2, hence the sum of the infinite series is 2.

For Problems 6-14, keep in mind our convention that $a_k = b_k - b_{k+1}$, where the a_k are the terms of the series. It is also common for people to define a sequence c_k such that $a_k = c_{k+1} - c_k$. If you did that, your "b_k" sequence will be the negative of the ones that appear here. Also, because b_k is defined by a difference equation, answers can also differ by a constant.

6. $b_k = -(-1)^k/k$. (To verify, compute $b_k - b_{k+1} = -(-1)^k/k + (-1)^{k+1}/(k+1) = (-1)^{k+1}\dfrac{2k+1}{k(k+1)}$.)

We have $\dfrac{3}{1\cdot 2} - \dfrac{5}{2\cdot 3} + \ldots + (-1)^{n+1}\dfrac{2n+1}{n(n+1)} = 1 - (-1)^n/(n+1)$. The series converges to $b_1 = 1$.

7. $b_k = \dfrac{1}{2k(k+1)}$. We have $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \ldots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$.
The series converges to $b_1 = 1/4$.

8. $b_k = \dfrac{1}{3k(k+1)(k+2)}$. We have $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{n(n^2+6n+11)}{18(n+1)(n+2)(n+3)}$.
The series converges to $b_1 = 1/18$.

9. $b_k = -\sqrt{k}$. We have $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k+1}} = \sqrt{n+1} - 1$. The series does not converge.

10. $b_k = -k!$. We have $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$.
The series does not converge.

11. $b_k = -\dfrac{k(k-1)(2k-1)}{6}$. We have $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
The series does not converge.

12. $b_k = \dfrac{\cos(2k-1)}{2\sin 1}$. We have $\sin 2 + \sin 4 + \sin 6 + \ldots + \sin(2n) = \dfrac{\cos 1 - \cos(2n+1)}{2\sin 1}$.
The series does not converge.

13. $b_k = 1/k!$. We have $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$. The series converges to 1.

14. $b_k = -\arctan(2k - 1)$. We have $\displaystyle\sum_{k=1}^{n} \arctan\frac{1}{2k^2} = \arctan\frac{n}{n+1}$. The series converges to $\pi/4$.

15. The terms a_k satisfy $a_k = c_k - c_{k+1}$ where $c_k = b_k - b_{k+1}$. Therefore the nth partial sum $\displaystyle\sum_{k=1}^{n} a_k$ is equal to $c_1 - c_{n+1} = b_1 - b_2 - b_{n+1} + b_{n+2}$.

(Note: There was no Problem 16 in this Summer Fun problem set!)

17. Instead of following the hint, we look for a pattern in the answers to Problems 7 and 8.

The "b_k" for Problem 7 is $\dfrac{1}{2k(k+1)}$. The "b_k" for Problem 8 is $\dfrac{1}{3k(k+1)(k+2)}$.

These suggest that if we let $b_k = \dfrac{1}{nk(k+1)(k+2)\cdots(k+(n-1))}$, then $a_k = b_k - b_{k+1}$, and, indeed, this is true (please check it!).

Thus, the series converges to $b_1 = \dfrac{1}{n \cdot n!}$.

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Induction

by Girls' Angle Staff

1. A. The first odd number is 1 and its sum is 1. Since $1 = 1^2$, the base case is established.

B. Fix some positive integer N. Assume that $1 + 3 + 5 + \ldots + (2N - 1) = N^2$. We will show that

$$1 + 3 + 5 + \ldots + (2N - 1) + (2N + 1) = (N + 1)^2.$$

Add $2N + 1$ to both sides of the equation $1 + 3 + 5 + \ldots + (2N - 1) = N^2$ to get

$$1 + 3 + 5 + \ldots + (2N - 1) + (2N + 1) = N^2 + (2N + 1).$$

Now observe that $N^2 + (2N + 1) = (N + 1)^2$, as desired.

2. Here's a table of the first few positive perfect cubes and their sums:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**
***n*th positive perfect cube**	1	8	27	64	125	216	343
sum of first *n* positive perfect cubes	1	9	36	100	225	441	784

Notice that the numbers in the bottom row are the squares of 1, 3, 6, 10, 15, 21, and 28, and this sequence is the sum of the first n positive integers, which is given by the formula $n(n + 1)/2$. So we conjecture that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (n(n + 1)/2)^2$.

The base case, when $n = 1$, is $1^3 = (1(1+1)/2)^2 = (1)^2$, which is true.

Fix a positive integer N and assume that $1^3 + 2^3 + 3^3 + \ldots + N^3 = (N(N + 1)/2)^2$. If we add $(N + 1)^3$ to both sides, we get $1^3 + 2^3 + 3^3 + \ldots + N^3 + (N + 1)^3 = (N(N + 1)/2)^2 + (N + 1)^3$. To establish the inductive step, we have to show that the right-hand side of this equation is equal to $((N + 1)(N + 2)/2)^2$. We compute

$$\begin{aligned}
(N(N + 1)/2)^2 + (N + 1)^3 &= (N + 1)^2(N^2/4 + N + 1) \\
&= (N + 1)^2(N^2 + 4N + 4)/4 \\
&= (N + 1)^2(N + 2)^2/4 \\
&= ((N + 1)(N + 2)/2)^2
\end{aligned}$$

as desired.

3. Let P_n be the statement that $(\sqrt{2} + 1)^n$ can be written as $a + b\sqrt{2}$, where a and b are integers. We prove P_n be induction on n. The base case P_1 is true since $\sqrt{2} + 1$ has the desired form (with $a = b = 1$). Let N be a fixed positive integer and assume that P_N is true. Thus, we can write $(\sqrt{2} + 1)^N = a + b\sqrt{2}$, where a and b are integers.

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We have $(\sqrt{2}+1)^{N+1} = (a+b\sqrt{2})(\sqrt{2}+1) = 2b + a + (a+b)\sqrt{2}$.

Since a and b are integers, so are $2b + a$ and $a + b$. Therefore P_{N+1} is also true and the inductive step is proven.

4. B. A 2 by 1 rectangle only has room for one 2 by 1 domino, so $N_1 = 1$. A 2 by 2 rectangle can be tiled in two ways: 2 horizontal or 2 vertical dominos. So $N_2 = 2$. Since $F_1 = 1$ and $F_2 = 2$ (as defined in the problem statement), we've established the base cases $N_1 = F_1$ and $N_2 = F_2$.

C. Let $k > 1$ and consider a 2 by $k + 1$ rectangle tiled by 2 by 1 dominos. Look at its right end. There are two possibilities. Either the right side of the rectangle is the side of a single, vertically oriented domino or it is the union of the widths of two horizontally oriented, stacked dominos. Thus, the set of tilings of a 2 by $k + 1$ rectangle can be split into two disjoint subsets depending on whether the right edge is the side of a single vertical domino (call it subset S_1), or of two stacked horizontal dominos (call it subset S_2).

The subset S_1 contains exactly N_k tilings because every tiling of a 2 by k rectangle can be extended to a different tiling of a 2 by $k + 1$ rectangle by attaching a single vertically oriented domino to its right end, and every tiling in S_1 can be so obtained.

The subset S_2 contains exactly N_{k-1} tilings because every tiling of a 2 by $k - 1$ rectangle can be extended to a different tiling of a 2 by $k + 1$ rectangle by attaching two stacked, horizontally oriented dominos to its right end, and every tiling in S_2 can be so obtained.

Therefore, $N_{k+1} = N_k + N_{k-1}$ for $k > 1$.

Since N_k satisfies the same recurrence relation as that which defines the Fibonacci sequence and the first two terms of N_k start off just like the Fibonacci sequence, we conclude, by induction, that $N_k = F_k$ for all $k > 0$.

5. Hint: Consider counting the number of subsets that contain only odd numbers instead.

6. A. When x_1 and x_2 are real numbers, $(x_1 - x_2)^2 \geq 0$ with equality if and only if $x_1 = x_2$. Expanding, we get $x_1^2 - 2x_1x_2 + x_2^2 \geq 0$. Now add $4x_1x_2$ to both sides: $x_1^2 + 2x_1x_2 + x_2^2 \geq 4x_1x_2$. Now take the principle square root of both sides: $x_1 + x_2 \geq 2\sqrt{x_1x_2}$. Divide both sides by 2 and see that P_2 is true.

B.
$$\begin{aligned}
\frac{x_1+x_2+x_3+\ldots+x_{2n}}{2n} &= \frac{x_1+x_2+x_3+\ldots+x_n}{2n} + \frac{x_{n+1}+x_{n+2}+x_{n+3}+\ldots+x_{2n}}{2n} \\
&= \frac{1}{2}\left(\frac{x_1+x_2+x_3+\ldots+x_n}{n} + \frac{x_{n+1}+x_{n+2}+x_{n+3}+\ldots+x_{2n}}{n}\right) \\
&\geq \frac{1}{2}\left(\sqrt[n]{x_1x_2x_3\cdots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}\cdots x_{2n}}\right) \\
&\geq \sqrt[2n]{x_1x_2x_3\cdots x_{2n}}
\end{aligned}$$

where there is equality in the first inequality if and only if $x_1 = x_2 = x_3 = \ldots = x_n$ and $x_{n+1} = x_{n+2} = x_{n+3} = \ldots = x_{2n}$. Therefore, there can be equality throughout only if the first n of the x_k are equal, say to x, and the last n of the x_k

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are equal, say to y, and, using P_2 in the last inequality, also $x = y$, which means that all the x_k are equal.

Hence, P_2 and P_n imply P_{2n}.

C. Given $n - 1$ positive real numbers $x_1, x_2, x_3, \ldots, x_{n-1}$, let a be their arithmetic mean. We compute

$$\begin{aligned} a = \frac{x_1 + x_2 + x_3 + \ldots + x_{n-1}}{n-1} &= \frac{n(x_1 + x_2 + x_3 + \ldots + x_{n-1})}{n(n-1)} \\ &= \frac{n(x_1 + x_2 + x_3 + \ldots + x_{n-1})/(n-1)}{n} \\ &= \frac{x_1 + x_2 + x_3 + \ldots + x_{n-1} + (x_1 + x_2 + x_3 + \ldots + x_{n-1})/(n-1)}{n} \\ &= \frac{x_1 + x_2 + x_3 + \ldots + x_{n-1} + a}{n} \\ &\geq \sqrt[n]{x_1 x_2 x_3 \cdots x_{n-1} a}\ . \end{aligned}$$

Equality holds if and only if all the x_k are equal (note in this case, $a = x_k$, for all k).

If we now divide throughout by $\sqrt[n]{a}$, we get $a^{(n-1)/n} \geq \sqrt[n]{x_1 x_2 x_3 \cdots x_{n-1}}$. Raising both sides to the $n/(n - 1)$ power yields P_{n-1}.

D. The inductive step in part C can be used to deduce P_n once it is known that P_N is true for any $N > n$. Using the induction step proved in part B, we can get P_n for all n of the form 2^k. Since 2^k increases without bound as k tends to infinity, for any n we can find k so that $n < 2^k$.

7. (Emily and Jasmine) Let $T_n(x)$ be the Chebyshev polynomials of the first kind. These are defined recursively as follows: $T_0(x) = 1$, $T_1(x) = x$, and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n > 1$.

A. We'll not only show that the leading coefficient of $T_n(x)$ is 2^{n-1}, but also that the degree of $T_n(x)$ is n for all positive integers n. Since $T_1(x) = x$, our claim is true for $n = 1$.

Using the recursive formula, we find that $T_2(x) = 2x(x) - 1 = 2x^2 - 1$, so, by inspection, our claim is also true for $n = 2$.

For the inductive step, we show that our claim is true for $n = N + 1$ if we assume it is true for $0 < n \leq N$, where $N > 1$. By definition, $T_{N+1}(x) = 2xT_N(x) + T_{N-1}(x)$. The degree of $2xT_N(x)$ is one more than the degree of $T_N(x)$, which, by our inductive hypothesis, is N, so the degree of $2xT_N(x)$ is $N + 1$. The degree of $T_{N-1}(x)$, also by our inductive hypothesis, is $N - 1$. We deduce that the degree of $T_{N+1}(x)$ must be $N + 1$. Its leading coefficient is twice that of the leading coefficient of $T_N(x)$, which, by our induction hypothesis is 2^{N-1}. Hence, the leading coefficient of $T_{N+1}(x)$ is 2^N. This proves the inductive step.

B. We claim that the constant terms of $T_n(x)$ cycle through the sequence 1, 0, -1, 0, starting with 1.. Since the constant terms of $T_0(x)$ and $T_1(x)$ are 1 and 0, respectively, our claim is true for n = 0 and 1. The recursive formula $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ shows that the constant term of $T_n(x)$ is the negative of the constant term of $T_{n-2}(x)$.
By induction, it follows that the constant terms of $T_n(x)$ for even n are 0 since -0 = 0 and

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the constant terms of $T_n(x)$ for odd n alternate between 1 and -1.

C. We claim that $\cos nx = T_n(\cos x)$ for all nonnegative integers n and prove this by induction on n. When $n = 0$, the formula is $\cos 0 = T_0(\cos x)$. Since $\cos 0 = 1$ and $T_0(x) = 1$, this is true. When $n = 1$, the formula is $\cos x = T_1(\cos x)$. Since $T_1(x) = x$, this is also true. Hence, our claim is true for $n = 0$ and $n = 1$.

Now fix a positive integer $N > 0$. Assume that our claim is true for all $0 \le n \le N$. We shall prove that our claim is also true for $n = N + 1$. We use the trigonometric identity $2\cos(nx)\cos(x) = \cos((n + 1)x) + \cos((n - 1)x)$. By our induction hypothesis, we can rewrite this as $2\,T_n(\cos x)\cos x = \cos((n + 1)x) + T_{n-1}(\cos x)$. Rearranging terms, this is equivalent to $\cos((n + 1)x) = 2\,T_n(\cos x)\cos x - T_{n-1}(\cos x)$. By definition of the Chebyshev polynomials of the first kind, $2\,T_n(\cos x)\cos x - T_{n-1}(\cos x) = T_{n+1}(\cos x)$. Hence $\cos((n + 1)x) = T_{n+1}(\cos x)$, as desired.

D. We claim that $T_n(x) - 1 = 2^{n-1}\prod_{k=0}^{n-1}\left(x - \cos\left(\frac{2\pi k}{n}\right)\right)$. By substituting $x = 2\pi k/n$ into the formula $\cos nx = T_n(\cos x)$, we see that $\cos 2\pi k/n$ is a root of the polynomial $T_n(x) - 1$. This gives us $\lfloor (n+1)/2 \rfloor + 1$ distinct roots by taking $0 \le k \le \lfloor (n+1)/2 \rfloor$. (For integers k outside this range, we don't get new values of $\cos 2\pi k/n$.) However, when $0 < x = 2\pi k/n < \pi$, the formula $\cos nx = T_n(\cos x)$ shows $T_n(y) \le 1$ for values of y near to and surrounding $\cos x$. Therefore, for such x, $\cos x$ is a multiple root of $T_n(x) - 1$. Because the degree of $T_n(x)$ is n, a count of the roots with multiplicity shows that each multiple root has multiplicity exactly 2. Our claim follows.

Vieta's formulas tell us that the product of the roots of $T_n(x)$ is equal to $(-1/2)^n$ times the constant term of $T_n(x)$. Using part B, we conclude that $\prod_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right)$ is 0 if n is 0, modulo 4, $1/2^{n-1}$ if n is 1 or 3, modulo 4, and $-1/2^{n-2}$ if n is 2, modulo 4.

8. From Problem 3, we know that $(\sqrt{2} + 1)^n = a_n + b_n\sqrt{2}$, where a_n and b_n are integers. We shall prove by induction on n that $a_n^2 - 2b_n^2 = \pm 1$. The result would follow.

For the base case, note that $a_1 = b_1 = 1$, so $a_1^2 - 2b_1^2 = 1 - 2 = -1$.

Let N be a positive integer and assume that $a_N^2 - 2b_N^2 = \pm 1$.

Note that $a_{N+1} + b_{N+1}\sqrt{2} = (\sqrt{2} + 1)^{N+1} = (a_N + b_N\sqrt{2})(\sqrt{2} + 1) = a_N + 2b_N + (a_N + b_N)\sqrt{2}$. Thus, $a_{N+1} = a_N + 2b_N$ and $b_{N+1} = a_N + b_N$.

We compute

$$\begin{aligned} a_{N+1}^2 - 2b_{N+1}^2 &= (a_N + 2b_N)^2 - 2(a_N + b_N)^2 \\ &= a_N^2 + 4a_Nb_N + 4b_N^2 - 2a_N^2 - 4a_Nb_N - 2b_N^2 \\ &= -a_N^2 + 2b_N^2 \\ &= -(a_N^2 - 2b_N^2) \end{aligned}$$

Since $a_N^2 - 2b_N^2 = \pm 1$, we see that $a_{N+1}^2 - 2b_{N+1}^2 = \pm 1$, as desired.

Can you show that $a_{n+1} = 2a_n + a_{n-1}$ for $n > 0$?

The Symmetric Group

by Girls' Angle Staff

1. $n!$. The 24 elements of S_4 are 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, and 4321.

2. The map pq also maps $\{1, 2, 3, ..., n\}$ to itself. If $p(q(k)) = p(q(j))$, then $q(k) = q(j)$, since p is one-to-one and, hence, $k = j$, since q is also one-to-one. Therefore pq is one-to-one and since $\{1, 2, 3, ..., n\}$ is finite, pq must also be onto, hence it is a permutation in S_n.

3. $p(1(k)) = p(k)$ and $1(p(k)) = p(k)$.

4. $((pq)r)(k) = (pq)(r(k)) = p(q(r(k)))$. $(p(qr))(k) = p((qr)(k)) = p(q(r(k)))$. Hence, $(pq)r = p(qr)$.

5. Let $1 \le k \le n$. Since p is one-to-one and onto, there exists a unique j such that $p(j) = k$. Define $u(k) = j$. If $u(x) = u(y)$, then $x = p(u(x)) = p(u(y)) = y$. Hence, u is one-to-one and since u is defined on a finite set, it is also onto.

6. Suppose $pp_1 = pp_2$. Then $p^{-1}pp_1 = p^{-1}pp_2$, which shows that $p_1 = p_2$. For Cayley graphs of S_n, this implies that no two edges of the same color will point to the same permutation.

7. If $p = 1$, there is nothing to show, so assume $p \ne 1$. Since H is closed under composition, p^k is in H for all positive integers k. Since there are a finite number of permutations, there must be $k < j$ with $p^k = p^j$. Using Problem 6, this means that $1 = p^{j-k}$. Since $p \ne 1$, we in fact have $j - k > 1$. Hence, 1 and $p^{-1} = p^{j-k-1}$ are in H.

8. (1623)(57).

9. $H = \{1, (123), (132)\}$. H is a proper subgroup with 3 elements.

10. Observe that $(a_1a_2a_3...a_k) = (a_1a_2)(a_2a_3)(a_3a_4) \cdots (a_{k-1}a_k)$, so it suffice to show that every transposition (ab), with $a < b$, is a product of transpositions of the form $(k\ k+1)$. Now observe that

$$(a\ b) = (a\ a+1)(a+1\ a+2)(a+2\ a+3) \cdots (b-1\ b)(b-2\ b-1)(b-3\ b-2) \cdots (a\ \ a+1).$$

11. Observe that $(123...n)^k(12)(123...n)^{n-k} = (k+1\ k+2)$. Since we can get all transpositions of the form $(a\ a+1)$, using Problem 10, we can get any permutation.

12. Hint: Define a map σ from S_n to $\{-1, 1\}$ by setting $\sigma(p) = (-1)^N$ where p is in S_n and N is equal to the number of pairs (i, j) with $i < j$ such that $p(i) > p(j)$. Show that $\sigma(pq) = \sigma(p)\sigma(q)$. Show that A_n consists of permutations p in S_n such that $\sigma(p) = 1$.

13. $H = \{1, (12)(34), (13)(24), (14)(23)\}$.
$H(12) = \{(12), (34), (1423), (1324)\}$.
$H(234) = \{(234), (124), (132), (143)\}$.
$H(1324) = \{(1324), (1423), (34), (12)\}$.

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These right cosets all contain 4 elements and they are either disjoint or equal to each other.

14. If p is in H, then Hp is a subset of H since H is closed under composition. If $xp = yp$, then $x = y$ so Hp has the same size as H. Hence $H = Hp$. Since 1 is in H, p is in Hp. So if p is not in H, we cannot have $Hp = H$.

15. Suppose x is in Hp_1 and Hp_2, say $x = hp_1 = h'p_2$. Then $p_2 = h'^{-1}hp_1$. Since right multiplication by $h'^{-1}h$ permutes H, we see that $Hp_2 = Hh'^{-1}hp_1 = Hp_1$. For any x in S_n, x is contained in Hx.

16. (For the size, see the solution to Problem 14.) Since all right cosets of H have the same size and partition S_n, the number of elements in S_n is equal to the number of elements in H times the number of right cosets of H, i.e., $|S_n| = |H|\,[S_n : H]$.

19. Observe that $m_1(x) = 1(x) = x$. Also, $m_p(m_q(x)) = m_p(q(x)) = p(q(x)) = (pq)(x) = m_{pq}(x)$.

20. Since $m_p(x) = y$, $m_{p^{-1}}(m_p(x)) = m_{p^{-1}}(y)$. But, $m_{p^{-1}}(m_p(x)) = m_{p^{-1}p}(x) = m_1(x) = x$. Thus, $m_{p^{-1}}(y) = x$.

21. Since $1x = x$, 1 is in Stab(x). Suppose p and q are in Stab(x). Then $p(x) = x$ and $q(x) = x$. Hence, $(pq)(x) = p(q(x)) = p(x) = x$, so pq is also in Stab(x). Hence, Stab x is a subgroup of S_n. Note that if p and q are in the same left coset r Stab x of Stab x, then $p(x) = q(x)$ (and $p(x)$ and $q(x)$ are also equal to $r(x)$). Therefore, we can define a map from the set of left cosets of Stab x into Orb(x) by sending r Stab x to $r(x)$. This map is onto and we will show that is one-to-one. Suppose $p(x) = q(x)$. Then $q^{-1}p$ is in Stab x. Hence $q^{-1}p$ Stab x = Stab x, so p Stab $x = q$ Stab x. We conclude that $|\text{Orb}(x)| = [S_n : \text{Stab}(x)]$.

22. A. First, $1(T) = \{1(t) \mid t \in T\} = \{t \mid t \in T\} = T$. Second,

$$p(q(T)) = p(\{q(t) \mid t \in T\}) = \{p(q(t)) \mid t \in T\} = \{(pq)(t)) \mid t \in T\} = (pq)(T).$$

B. Let p be in Stab(T). Then p permutes the elements of T. This implies that T also permutes the elements not in T. Furthermore, any permutation in S_n that permutes the elements of T is in Stab(T). Since there are $k!$ permutations of T and $(n - k)!$ permutations of the elements not in T, we conclude that $|\text{Stab}(T)| = k!(n - k)!$.

C. Let S and T in X_k. Suppose $S = \{s_1, s_2, s_3, \ldots, s_k\}$ and $T = \{t_1, t_2, t_3, \ldots, t_k\}$. Extend the sequence s_i to contain all elements of $\{1, 2, 3, \ldots, n\}$ and do the same for t_i. Let p be the permutation in S_n such that $p(s_i) = t_i$. Then $p(S) = T$. Thus, X_k is a subset of Orb(T). Finally, note that $p(T)$ has k elements for all p in S_n. Thus, X_k = Orb(T).

D. Using Problem 21, we conclude that $|X_k| = [S_n : \text{Stab}(T)] = \dfrac{n!}{k!(n-k)!}$.

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Derivatives

by Ken Fan

1. (Leibniz's rule) The nth derivative of $f(x)g(x)$ with respect to x is given by

$\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x)$. We prove this by induction on n. When $n = 1$, the formula coincides with the product rule. Suppose the formula is true for $n = N$, where N is a positive integer. Then the $(N + 1)$st derivative of $f(x)g(x)$ with respect to x is equal to the derivative, with respect to x, of $\sum_{k=0}^{N}\binom{N}{k}f^{(N-k)}(x)g^{(k)}(x)$, which is $\sum_{k=0}^{N}\binom{N}{k}(f^{(N-k+1)}(x)g^{(k)}(x)+f^{(N-k)}(x)g^{(k+1)}(x))$. After rearranging terms, this is equal to $\sum_{k=0}^{N+1}\left(\binom{N}{k}+\binom{N}{k-1}\right)f^{(N-k+1)}(x)g^{(k)}(x)$, where we adopt the convention that $\binom{N}{-1}=0$ and $\binom{N}{N+1}=0$. Using the fact that $\binom{N+1}{k}=\binom{N}{k}+\binom{N}{k-1}$, the inductive step follows.

2. When you zoom in on a point on the graph of a differentiable function, the graph looks more and more like part of a straight line, but if the graph isn't precisely a straight line, there will be a deviation from the straight line, even if it is imperceptible to the eye. When you zoom in a lot, you are looking at a very small piece of the graph. The horizontal coordinate may be confined to a teeny amount of the number line. Changes in the slope of the tangent as you traverse this snippet of the graph can be quite large *per unit length*, but since the zoomed in section covers a very small portion of the number line, the change in slopes can still be imperceptible.

For example, consider cos x. We know that the derivative of cos x, with respect to x, is –sin x. If we zoom in on the graph of cos x near the point (0, 1), the graph will look more and more like a snippet of a horizontal line. In the range $-1/100 < x < 1/100$, for instance, the value of cos x remains between 0.99995 and 1. This deviation is so small that if you make a graph of cos x over the range of values $-1/100 < x < 1/100$ at a scale where the physical distance between -1/100 and 1/100 on your horizontal axis is about the width of a standard sheet of paper, the graph itself will fit within the thickness of a horizontal line drawn with a ballpoint pen.

From -1/100 to 1/100, the slope of the tangent varies from sin(-1/100) to sin(1/100), or, approximately, from -1/100 to 1/100. That's a very small change in the slope from the left endpoint to the right endpoint of the graph. However, the *rate of change* of this slope, i.e., the value of the second derivative, is approximately (1/100 – (-1/100))/(1/100 – (-1/100)) = 1.

3. Hint: Use Vieta's formulas.

4. Suppose $p(x)$ has a multiple root at $x = r$. Then $p(x) = (x - r)^2q(x)$ where $q(x)$ is a polynomial. Using the product rule, $p'(x) = 2(x - r)q(x) + (x - r)^2q'(x)$. Thus $(x - r)$ is a factor of both $p(x)$ and $p'(x)$. Conversely, suppose that $p(x)$ and $p'(x)$ share a common factor. Then there exists r such that $(x - r)$ is a factor of both $p(x)$ and $p'(x)$.
We claim that r is a multiple root of $p(x)$. If not, then $p(x) = (x - r)q(x)$ where $(x - r)$ is not a factor

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of $q(x)$. Using the product rule, we find $p'(x) = q(x) + (x - r)q'(x)$. Hence, $p'(r) = q(r) \neq 0$, contradicting the fact that $p'(r) = 0$.

5. A complete answer to this question could be quite long. We'll instead confine ourselves to showing that when the derivative is defined as in the statement of the problem, the derivative of a sum is the sum of the derivatives.

Let f and g be two functions that map the real numbers to itself and assume that the derivative of $f(x)$ and $g(x)$ exist at $x = z$. Let m_f and m_g be the derivatives of $f(x)$ and $g(x)$ (with respect to x) at $x = z$, respectively. We will show that the derivative of $f(x) + g(x)$ exists at $x = z$ and is equal to $m_f + m_g$.

Fix $\varepsilon > 0$. We must show that there exists $\delta > 0$ such that

$$| f(z + h) + g(z + h) - ((m_f + m_g)h + f(z) + g(z)) | < \varepsilon|h|$$

for all $0 < |h| < \delta$.

By definition, since $\varepsilon/2$ is also greater than 0, there exists $\delta_f > 0$ and $\delta_g > 0$ such that

$$| f(z + h) - (m_f h + f(z))| < (\varepsilon/2)|h| \text{ for all } 0 < |h| < \delta_f$$
$$\text{and } | g(z + h) - (m_g h + g(z))| < (\varepsilon/2)|h| \text{ for all } 0 < |h| < \delta_g.$$

Let $\delta = \min(\delta_f, \delta_g)$. Then, for all $0 < |h| < \delta$, we have

$$\begin{aligned}
&| f(z + h) + g(z + h) - ((m_f + m_g)h + f(z) + g(z)) | \\
&\quad = | f(z + h) - (m_f h + f(z)) + g(z + h) - (m_g h + g(z)) | && \text{(rearranging terms)} \\
&\quad \leq | f(z + h) - (m_f h + f(z)) | + | g(z + h) - (m_g h + g(z)) | && \text{(triangle inequality)} \\
&\quad < (\varepsilon/2)|h| + (\varepsilon/2)|h| \\
&\quad = \varepsilon|h|
\end{aligned}$$

as desired.

For a complete treatment, consult a book on real analysis, such as *Introduction to Calculus and Analysis, Volume I* by Courant and John or *Principles of Mathematical Analysis* by Rudin.

6. This is not true. Two functions may change in tandem, yet be separated by a constant everywhere. For example, consider $f(x) = x$ and $g(x) = x + 1$. However, if you know the derivative everywhere *and* the value of the function at a single point, then you can recover the function. The operation of recovering a function from its derivative is called **integration**.

7. The double-angle formula for sine is $\sin(2x) = 2\sin(x)\cos(x)$. Using the chain rule, the derivative of $\sin(2x)$ with respect to x is $2\cos(2x)$. Using the product rule, the derivative of $2\sin(x)\cos(x)$ with respect to x is $2\cos(x)\cos(x) - 2\sin(x)\sin(x)$. Therefore, $\cos(2x) = \cos^2(x) - \sin^2(x)$.

Summer Fun!

The half-angle formula for sine is $\sin(x/2) = \sqrt{(1-\cos x)/2}$. Using the chain rule, the derivative of $\sin(x/2)$ with respect to x is $\cos(x/2)/2$. Using the chain rule, the derivative of $\sqrt{(1-\cos x)/2}$ with respect to x is $\dfrac{\sin(x)/2}{2\sqrt{(1-\cos x)/2}}$. Therefore,

$\cos(x/2) = \dfrac{\sin x}{\sqrt{2(1-\cos x)}}$. If we multiply the top and bottom of this last expression by $\sqrt{1+\cos x}$

and simplify, we can express this identity in the more familiar form $\cos(x/2) = \sqrt{\dfrac{1+\cos x}{2}}$.

8. Let $f(x) = \arctan x$. By definition, $f(\tan x) = x$. Differentiating both sides with respect to x, we find $f'(\tan x)/\cos^2 x = 1$, hence $f'(\tan x) = \cos^2 x$. If we substitute $\arctan y$ for x, we find that $f'(y) = \cos^2(\arctan y)$. A right triangle with legs of length 1 and y has a hypotenuse of length $\sqrt{1+y^2}$. Therefore $\cos(\arctan y) = 1/\sqrt{1+y^2}$ and $f'(y) = \dfrac{1}{1+y^2}$. Thus, $f'(0) = 1$.

Note that $\dfrac{1}{1+x^2} = i\left((x+i)^{-1} - (x-i)^{-1}\right)/2$, where i is a square root of -1.

We begin computing higher derivatives of $f(x)$ and look for a pattern:

$$f^{(2)}(x) = i(-(x+i)^{-2} + (x-i)^{-2})/2.$$
$$f^{(3)}(x) = i(2(x+i)^{-3} - 2(x-i)^{-3})/2.$$
$$f^{(4)}(x) = i(-3\cdot 2(x+i)^{-4} + 3\cdot 2(x-i)^{-4})/2.$$

Evidently, for $n > 0$, $f^{(n)}(x) = (-1)^n i(n-1)!(-(x+i)^{-n} + (x-i)^{-n})/2$. We prove this by induction. The formula is true for $n = 1$ by design. The inductive step follows from the fact that the derivative of $(-1)^n i(n-1)!(-(x+i)^{-n} + (x-i)^{-n})/2$ with respect to x is

$$(-1)^n i(n-1)!(n(x+i)^{-(n+1)} - n(x-i)^{-(n+1)})/2 = (-1)^{n+1} in!(-(x+i)^{-(n+1)} + (x-i)^{-(n+1)})/2.$$

Hence $f^{(n)}(0) = (-1)^n i(n-1)!(-i^{-n} + (-i)^{-n})/2$. It's convenient to simplify this expression according to cases that depend on the remainder n leaves upon division by 4. If n is even (i.e., if n is 0 or 2, modulo 4), this expression reduces to 0. If n is 1, modulo 4, the expression simplifies to $(n-1)!$ and if n is 3, modulo 4, the expression simplifies to $-(n-1)!$.

If you know about Taylor expansions, this computation shows that

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - x^{11}/11 + \ldots.$$

9. The equation is equivalent to $f(x)f'(x) = 1$. Notice that the derivative, with respect to x, of $f^2(x)/2$ is $f(x)f'(x)$. Hence, $d/dx\, f^2(x)/2 = 1$. Any function of the form $x + c$, where c is a constant has derivative equal to 1. Thus $f^2(x)/2 = x + c$ and $f(x) = (2(x + c))^{1/2}$. In order to be defined for all $x \geq 0$, we take $c \geq 0$.

Summer Fun!

Calendar

Session 17: (all dates in 2015)

September	17	Start of the seventeenth session!
	24	
October	1	
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	
	10	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls’ Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don’t like math, what don’t you like? If you love math, what do you love? What would you like to get out of a Girls’ Angle Membership?

____-

The $36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $36 for a 1-year Girls’ Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton Univeresity
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, The Dartmouth Institute
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, assistant professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, associate professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $36 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____