Girls' Angle Bulletin

August/September 2016 • Volume 9 • Number 6

To Foster and Nurture Girls' Interest in Mathematics

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From the Founder

As summer wraps up, here's a big welcome to all new and returning Girls' Angle members! We're looking forward to working with you this fall and can't wait to see what kinds of mathematical journeys we end up exploring together.

- Ken Fan, President and Founder

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Saturday Chocolate Bar at The Langham, Boston*, courtesy of our friends at the Langham Hotel – classic British elegance and style in a historic Boston landmark.

An Interview with Ana-Maria Castravet

Trust yourself.
Follow your intuition.

Ana-Maria Castravet is an Associate Professor of Mathematics at Northeastern University. She earned her doctoral degree in mathematics at the Massachusetts Institute of Technology.

Ken: When did becoming a mathematician become a goal for you? What turned you on to mathematics?

Ana-Maria: I liked solving mathematics problems from an early age, maybe third grade or so. I don't think I made a goal of becoming a mathematician, I just let things flow, and somehow this is where I arrived. I liked the clarity of mathematics. Given a set of data, one had to derive something else, with no ambiguity involved.

Ken: How did you learn mathematics? What are some of the best ways you found to study the subject?

Ana-Maria: With the help of my older sister, I did try to learn ahead of time the mathematics in the school manuals. I went to math competitions back in Romania. In those days mathematics seemed however just a bag of tricks. It was a revelation in my first year of undergraduate studies (at the University of Bucharest) that this isn't the case, that there are theorems connecting all the tricks. As with learning any subject, you need to trust your intuition and ask yourself questions. Try answering those questions. Don't take others' answers for granted. One learns a lot of mathematics by discussing with others – things you will most likely never learn by yourself.

Ken: What is algebraic geometry?

Ana-Maria: Algebraic geometry is the study of algebraic varieties, which are solution sets (zeroes) of polynomial equations in several variables. You study the geometry of the shape of the solution set (or if solutions even exist) using the equations, so reducing the question to algebra. To give an example, consider the equation $x - y + 1 = 0$ in the variables x and y. The solution set is a line in the xy-plane. All lines in the plane can be described by an equation of the form $ax + by + c = 0$, for some real numbers a, b, and c. So all lines are algebraic varieties. So are circles (for example, $x^2 + y^2 - 1 = 0$), ellipses (for example, $x^2/4 + y^2/9 - 1 = 0$), parabolas (for example, $y - x^2 = 0$), and hyperbolas (for example, $x^2 - y^2 - 1 = 0$). As we increase the degree and/or the number of variables, one can get many different shapes. There is also the issue whether we ask for solutions which are real, rational, or complex numbers. For example, $x^2 + y^2 + 1 = 0$ has no real solutions, but has infinitely many complex solutions.

Ken: Can you please explain a problem from algebraic geometry that is accessible to high school students?

Ana-Maria: A very important and hard problem in algebraic geometry is whether a given algebraic variety X can be parametrized algebraically, that is, if one can describe "most" solutions as polynomials, or more generally rational functions (quotients of polynomials) in some variables. Such a parametrization is desirable because polynomials are fairly simple functions and it is useful to have solutions described this way.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Ana-Maria Castravet and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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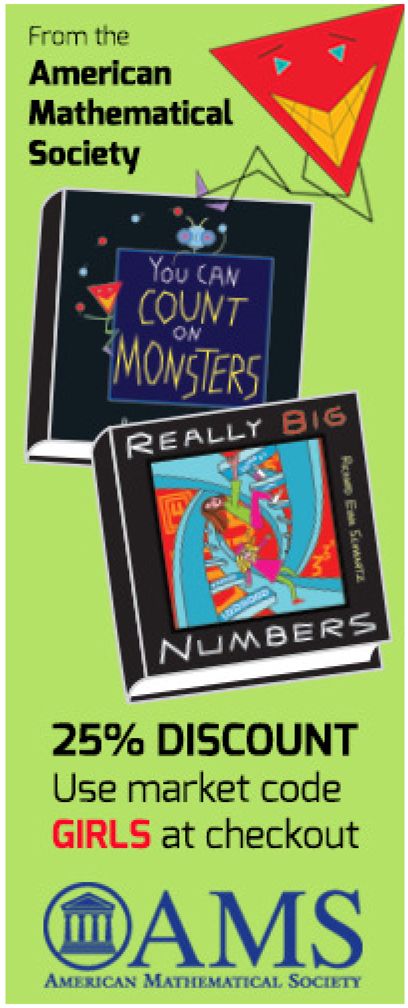

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Scaling in a Sticky Place: The Fluid Mechanics of the Chocolate Fountain, Part 1

by Helen Wilson

The first time I saw a chocolate fountain in action was at a wedding fair.[1] I was instantly fascinated by the inward-falling sheets of chocolate. Why doesn't the chocolate fall straight downwards? I even got the owner of the stall to turn the fountain off and on again so I could see the startup: the chocolate does fall inwards even before it's reached the bottom.

Photo courtesy of Adam Townsend.

A chocolate fountain.

That got me thinking about the chocolate fountain as a whole. I'm a specialist in non-Newtonian fluid mechanics – that is, the flow of fluids that are in some way not as simple as air, water, or syrup. There are many ways a material can deviate from the simple model, usually associated with something complicated going on at a microscopic scale within the fluid. Some liquids (e.g. milk) are actually an emulsion: droplets of one liquid suspended in another (in milk the suspended droplets are the fat). Others have solids suspended in them – muds, for instance (which can be considered liquids as they do flow) are a suspension of soil particles in an aqueous background liquid. And many synthetic and biological fluids – ranging from shampoo to molten plastics and even saliva – have long polymer molecules moving around within the fluid.

Dr. Helen Wilson is Professor of Applied Mathematics at the University College London.

Molten chocolate is another complex fluid. Chocolate is a mixture of cocoa solids, sugar crystals, and fats. At the temperatures found in a running chocolate fountain (around 40° C), the fats have melted but the cocoa solids and the sugar are still in their solid form, giving us a suspension of two different kinds of solid particles. The shape of the sugar particles depends on whether or not the chocolate has been tempered during manufacture, by raising it to a very high temperature and then quickly cooling it. Add to that complexity the strange fact that

[1] Wedding fairs are horrible (to my mind, at least) events where lots of different businesses assemble to try to persuade you to spend more money than you can sensibly spare to make your wedding day "the best day of your life". Avoid them if you can – unless they are your only opportunity to sample a chocolate fountain.

the behavior of the cocoa butter depends on the year of harvest, and we have rather an unpredictable non-Newtonian fluid on our hands.

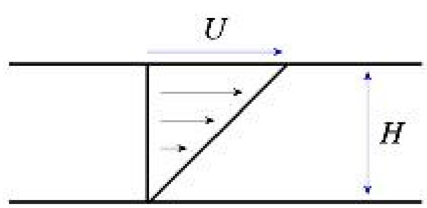
Faced with modelling a fluid this complex, it's often best to take an engineering approach and simply measure the behavior of the fluid in a simple flow, rather than trying to predict it. The property we usually try to measure is the fluid's viscosity or thickness. For a Newtonian fluid at a controlled temperature and pressure, this is a single number – here are a few of them:

Fluid	Viscosity (Pa s)
Water	10^{-3}
Olive oil	10^{-1}
Glycerol	10^{0}
Liquid honey	10^{1}
Golden syrup	10^{2}
Glass	10^{40}

Table 1: Viscosities of a range of Newtonian fluids, at a pressure of 1 atmosphere and room temperature, 20.2° C. The SI unit of viscosity is the Pascal second; 1 Pa s = 1 kg m^{-1} s^{-1}. Taken from [3].

The viscosity is an essential property of the fluid because it determines how fast energy is dissipated, during flow, by the friction within the fluid. This dissipation balances against the rate at which energy is supplied (by gravity, say, or some forcing like a propellor) to determine how fast the fluid will flow.

The standard way to measure the viscosity of a fluid is to apply a very simple flow called shear flow, which is created by sliding one plate over another with the fluid confined between them:



We can measure the force F required to drive the top plate at a given speed U. For a Newtonian fluid, we always find that the force is proportional to the velocity, $F = \alpha U$, and if we name the area of the plates as A and the shear rate $\dot{\gamma} = U / H$, we can extract the fluid's viscosity:

$$\eta = \frac{\alpha}{A} = \frac{F}{A\dot{\gamma}H}.$$

One of the most common differences between the behaviour[2] of a non-Newtonian fluid and a Newtonian fluid is that viscosity for a non-Newtonian fluid is often not a constant. Instead it depends on how fast the applied shear flow is: specifically, it depends on the shear rate $\dot{\gamma}$. Because in more complex flows, the shear rate is likely to be different in different flow regions, this can have a big effect on how the fluid flows. So knowledge of experimental behaviour of a

[2] "Behaviour" is the British spelling of "behavior". Dr. Wilson is from the United Kingdom. There are other British spellings in this article. Can you find them?

non-Newtonian fluid in shear flow is a vital ingredient in any mathematical modelling of the flow.

Luckily for us (I'm absolutely not an experimental scientist) people have done this with chocolate before. There are many papers available on chocolate rheology[3], all of which attempt to answer our key question

How does the viscosity of chocolate depend on the shear rate?

Of course, because of the differences in composition of different chocolates, and the idiosyncrasy of the harvest, these papers give many different answers to the question. I've chosen here one answer (taken from page 11 of [1]) which is my favourite simply because it is mathematically clean:

$$\eta = 65\dot{\gamma}^{-2/3} \text{ Pa s}$$

where the shear rate $\dot{\gamma}$ is measured in reciprocal seconds. As the shear rate increases, the viscosity decreases: so chocolate is a shear-thinning fluid.

What other non-Newtonian fluids are there? The range is wide, but here are a few of my favourites.

Oobleck is a new name for an old thing. If you mix 2 parts cornflour (cornstarch) with 1 part water you get something between a liquid and a solid. It's not easy to mix it smoothly (if you are actually wanting to use it to make gravy, you would add more water) but once you have, you can pour it as a liquid. But if you stir it too fast it will crack like a solid; and (great for demos) if you punch straight down into a big bowl of it you don't get wet. If you make enough of it you can run across the surface; but if you stand still you will sink. YouTube is the best place to see it in action [4]. This is a shear-thickening material: when you apply fast flow rates to it, it gets very thick indeed (to the point of behaving like a solid).

Toothpaste is an example of a material with a so-called yield stress. This means that under very light forcing (such as gravity, while it's sitting on your toothbrush waiting for some action) it doesn't flow at all. It's designed like this precisely so it doesn't run off the brush while you're putting the lid back on the tube; but once you start brushing (applying a fast shear flow) the paste flows quite easily to distribute around your teeth. It's an extreme example of a shear-thinning fluid (as the viscosity looks almost infinite when there is no flow).

Blood is also shear thinning. At very low flow rates it needs to be thick so that if you've been cut, the blood will hang around at the site of the wound long enough for clotting to start; but we need it to be thin while it's being pumped around the body otherwise the strain on the heart would be too much. Here it's the deformable blood cells that provide most of the fluid complexity.

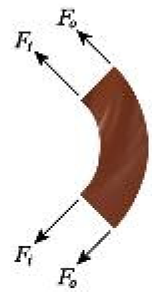
Volcanic lava is also shear-thinning; in fact its behaviour (at larger scales, higher temperatures, and larger overall viscosity) is rather similar to that of molten chocolate. In fact this is not so surprising: it has unpredictably shaped rocks suspended in it just as the chocolate has unpredictably shaped sugar crystals.

So now we have a model for the viscosity of chocolate (or indeed lava). Next time we'll see how we can use it in some mathematics.

But before I conclude Part 1, remember that question at the beginning about the sheets falling inwards? We can answer that without even using any fluid mechanics – or any equations

[3] Rheology (not a typo for theology) is the study of the deformation and flow of matter. It's a growing scientific discipline as we gradually realise that almost everything we process in industry is non-Newtonian.

– at all. It’s pure surface tension: water would (and does) do the same thing. Think of a small horizontal section through the falling sheet. The cross-section is an annulus[4] of chocolate. Now look at the surface tension forces acting on an angular segment of that:



We don’t even have to calculate here! As long as we know that the magnitudes of the two inner forces are equal, and the magnitudes of the two outer forces are equal, we can see that the vector sum of the inner forces will act leftwards and so will the vector sum of the outer forces. So all the surface tension acts to pull this segment to the left, which means this section of chocolate will move leftwards over time. Because we chose a section at the right hand side of the circle that means it will move towards the centre as time passes.

Zoom back out to the 3D sheet of falling chocolate. Each segment moves towards the centre over time. But, of course, each segment is also falling as time passes; so the result is that the radius at the bottom is smaller than at the top. The sheet has been pulled inwards by surface tension.

Acknowledgments

This work was originally an MSci project that I supervised; Adam Townsend (adamtownsend.com) did all of the work. It is now an Open Access article [2].

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[1] *Melting chocolate*. S. Radosasvljevic, A. Schlunk, R. Tanner, and S. Nasseri. Technical report, CRC for Polymers, Sydney, February 2000.

[2] *The fluid dynamics of the chocolate fountain*. A. K. Townsend and H. J. Wilson. European Journal of Physics **37**:105803 (2016). iopscience.iop.org/article/10.1088/0143-0807/37/1/015803

[3] *The Engineering ToolBox.* engineeringtoolbox.com

[4] *Biking across a Pool of Cornstarch*. Hard Science. www.youtube.com/watch?v=BleCJJAKkgw

This content was supported in part by a grant from MathWorks.

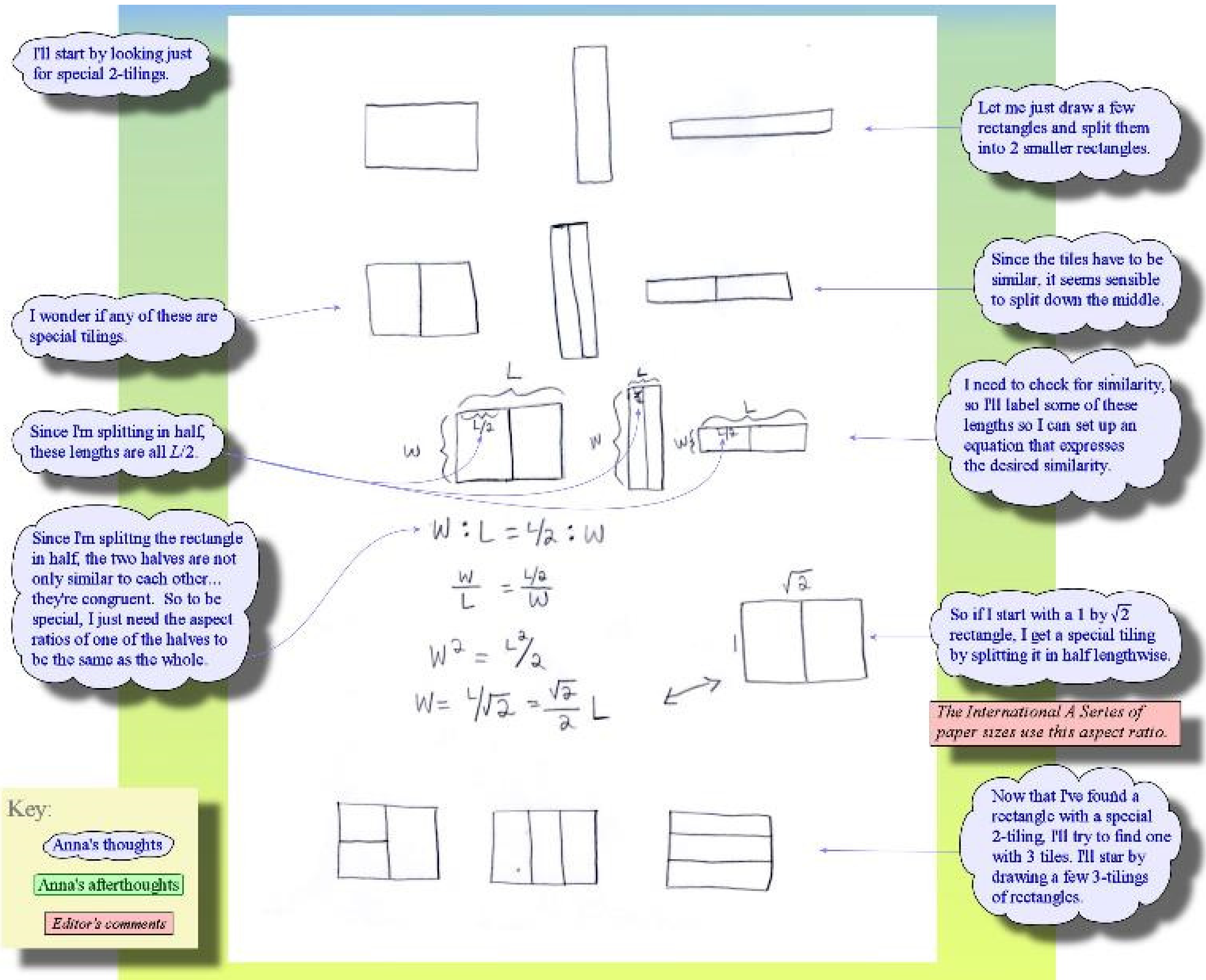
[4] An annulus is just a fancy name for a disc with a circular hole in the middle. It’s Latin for “little ring”.

Anna's Math Journal

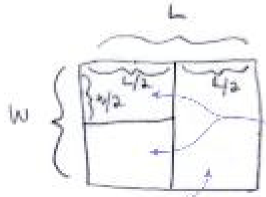
By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna tackles Problem 2 from the Similar Tiles Summer Fun problem set in the previous issue.



I'm intrigued by this design. I wonder if it's possible to get a special 3-tiling by halving a half of the original rectangle.



I'll label lengths and set up similarity equations.

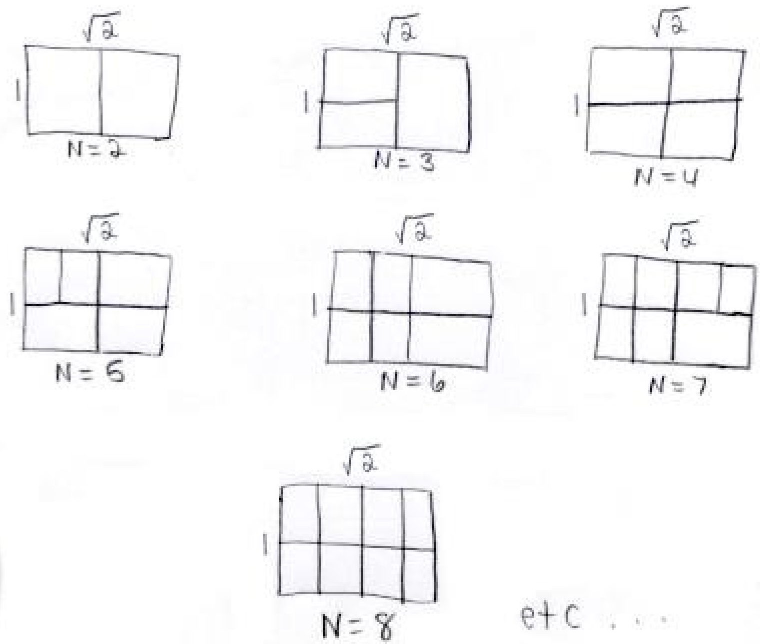
$$\frac{L}{W} = \frac{L/2}{W/2}$$

The smaller tiles each have half the length *and* width of the original, so they're definitely similar to each other and the original.

So I just need the larger of the 3 tiles to be similar to the original.

$$W : L = L/2 : W$$

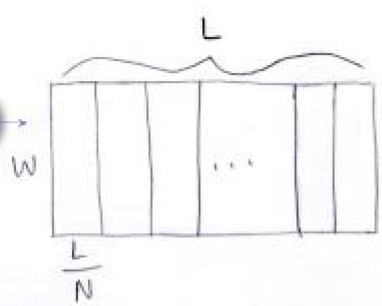
Wait! How silly of me! This is the same equation I just solved to find a special 2-tiling. All I've done to get a 3rd tile is to take my special 2-tiling and halve one of the tiles!



All of this makes perfect sense now. If I start with a 1 by root 2 rectangle, I can just keep on halving tiles lengthwise to obtain proper special tilings with any number of tiles because all tiles will retain the 1 by root 2 aspect ratio. Neat!

Combinatorially, I wonder how many different special N-tilings of a 1 by root 2 rectangle I can get by halving in this manner. Hm, I'll have to think about that someday.

I wonder if there is a special N-tiling where I just split the rectangle into N equal pieces like this.



$$W : L = \frac{L}{N} : W$$

$$\frac{W}{L} = \frac{L/N}{W}$$

$$W^2 = L^2/N$$

$$W = \frac{1}{\sqrt{N}} L$$

There is! I just have to start with a 1 by root N rectangle.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

ABB 8/29/16

The Problem of Thirty Birds

by Matthew de Courcy-Ireland

Fibonacci's *Liber Abaci* – "Book of calculation" – is a math textbook published in the year 1202. It taught a wide audience how to do arithmetic. It contains many amusing problems, such as this one which appeared in my Summer Fun problem set in the previous issue of this *Bulletin*:

> Bernadette buys 30 birds using 30 silver coins. There are three types of birds: a partridge costs 3 coins, a dove costs 2 coins, and a sparrow costs half a coin (meaning two sparrows for a silver – half-coins are not in circulation). How many birds must Bernadette have bought?

Suppose you tried to solve the problem by writing down every possible way of purchasing up to 30 birds of each kind, computing how much each way costs, and then locating those where the cost turns out to be 30 silver coins. This would involve checking a large number of cases. Indeed, if Bernadette checks all the possible ways to buy anywhere from 0 to 30 birds of each of the 3 types of birds, that would amount to $31 \times 31 \times 31 = 29791$ possibilities! But we remember that in the case of the sparrow, only the even numbers between 0 and 30 are allowed. So there are 31 possibilities for the number of partridges or doves and 16 possibilities for the number of sparrows. This reduces the number of possibilities to try to $31 \times 31 \times 16 = 15376$. There are also some easy ways to take the respective costs of the birds into account: With only 30 silver coins, and a price of 3 coins per partridge, Bernadette can in fact only afford 10 partridges at the most. Likewise, she can afford at most 15 doves at a cost of 2 coins apiece. The low cost of sparrows provides no additional constraint but at least we still get to skip the cases involving an odd number of sparrows. Altogether, there are $11 \times 16 \times 16 = 2816$ cases to be checked when we take into account the individual prices.

But, hold on! That computation ignores the fact that the *total* number of birds Bernadette purchases is 30. Algebra allows us to impose this constraint and narrow the search tremendously. Introduce variables p, d, and s for the number of partridges, doves, and sparrows that Bernadette purchases at the market. The information that Bernadette bought 30 birds in total is expressed by the equation

$$p + d + s = 30.$$

For each value of p, the number of solutions to $d + s = 30 - p$ is equal to $(30 - p) + 1$, since d can run through any of the values from 0 to $30 - p$, and once d is determined, so is s. There are precisely $(30 - p) + 1$ integers between 0 and $30 - p$, inclusive. Hence, to get the total number of solutions to $p + d + s = 30$, in nonnegative integers, we can add up the values of $(30 - p) + 1$ for each value of p from 0 to 30:

$$31 + 30 + 29 + \ldots + 3 + 2 + 1 = 31(32)/2 = 496.$$

But we remember that in the case of the sparrow, only the even numbers between 0 and 30 are allowed and Bernadette can only afford to purchase at most 10 partridges and 15 doves. So, for each possibility for the number of partridges p, between 0 and 10, inclusive, the number of doves and sparrows will have to amount to the remainder, which can range between 20 and 30 birds. At most 15 of these 20 to 30 birds can be doves, so of the $30 - p$ doves and sparrows, at least $15 - p$ of them must be sparrows. The number of sparrows will be an even number between

$15 - p$ and $30 - p$, inclusive. Since there are 16 numbers between $15 - p$ to $30 - p$ (including the first and last), there are 8 even numbers in that range, and so there are 8 possible numbers of sparrows. We conclude that for each of the values of p from 0 to 10, there will be 8 cases to consider for a total of 88 cases to be checked when we take into account the individual prices. This is still a lot of checking to do, so it would be nice if we could somehow impose the constraint that the three costs have to add up to 30 as well.

The financials of the matter are accounted for by the equation

$$3p + 2d + 0.5s = 30.$$

If we combine this equation with the equation that says that Bernadette purchases a total of 30 birds, namely $p + d + s = 30$, we can solve for two of the variables in terms of the third one. Each of the three choices (for which variable to express the others in terms of) makes it easy to see its own kind information. The result of isolating variables is that

$$\begin{array}{ccccccc} s/2 & = & 10 + p/3 & = & 12 - d/5 & = & s/2 \\ d & = & 10 - 5p/3 & = & d & = & 60 - 5s/2 \\ p & = & p & = & 6 - 3d/5 & = & 3s/2 - 30 \end{array}$$

The bottom set of equations perhaps makes it easiest to see that d must be divisible by 5 since that is the only way that $6 - 3d/5$ can be an integer. And the middle set of equations tell us that p must be divisible by 3 since that is the only way $10 - 5p/3$ can be an integer. This could all be useful information for solving other problems of this type, but it turns out that the top set of equations are enough to handle the original 30 birds problem. Bernadette isn't some sort of bird bank loaning to the merchants, so she didn't buy a negative number of partridges or doves. Therefore the equations $s/2 = 10 + p/3$ and $s/2 = 12 - d/5$ imply that $10 \le s/2 \le 12$. So there are only 3 cases to check! The number s of sparrows must be 20, 22, or 24, yielding solutions of

- 20 sparrows, 10 doves, and no partridges
- 22 sparrows, 5 doves, and 3 partridges
- 24 sparrows, no doves, and 6 partridges

The solution is almost unique. In the time of Fibonacci, it is likely that people would have been interested only in solutions in positive integers, and, in that case, the solution would be unique. Nevertheless, it is worth noting that Bernadette can choose not to buy birds of some type.

It is instructive to play around with the numerical values given in the problem. What if instead of using 30 silver coins to buy 30 birds costing 3, 2, or ½ coins depending on the type of bird, Bernadette bought B birds using C coins with prices P (for partridges), D (for doves), and S (for sparrows)? Can you find a relationship between B, C, P, D, and S that will guarantee a unique solution to the problem? With the help of algebra, yes you can!

Let p, d, and s be the respective numbers of partridges, doves, and sparrows that Bernadette buys. The problem involves solving the system of equations

$$\begin{aligned} p + d + s &= B \\ Pp + Dd + Ss &= C \end{aligned}$$

If two types of bird have the same price, then there is some ambiguity right away where we cannot tell how many birds of those two types were purchased. Since we're seeking parameters that lead to a unique solution, we shall therefore assume that $P > D > S$. The result of eliminating variables is that

$$\begin{array}{ccccccc} p & = & \frac{C-DB}{P-D}+\frac{D-S}{P-D}s & = & \frac{C-SB}{P-S}+\frac{S-D}{P-S}d & = & p \\[2ex] d & = & \frac{PB-C}{P-D}+\frac{S-P}{P-D}s & = & d & = & \frac{C-SB}{D-S}+\frac{S-P}{D-S}p \\[2ex] s & = & s & = & \frac{PB-C}{P-S}+\frac{D-P}{P-S}d & = & \frac{DB-C}{D-S}+\frac{P-D}{D-S}p \end{array}$$

It is reassuring to check that this recovers the solution to the original problem of 30 birds when $B = C = 30$, $P = 3$, $D = 2$, and $S = 1/2$. (Please do check this!)

The assumption that $P > D > S$ guarantees that all the denominators above are positive, so that we avoid dividing by 0 and also have an easier time identifying whether quantities are positive or negative. The positive quantities have been highlighted in green, and the negative ones in red. These signs are determined by the ordering $P > D > S$ and the additional steps that

$$C = Pp + Dd + Ss \le Pp + Pd + Ps = P(p + d + s) = PB,$$

and, likewise

$$C = Pp + Dd + Ss \ge S(p + d + s) = SB.$$

Using the fact that p, d, and s are nonnegative together with a close examination of the signs in the equations above allows us to locate p and s in these intervals:

$$\begin{aligned} \frac{C-DB}{P-D} \le\ & p \le \frac{C-SB}{P-S}, \\ \frac{DB-C}{D-S} \le\ & s \le \frac{PB-C}{P-S}. \end{aligned}$$

For d, we get two upper bounds instead of an upper bound and a lower bound:

$$d \le \min\left(\frac{PB-C}{P-D}, \frac{C-SB}{D-S}\right).$$

Notice that

$$\frac{PB-C}{P-D} - \frac{C-SB}{D-S} = \frac{(D-S)(PB-C)-(C-SB)(P-D)}{(P-D)(D-S)} = \frac{(P-S)(DB-C)}{(P-D)(D-S)}.$$

This tells us that which bound on d is better depends on whether $DB - C$ is positive or negative – in other words, whether the moderately priced bird is still pretty expensive or whether it's a really good deal. This also determines whether our lower bounds on p and s are negative, in which case they do not improve on the information given initially that $p \ge 0$ and $s \ge 0$.
If $DB - C \ge 0$, meaning the moderately priced bird is still a bit pricey, then our best estimate so far is

$$0 \le d \le \frac{C-SB}{D-S}.$$

In the other case, where $DB - C < 0$, we have

$$0 \le d \le \frac{PB-C}{P-D}.$$

The problem has a unique solution provided this interval contains a unique integer d such that the corresponding p and s are also integers. The easiest case to see when p and s are integers is when all of the parameters B, C, P, D, and S are whole numbers and $C - SB$ and $PB - C$ are divisible by $P - S$ but $S - D$ is not. The latter condition is reasonable since we have ordered $P > D > S$, making $D - S$ smaller than $P - S$ so that $S - D$ will not be evenly divisible by $P - S$. Then the formula for p and s in terms of d shows that d must be an integer multiple of $P - S$. In order to have a unique solution – or rather three solutions, as in the thirty birds problem, where some type of bird may go unpurchased – we should have (in the case where $DB - C > 0$):

$$\frac{C-SB}{D-S} = 2(P-S).$$

This way, the length of the interval of possible values of d is twice the spacing between allowed values of d. This yields one solution where $d = 0$ and no doves are purchased, another solution at the other endpoint so that $p = 0$ and no partridges are purchased, and an intermediate solution at the midpoint of the interval, where $d = P - S$. The case where $DB - C \le 0$ can be handled the same way by using $(PB - C)/(P - D)$ as the upper bound on d instead of $(C - SB)/(D - S)$. The cases where some of the prices are not whole numbers or where the divisibility conditions do not hold are trickier but similar: d has been located in an interval of known length and must vary in increments of $P - S$ in such a way as to cancel out any denominators that might prevent p and s from being integers. The condition for a unique solution is that this interval contain a unique point in that progression of step-size $P - S$.

Summer Fun!

In the last issue, we presented the 2016 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil! Move your mind! You might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

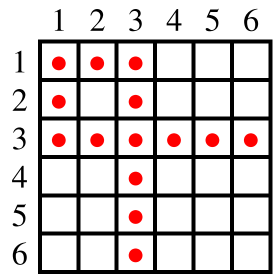
Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Summer Fun!

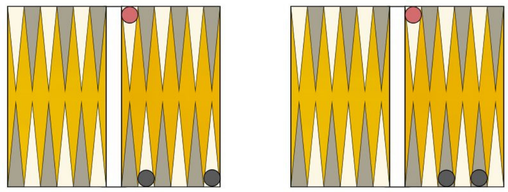
Backgammon Strategy

by Aaron Levy



1. The probability she rolls 6-2 is 2/36 = 1/18. The probability she rolls 6-2 or 4-1 is 4/36 = 1/9.

2. A 3 on either dice will work. That is the 3rd row and the 3rd column in the die-roll grid at left: this amounts to 11 rolls. Also a 2-1 will work (and a 1-2). This is another 2 rolls. Also a 1-1 will work. This is 14 possible rolls out of 36, or 7/18.



3. First consider the left of the two configurations shown at right. The only rolls thereafter which will *not* allow her to remove both checkers lie in the upper left 4×4 square in the die-roll grid, excluding 2-2, 3-3, and 4-4. Thus, the probability of winning on the next roll is 23/36. In the rightmost configuration, the only rolls that do not work are in the first row and the first column and 2-3. That means the probability of winning on the next roll is also 23/36. So it doesn't matter which she chooses.

4. She can either leave checkers on the 6th and the 2nd or on the 5th and the 3rd. In the first case, only 13 rolls win on the next turn. In the second case, 14 rolls win on the next turn. Thus, she is better off leaving checkers on the 5th and the 3rd spaces.

5. Yes. 19 rolls win and 17 lose. Her probability of winning exceeds 1/2, so she should double.

6. She should take the double when she has more than a 25% chance to win the game. Suppose her probability of winning is p. In N games, if she takes all of them, she should expect to win $2Np$ points and lose $2N(1-p)$, for an expected point difference (between winning and losing) of $2N(2p-1)$. If she drops all of them, she loses N points, for an expected difference of $-N$. Thus she should take whenever $2N(2p-1) > -N$, which happens when $p > 1/4$. Thus if there is one roll remaining, she should take if there are fewer than $¾ \times 36 = 27$ rolls that win for black. If black has 27 winning rolls exactly, the expected value is the same.

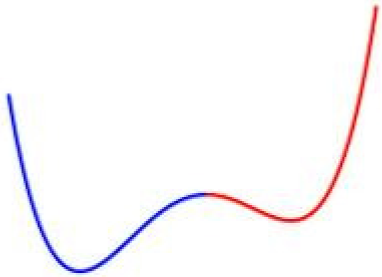
7. If black offers, then her expected point difference is $2N(qp - q(1-p) + (1-q)/2)$. If black does not offer, then her expected point difference is $N(p - (1-p))$. So black should offer a double when $2N(qp - q(1-p) + (1-q)/2) > N(p - (1-p))$, i.e., when $p(4q-2) > 3q-2$. When $q = 1$, this reduces to $p > 1/2$. When $2/3 < q < 1$, this condition is equivalent to $p > X$ for some $X < 1/2$. When $0 \le q \le 2/3$, this inequality will always hold. This says that when red is timid, and only takes a small fraction of the time, black should be more aggressive with doubling.

8. The probability that black hits red is $160{,}757/1{,}259{,}712 \approx 0.1276$. The probability that red hits black is $17{,}021/69{,}984 \approx 0.2432$. The probability that the checkers pass in peace is $792{,}577/1{,}259{,}712 \approx 0.6292$. Notice that the first person to move is at a disadvantage. If you solved this problem, please send in your workings to girlsangle@gmail.com. We'd love to see it!

Summer Fun!

Greedy Algorithms

by Zachary Sethna

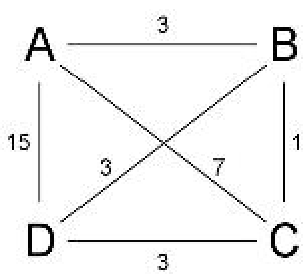


Before we get started, let's take a moment to think about the advantages and disadvantages of the greedy algorithms that we'll be looking at. Greedy algorithms always make a local choice instead of a global choice (e.g. in Problem 1, picking the highest value coin we can at each chance as opposed to listing out all possible choices of coins and picking the best). This makes greedy algorithms fast, easy to come up with, and generally give good local solutions, however they will frequently fail to find the global optimum if there are local optimums. To get a sense of this, suppose we are trying to find the minimum of the function whose graph is shown above left.

By eye, one can see that the minimum is the minimum in the blue region. However, let's apply the method of steepest descent, which is to start at a point, and then always take steps in the direction that takes us down the most (that is, the "greedy" direction). Note that if we start anywhere in the red region we will end up at the minimum in the red region, which is a local minimum but not a global one. In general, we will have to carefully decide whether or not the greedy algorithm finds the optimal solution. There are important classes of problems where greedy algorithms are proven to work (e.g. finding minimum spanning trees in a graph), but many where it will frequently fail (e.g. the traveling salesman problem which we analyze later). In Problems 1 and 2 the number of possible choices you can make are purposefully small so that you can enumerate all possible legal choices (i.e. taking a global perspective) and then compare to the greedy choice to see some examples where greedy choices work and where they fail. In Problem 3 you get to prove that a greedy algorithm works for a particularly elegant problem.

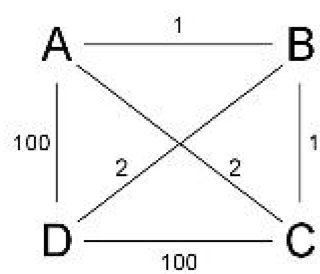
1. For standard coinage, the greedy solution is the optimal one, thus for \$1.57, the minimum number of coins is 6: 1 dollar coin, 2 quarters, 1 nickel, 2 pennies.

For the non-standard coinage the greedy solution requires 8 coins: 1×\$1.00, 1×\$0.10, 8×\$0.01, whereas the optimal solution uses only 5 coins: 1×\$1.00, 2×\$0.07, 2×\$0.01.



2. For the setup shown at left, the most efficient solution is the greedy solution, specifically A → B → D → C → A which costs 3 + 3 + 3 + 7 = 16.

An example where the greedy path is the *longest path* is shown at right.



Now, the greedy path is A → B → C → D → A costing 1 + 1 + 100 + 100 = 202 which is the maximum possible cost. In fact, a minimum path, such as A → D → B → C → A, only costs 100 + 2 + 1 + 2 = 105.

Summer Fun!

3. We first try using the greedy algorithm on some examples to see if we can identify a pattern:

$$\begin{aligned}
\frac{4}{5} &= \frac{1}{2}+\frac{3}{10}=\frac{1}{2}+\frac{1}{4}+\frac{1}{20}\\
\frac{5}{6} &= \frac{1}{2}+\frac{1}{3}\\
\frac{6}{7} &= \frac{1}{2}+\frac{5}{14}=\frac{1}{2}+\frac{1}{3}+\frac{1}{42}\\
\frac{12}{13} &= \frac{1}{2}+\frac{11}{26}=\frac{1}{2}+\frac{1}{3}+\frac{7}{78}=\frac{1}{2}+\frac{1}{3}+\frac{1}{12}+\frac{1}{156}\\
\frac{22}{23} &= \frac{1}{2}+\frac{21}{46}=\frac{1}{2}+\frac{1}{3}+\frac{17}{138}=\frac{1}{2}+\frac{1}{3}+\frac{1}{9}+\frac{5}{414}=\frac{1}{2}+\frac{1}{3}+\frac{1}{9}+\frac{1}{83}+\frac{1}{34362}
\end{aligned}$$

Notice after each step the numerator of the remaining fraction (when expressed in reduced terms) is *strictly decreasing* (indicated by the red numerators above). If this is true, then the greedy algorithm will always work in at most m steps where our original fraction is m/n.

Let's prove it!

Say we have the fraction m/n. The greedy algorithm would tell us to find the *smallest* integer k such that $m/n > 1/k$. The remaining amount would then be $m/n - 1/k = (mk - n)/(nk)$. Although both $mk - n$ and nk are integers, they may share a common factor. Let r be the greatest common factor of $mk - n$ and nk. Then if we let $m' = (mk - n)/r$ and $n' = nk/r$, the remaining fraction, in lowest terms, is m'/n'.

We will now do a proof by contradiction. If you are not familiar with proofs by contradiction, what we will do is assume the logical complement of what we wish to prove and show that it leads to a contradiction, proving that our assumption is false. Since we want to show that $m' < m$, we will assume that $m' \geq m$ and see where it takes us:

$$\begin{aligned}
mk - n = m'r &\geq mr \qquad \text{(by our assumption that } m' \geq m\text{)}\\
mk - mr &\geq n\\
m(k-r) &\geq n\\
\frac{m}{n} &\geq \frac{1}{k-r}
\end{aligned}$$

But, this contradicts the fact that we picked the *smallest* possible k for our greedy algorithm. Thus, we know that our assumption that $m' \geq m$ is false, so it must be that $m' < m$.

Summer Fun!

Probability Puzzles with Bags of Coins

by Lauren McGough

We will sometimes denote the probability of some event E by $P(E)$. Also, we will sometimes use "G" to stand for a gold coin and "S" to stand for a silver coin.

Problem 1. A. The probability of pulling the gold coin is 1/6, i.e. P(the first coin is G) = 1/6.

B. With replacement, the probability of pulling the gold coin twice is $1/6 \times 1/6 = 1/36$. The probability of pulling the gold at least once is the probability of pulling it on the first try, plus the probability of getting silver on the first try, but gold on the second try: $1/6 + (5/6 \times 1/6) = 11/36$.

C. Without replacement, we can't pull the gold twice, but we can pull it at least once in the first two pulls and the probability of doing so is equal to the probability of pulling it first plus the probability of pulling it second: $1/6 + (5/6 \times 1/5) = 1/3$. The probability of pulling silver twice (in the first 2 pulls) is the probability of pulling a silver first *and* second, or $5/6 \times 4/5 = 2/3$. Note that this is also the same as not pulling gold in the first two tries; therefore, the probability of pulling silver twice in the first 2 pulls is also 1 minus the probability of pulling gold in the first 2 pulls. This explains why the two probabilities we just computed sum to 1.

D. Let's determine the probability of pulling the gold coin if we pull out k coins, with replacement. This probability is 1 minus the probability of having all k coins come up silver. The probability of getting k silver coins, with replacement, is $(5/6)^k$. Therefore, the probability of pulling the gold coin if we pull out k coins, with replacement, is $1 - (5/6)^k$. The smallest k for which this is at least ½ happens to be $k = 4$.

Without replacement, we again compute the probability of a pulling a string of k silver coins: $5/6 \times 4/5 \times \ldots \times (6 - k)/(6 - k + 1) = (6 - k)/6$. Note that $k \leq 6$ since, without replacement, we cannot pull more than the total number of coins. Hence, the probability of seeing the gold coin after pulling k coins, without replacement, is $1 - (6 - k)/6 = k/6$. The smallest k for which this is at least ½ is $k = 3$. (Can you find a simple way to see this?)

E and F. This table summarize the various probabilities (assuming $N > 1$):

		with replacement		without replacement	
Set Up	**$P(G)$**	**$P(GG)$**	**$P(GG, GS, SG)$**	**$P(GG)$**	**$P(GG, GS, SG)$**
3 G, 3 S	1/2	1/4	3/4	1/5	4/5
3 G, 7 S	3/10	9/100	51/100	1/15	8/15
g G, $N - g$ S	g/N	$(g/N)^2$	$1 - (N - g)^2/N^2$	$\frac{g(g-1)}{N(N-1)}$	$1 - \frac{(N-g)(N-g-1)}{N(N-1)}$

2. A. 2/3; the only bag that does not contain gold is the one with 2 silver coins.

Summer Fun!

B. I could be holding the bag with 2 golds, or 1 gold and 1 silver, but not the one with 2 silvers.

C. 1 or 2.

D. Name the gold coins for the sake of discussion (although they are physically identical): G_1 is the gold coin in the bag with a gold and silver coin, and G_2 and G_3 are the gold coins in the bag with 2 gold coins. Each particular coin has an equal likelihood of being drawn. If the coin I happen to pick is gold, then I know I am in one of the 3 equally probable cases that results in picking a gold coin. Hence the probability of picking any specific gold coin is 1/3.

E. Of the 3 equally likely cases of picking G_1, G_2, or G_3, one of these corresponds to the situation where I was given the bag with a gold and silver coin, and two correspond to the situation where I was given the bag with 2 gold coins. Therefore, the probability that the other coin in the bag is gold is 2/3. Pulling out *one* gold coin means it is twice as likely that the other coin is also gold!

3.There are 4 cubbies: silver top – silver bottom (*S*-*S*), gold top – silver bottom (*G*-*S*), silver top – gold bottom (*S*-*G*), gold top – gold bottom (*G*-*G*). Peeking in the top drawer, we have four equally likely outcomes: we could see the top "*S*" from *S*-*S*, the "*G*" from *G*-*S*, the "*S*" from *S*-*G* or the top "*G*" from *G*-*G*. *Two* of these possibilities are *G*. The *G* from *G*-*S* does not have a gold in the bottom drawer, and the top *G* from *G*-*G* does. Since these two cases are equally likely, the probability that the coin in the bottom drawer is *G* is therefore 1/2. What's the difference between this and the previous case? It all hinges on the fact that a "top" coin is different from a "bottom" coin. That is, in the *G*-*G* cubby, you can distinguish between the 2 gold coins (one's on top, the other's on the bottom), whereas you could not distinguish between the 2 gold coins in the bag. This takes away the "combinatorial" advantage that the *GG* bag enjoyed in problem 2.

4. This problem is equivalent to having 1001 equally likely bags with 0, 1, …, 1000 gold coins, and we are holding an unknown bag. The total number of gold coins in all the bags is 500,500, which is the sum of the numbers 1 through 1000. I break even or make money if the bag contains at least 501 gold coins. The total number of gold coins in bags that contain at least 501 gold coins is 501 + 502 + ... + 1000, which sums to 375,250. As in Problem 2, the probability of at least breaking even is thus $375{,}250/500{,}500 \approx 0.74975$, which is greater than 50%. So, yes, after seeing just one gold coin, it is worth it to buy the bag for $501.

5. C. In 2 of the 3 equally likely cases, switching caused me to win. That means if I switch, I have a 2/3 chance of winning, so, yes, I should switch.

D. The probability of picking the prize is $1/N$, and if I did pick the prize, switching would cause me to lose. But the probability of not picking the prize is $1 - 1/N$, and in this case, switching would enable me to win.

In other words, the probability of winning when I switch is equal to the probability of picking the wrong bag on my first try: as long as I didn't pick the prize with my first pick, then the remaining closed bag will be the prize. Therefore, I should switch!

Summer Fun!

The Thirty Birds, and other problems with integer constraints

by Matthew de Courcy-Ireland

1. A necessary condition is that m must be a prime number, but it isn't true that $2^m - 1$ is prime for all prime numbers m. There is a deceptive pattern at the beginning: $2^3 - 1 = 7$, $2^5 - 1 = 31$, and $2^7 - 1 = 127$, are all prime. But $2^{11} - 1 = 2047 = 23 \times 89$ is not prime. There is a special test available for deciding quickly whether a number of the form $2^p - 1$, where p is prime, is prime or not. It is called the Lucas-Lehmer test. Because of this test, the largest prime number known to humanity at any given time is usually a Mersenne prime. However, it is a notorious open problem to determine whether there are infinitely many primes p such that $2^p - 1$ is prime or whether this supply of enormous prime numbers eventually dries up. If your computer isn't too busy with your own experiments, you can have it assist in the Great Internet Mersenne Prime Search.

To prove that m must be prime in order for $2^m - 1$ to be prime, suppose that m factors as $m = ab$. Then we can factor $2^m - 1$ with the help of a geometric sum! The formula for evaluating a geometric sum is

$$r^{b-1} + r^{b-2} + \cdots + r + 1 = \frac{r^b - 1}{r - 1},$$

whose proof we will review in a moment. But first, let's see why it factors $2^m - 1$ for us:

$$2^m - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b = 1} + \ldots + 2^a + 1)$$

by applying the geometric sum formula above with 2^a as our choice of r. Both of these factors are larger than 1 as long as a and b themselves are larger than 1. So if m is not prime, then neither is $2^m - 1$.

To add up the geometric sum, observe that the sum we get when we multiply by r is very similar to the original sum. If we give $r^{b-1} + r^{b-2} + \ldots + r + 1$ the name S, then $rS - S = r^b - 1$, so division teaches us that

$$S = \frac{r^b - 1}{r - 1}$$

as claimed.

2. As in the previous case, we can derive a condition that m must satisfy: $2^m + 1$ cannot be prime unless m is a power of 2. The first few numbers of this form are indeed prime:

$$3, 5, 17, 257, 65537.$$

Summer Fun!

But the next one has a famous factorization found by Euler: $2^{32} + 1 = 641 \times 6700417$, disproving Fermat's speculation that all numbers of the form $2^{2^k} + 1$ are prime. The number 65537 is actually the largest known example of a prime of this form.

To show that m must be a power of 2, suppose to the contrary that it has an odd divisor. As above, we write $m = ab$, but now b is odd. Since b is odd, $(-1)^b = -1$ and we can argue as follows:

$$\begin{aligned} 2^{ab} + 1 &= 1 - (-1)^b(2^a)^b \\ &= 1 - (-2^a)^b \\ &= (1 - (-2^a))(1 + (-2^a) + \ldots + (-2^a)^{b-1}), \end{aligned}$$

using the geometric sum again. This means $2^m + 1$ is not prime because $2^a + 1$ divides it.

If you are comfortable with congruences and modular arithmetic, then you have access to an elegant way to understand both of the problems above. In the Mersenne case,

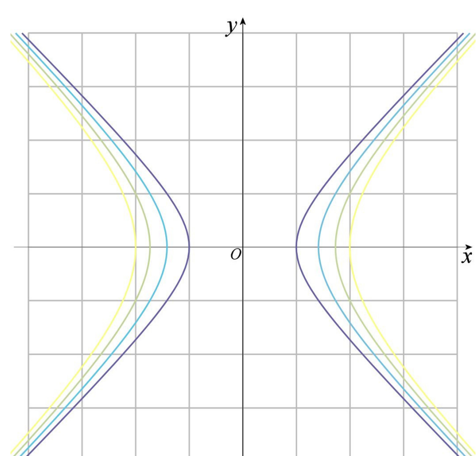
$$2^m - 1 = (2^a)^b - 1 \equiv 1^b - 1 \equiv 0 \pmod{2^a - 1},$$

which means that $2^a - 1$ divides $2^m - 1$. This agrees with what we found using the geometric sum. In the Fermat case, if $m = ab$ with b odd, we observe that

$$2^m + 1 = (2^a)^b + 1 \equiv (-1)^b + 1 \equiv 0 \pmod{2^a + 1},$$

which means that $2^a + 1$ is a factor of $2^m + 1$. So any m with an odd divisor b yields a composite Fermat number. The only case where it is possible to get a prime number is if m is a power of 2 so that it has only even divisors (other than 1).

3. From our study of conic sections, we recognize $x^2 - y^2 = N$ as the equation of a hyperbola regardless of the value of N. The effect of N is to scale the hyperbola. If our study of conic sections is too far in the future (or past) to help us, then we can still get some idea of the shape of this curve as follows: When $x = 0$, the equation reduces to $-y^2 = N$, which has no solutions for $N = 1, 2, 3, 4, \ldots$ or indeed any positive value of N. So the curve $x^2 - y^2 = N$ avoids the y-axis. It meets the x-axis where $y = 0$ in two points, namely $(\sqrt{N}, 0)$ and $(-\sqrt{N}, 0)$. As x and y take on larger and larger values, any fixed N becomes negligible so that the equation says $x^2 - y^2 \approx 0$.



The graphs of $x^2 - y^2 = N$, for $N = 1, 2, 3$, and 4.

Summer Fun!

In other words, very far from the origin, the curve is close to the pair of diagonal lines $x = y$ and $x = -y$. Also the expression $x^2 - y^2$ does not change if we change x to $-x$ or y to $-y$. This implies that the curve is symmetric about the x- and y-axes.

4. If (x, y) is a lattice point on this graph, then

$$1 = x^2 - y^2 = (x - y)(x + y).$$

Since x and y are integers, we have only two cases to consider:

$$\begin{aligned} x - y &= 1 \\ x + y &= 1 \end{aligned}$$

or

$$\begin{aligned} x - y &= -1 \\ x + y &= -1. \end{aligned}$$

Each case is a system of two linear equations in two unknowns with a unique solution. The resulting points are (1, 0) and (-1, 0). There are 2 lattice points on this curve.

5. How many lattice points are there on the graph of $x^2 - y^2 = N$, where N is a fixed integer?

The reasoning is similar for a general N as for $N = 1$, but more cases arise. As before, we write

$$N = x^2 - y^2 = (x - y)(x + y),$$

but now unique factorization leads to more cases. For each divisor d of N, we have a system of 2 linear equations for x and y:

$$\begin{aligned} x - y &= d \\ x + y &= N/d \end{aligned}$$

When $N = 1$, there are only 2 choices for d, namely 1 and -1. When $N = 2$, there are 4 choices for d: 1, -1, 2, and -2. Likewise, for $N = 3$ or indeed any prime number, there are 4 choices for d. When $N = 4$, there are 6 choices: ±1, ±2, and ±4. In general, there are twice as many choices as there are positive divisors of N because of the possibility of negative numbers. Each of these choices yields a unique lattice point on the hyperbola, which together form a complete list.

The number of divisors of N behaves erratically because it is a direct consequence of the prime factorization of N, but it is known to be quite small (relative to N). For example, one can prove that the number of divisors of N, for sufficiently large N, is smaller than $C_e N^e$, for any $e > 0$, where C_e is a number that depends only on e.

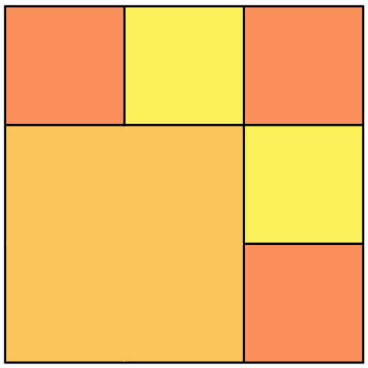
6-10. For the solution to Problems 6-10 concerning the "Thirty Birds" problem from Fibonacci's *Liber Abaci*, please see page 14.

Similar Tiles

by Ken Fan

1. One can obtain a special 4-tiling by splitting it in half both widthwise and lengthwise.

2. See *Anna's Math Journal* on page 12.



3. A special 6-tiling of a square.

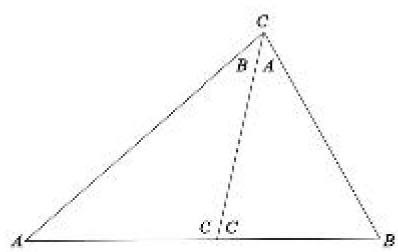
4. Special N-tilings of squares exists for all whole numbers N except for $N = 2$, 3, or 5. Notice that

$$n^2 - (n - 1)^2 = 2n - 1.$$

Thus, by dividing a square into n^2 congruent squares and then recombining $(n - 1)^2$ of them into a single square, we obtain a special tiling with $2n$ tiles, for any $n > 1$. If we have a special tiling of a square into $2n$ squares, we can split one of the squares into fourths to obtain a special tiling with $2n + 3$ squares. In this way, we get special N-tilings of squares for any N other than $N = 2$, 3, or 5. To be complete, we should show that special 2, 3, and 5-tilings are impossible, but we omit this step. If you have a proof, please send to girlsangle@gmail.com.

5. For any triangle, use 3 line segments to connect the midpoints of its sides in pairs. The result will be a special 4-tiling of the triangle.

6. This might be an open question. All rectangles and parallelograms admit proper special tilings, and there are other interesting cases, such as a trapezoid with one base of length twice that of the other.

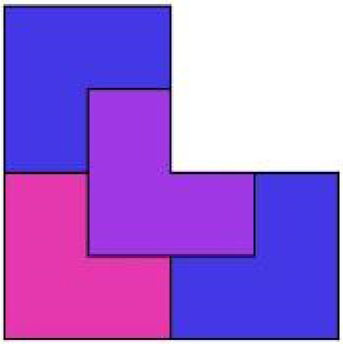


7. If two triangles are put together to form a new triangle, the two triangles must be joined so that they share an edge exactly for otherwise the resulting shape would have more than 3 sides. Therefore, a proper 2-tiling of a triangle can only be formed by splitting the triangle by a single line. Furthermore, this line must pass through a vertex of the triangle or else one of the tiles will be a quadrilateral. Label the vertices of the triangle A, B, and C. We'll use these labels to also refer to their corresponding angle measures. Suppose that C is the angle that is split by the 2 tiles as shown in the figure. The tip of the left tile at vertex A also shares the angle measure at A, and since the angle measure of the tip of the left tile at vertex C cannot measure C, the angle measure of the tip of the left tile at vertex C must measure B. By similar reasoning, the angle measure of the tip of the right tile at vertex C must measure A. Therefore, the 2 angles formed at the base by the dividing line must both measure C, which means that C must be a right angle. And, in fact, the altitude to the hypotenuse of any right triangle does split the right triangle into two similar right triangles. Thus, the triangles that admit a special 2-tiling are exactly the right triangles.

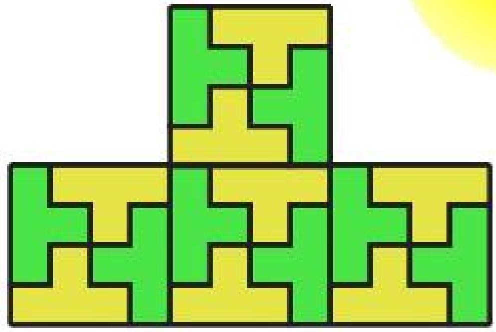
Summer Fun!

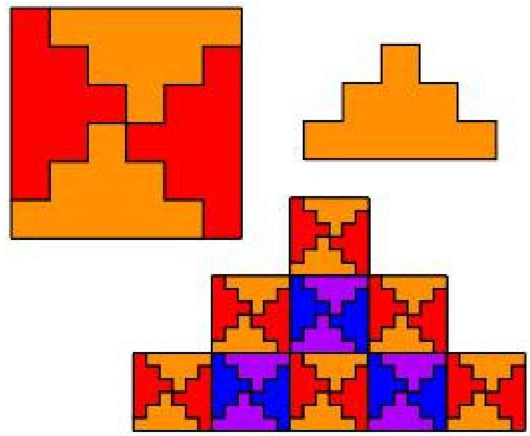
8. Start with a right triangle. Draw in altitudes to the hypotenuses of the right triangles that are created until you have the desired total number of triangles.

9.



10.





11. Yes. We can generalize the idea shown in the solution to problem 10. The idea is to build a shape out of squares, which can, itself, be used to tile a square. We can then rebuild the shape from similar copies of itself by piecing together square "packages" of the shape. The figure at right illustrates this process for a 12-sided shape. By increasing the number of steps in the "staircase", this technique will produce a shape with $4n$ sides ($n > 1$) that admits a proper special tiling.

12. Suppose a square has a super special tiling into N squares. Without loss of generality, we can assume that the tiles are unit squares. The side of our original square is lined with an integral number of tiles and so must have an integer length L. The area of the square is therefore L^2. On the other hand, the area is also N since the square is tiled by N unit tiles. Thus, $L^2 = N$. We conclude that N must be a perfect square. Conversely, any square can be split into a grid of congruent L by L squares for any positive integer L. To be proper, we also require that $N > 1$.

13-14. We will use the idea for the solution to Problem 11, generalized to the third dimension. We seek a non-prism 3D shape which has 2 properties:

1. The shape can be built out of perfect cubes.
2. Copies of the shape can be used to build a perfect cube.

We obtain such a shape as follows. Start with a perfect cube, 3 units on a side. Imagine this cube as consisting of $3^3 = 27$ unit cubes. From one face, remove the central unit cube. Attach this unit cube along one of its faces to the center of an adjacent face (oriented with all sides parallel or perpendicular to the original cube).

Notice that 4 such shapes can be linked to form a 6 by 6 by 3 slab, so 8 can be arranged to form a 6 by 6 by 6 cube.

Summer Fun!

Calendar

Session 19: (all dates in 2016)

September	15	Start of the nineteenth session!
	22	Jane Kostick, woodworker
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	
	8	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton Univeresity
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, The Dartmouth Institute
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, assistant professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, associate professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____