Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

From the Founder

The Girls' Angle club is a safe, friendly, welcoming place where we nurture ideas and invention. The basic strategy through the years has been the same: generate mathematical questions that inspire and encourage and assist members to find their answers. The result has been a kaleidoscope of mathematics that I feel privileged to have witnessed.

- Ken Fan, President and Founder

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Girls' Angle welcomes submissions that pertain to mathematics.

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Quintetra Star Nest. Designed and created by Jane and John Kostick. Photo courtesy of the Kosticks. Jane visited Girls' Angle this fall. See page 26.

An Interview with Dr. Brandy Wiegers

Dr. Brandy Wiegers is an Assistant Professor of Mathematics at Central Washington University. She earned her doctoral degree in applied mathematics from the University of California, Davis. She also serves as the Associate Director for the National Association of Math Circles.

Girls' Angle summer intern Sandy Pelkowsky conducted this interview.

Sandy: In college, you studied both mathematics and engineering. What attracted you to these two subjects? What ultimately led you to decide to become a mathematician?

Brandy: My engineering degree is in Biological Systems Engineering which is important because I discovered that by combining that engineering degree with mathematics I had the opportunity to do everything I loved - mathematics, chemistry, biology, and physics. I never had to make a choice, and instead had the chance to keep studying all the STEM fields from high school. I then had the opportunity to do several research experiences as an undergraduate where that diversity was really useful, but everyone I worked with kept valuing the mathematics that I added to the experience. For example, I did this awesome experience in Bermuda for three months where I was doing marine biology research (snorkeling to collect samples, and everything) and all my friends started to use me as the resident mathematician to help on their projects. I then took a mathematical biology course and found my true calling, a chance to bring my mathematical background and help other scientists explore their research more deeply.

Sandy: Please describe a mathematical concept or theorem that excited you when you were in school?

Brandy: The predator/prey models that I saw in my first mathematical biology course amazed me. I couldn't believe that such basic mathematics could explain the behavior of natural, wild animals. I also was completely entranced with mathematical fluids and thermodynamics. First they brought me in because they were so challenging. These were the hardest courses for engineers so at first I worked really hard to make sure I didn't fail the classes but soon I worked even harder to understand the material, because it was so interesting. I remember a graph in thermodynamics that showed four variables in one two dimensional Cartesian plane, using different lines and intersections. It looked like a crazy ink test that, when I finally understood, truly represented this beautiful representation of physical phenomenon. It was a form of art to me.

Sandy: Would you explain one or two of your favorite math problems that you've worked on and your contribution to them?

Brandy: My work in graduate school was focused on understanding the water that moves through the bottom one-centimeter of a plant root, working to understand where the water comes from that makes the plant grow. I did all this work using a numerical analysis approach, meaning I looked at all the variables and equations using a computer program and compared the results of different models to the real-world results. Doing this work I was able to prove a theory that my biology advisor had for fifteen years and a lot of people appreciated that, with 23 other publications citing this work in the last seven years. All of that said, there are two things that I remember the most about my graduate school work on this topic.

First, in the very first year, my advisor showed me the model they were

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Dr. Brandy Wiegers and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
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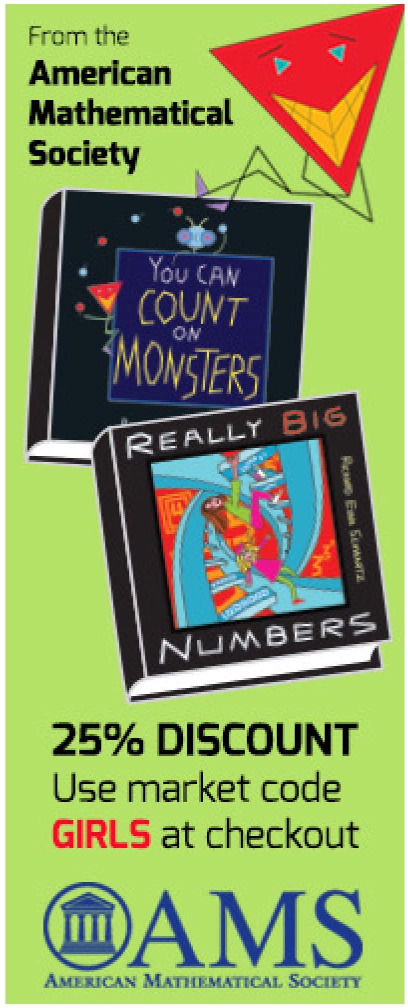

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Meditate$^{\text{Math}}$

You're given a line and a point not on it. In this situation, a common construction is to connect the point to the line with the line segment that is perpendicular to the given line. This is also referred to as "dropping a perpendicular from the point to the line." The point where the perpendicular meets the line is called the "foot" of the perpendicular.

In this installment of Meditate to the Math, we present 3 theorems that feature this construction. Find a comfortable place. Soak in these images. Dream up proofs of these theorems.

Theorem 1. In the xy-coordinate plane, consider the line given by the equation $ax + by + c = 0$, where a, b, and c are constants with a and b not both zero. The minimum distance of the point (X, Y) from the line is given by

$$\frac{|aX + bY + c|}{\sqrt{a^2 + b^2}}.$$

If you're stuck, here are some questions to meditate upon:

1. What is the graph of the function $f(x, y) = ax + by + c$? What are its level sets?
2. What is the distance between (x, y) and $(x + a, y + b)$
3. What is $f(x + a, y + b) - f(x, y)$?

Theorem 2. The altitudes of a triangle are concurrent.

The altitudes of a triangle are the 3 perpendiculars dropped from each vertex to the vertex's opposite side. If you're stuck, here are some questions to meditate upon:

1. If the sides of the gray triangle are parallel to corresponding sides of triangle ABC (see figure on next page), what more can you say about their relationship?
2. How do the altitudes of triangle ABC relate to the gray triangle?

The last theorem involves the concept of a **pedal triangle**. Let T be a triangle with interior point P. The first pedal triangle, T_1, with respect to P is the triangle whose vertices are the feet of the perpendiculars dropped from P to the sides of T. The second pedal triangle, T_2, with respect to P is the first pedal triangle of T_1 with respect to P. We can recursively define the nth pedal triangle, T_n, with respect to P as the first pedal triangle of T_{n-1} with respect to P, for $n > 1$.

Theorem 3. (J. Neuberg) Let T be a triangle with interior point P. Let T_3 be the third pedal triangle with respect to P. Then T is similar to T_3.

If you're stuck, here are some questions to meditate upon:

1. What do right triangles that share a hypotenuse also share?
2. How do various angles relate to each other?

Recall that inscribed angles in a circle that subtend the same arc have equal measure.

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Scaling in a Sticky Place: The Fluid Mechanics of the Chocolate Fountain, Part 2

by Helen Wilson

Photo courtesy of Adam Townsend.

We were looking at a chocolate fountain. Last time we saw that chocolate is a **shear-thinning** fluid, which becomes less viscous in faster flows. One model for its viscosity, η, is

$$\eta = 65\dot{\gamma}^{-2/3} \text{ Pa s}$$

where $\dot{\gamma}$ (measured in reciprocal seconds, s^{-1}) is the **shear rate**, the rate at which each layer of fluid slides over the one next to it.

We also looked at the falling curtain of chocolate, and saw that the main thing that causes it to fall inwards (rather than straight downwards) is surface tension.

The other main part of the chocolate flow is the pouring flow over each plastic dome. That's what I want to talk about now, because it is a lovely application of one of the key tools of applied mathematics: **scaling analysis**.

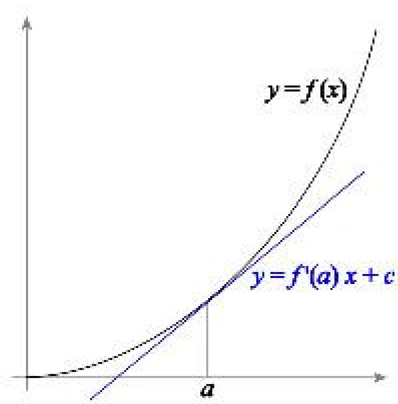
Of course, before we can start any kind of analysis, we need a set of equations to apply it to. We'll need some calculus to get us started. If you've not come across differentiation before, the fundamental idea is that it gives you the *rate of change* of something. If your interesting quantity y can be described in terms of its dependence on a parameter x via a function:

$$y = f(x)$$

then the rate of change of y with respect to x (meaning how fast y changes when you make a small change to x) is defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If you draw the graph of $y = f(x)$, then the slope of the tangent at $(a, f(a))$ is $f'(a)$ (see figure at right).



Now we'll be solving a real physical problem, in which our interesting quantity (a flow speed) will potentially depend on many different parameters: time t and position in three dimensions

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x, y, and z. We have a function of multiple variables $f(x, y, z, t)$. We will have to define a **partial derivative** to find the rate of change with respect to one variable at a time:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z, t) - f(x, y, z, t)}{h}.$$

Now that we have a notation, I can write down the governing equations for the steady, inertialess flow over the dome (which we assume is shaped like the top of a large sphere). **Be warned: these are going to be pretty hideous equations!** I'm going to use ϕ for the angle measured around the symmetry axis of the fountain, r for the distance from the centre of the large sphere, and θ for the angle between our position and the vertical (as seen from the centre of the sphere). This coordinate system (r, θ, ϕ) is called **spherical polar coordinates**, and it makes life simpler[1] whenever a problem has a sphere as a key part of its geometry. We write u_r, u_θ, and u_ϕ for the velocity components in the three directions. The symmetry of the flow lets us safely assume that $u_\phi = 0$ and all the velocities are independent of ϕ.

The governing equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{\sin\theta}\frac{\partial(u_\theta \sin\theta)}{\partial \theta} = 0 \tag{1}$$

$$\begin{aligned}\frac{\partial p}{\partial r} = {} & \frac{\partial}{\partial r}\left(\eta \frac{\partial u_r}{\partial r}\right) + 2\eta \frac{\partial}{\partial r}\left(\frac{u_r}{r}\right) + \frac{1}{2r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\eta \sin\theta \cdot \frac{\partial u_r}{\partial \theta}\right) \\ & + \frac{1}{2r\sin\theta}\frac{\partial}{\partial\theta}\left(\eta \sin\theta \cdot r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right) - \frac{\eta}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) - \rho g \cos\theta\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial p}{\partial \theta} = {} & \frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2 \eta \frac{\partial u_r}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\eta \frac{u_r}{r}\right) + \frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^4 \eta \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right) \\ & + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\eta \sin\theta \cdot \frac{\partial u_\theta}{\partial \theta}\right) - \frac{\eta}{r}\cot^2\theta \cdot u_\theta + \rho g r \sin\theta\end{aligned} \tag{3}$$

Here, p is pressure, ρ is the density of the chocolate and g is – you've guessed it – gravity.

Equation (1) is called the mass conservation equation. Equations (2) and (3) are the momentum equations. Nice, aren't they? No. No, they are completely incomprehensible! We don't really have a hope of making any sense of them as they stand.

It's a sticky problem. But one of the most fundamental tools of applied mathematics is going to come to our rescue. It's a piece of fantastic cunning called **scaling analysis**. It is messy and hand-wavy and no pure mathematician would call it mathematics at all: but it works.

The way it works is this: we try to ascertain the *typical magnitude* of each term in these equations. This comes about through a mix of mathematical and physical arguments. Then we compare them, and hope that some of them will be small enough to ignore.

[1] Honestly, these equations would be even worse when expressed in a different coordinate system.

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Acknowledgments

This work was originally an MSci project that I supervised; Adam Townsend (adamtownsend.com) did all of the work. It is now an Open Access article [2].

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Bibliography

[1] *The fluid dynamics of the chocolate fountain*. A. K. Townsend and H. J. Wilson. European Journal of Physics **37**:105803 (2016). iopscience.iop.org/article/10.1088/0143-0807/37/1/015803

[2] *Flow and instability of thin films on a cylinder and sphere*. D. Takagi and H. E. Huppert. Journal of Fluid Mechanics **647**:221-238 (2010).

In Search of Nice Triangles, Part 7

by Ken Fan | edited by Jennifer Silva

Emily and Jasmine continue their investigation into nice triangles. They've been using "nice" to denote angles that measure a rational multiple of π radians.

Recall that they are now looking for triangles with 3 nice angles and 2 sides of integer length.

They discovered that they can wrap up their "nice triangles journey" by determining all pairs of acute nice angles X and Y such that $\cos X = r \cos Y$, where r is a rational number.

They stumbled upon the notion of a "minimum" polynomial. As Emily recorded in her notebook: "The minimum polynomial (if it exists) for a number λ is a polynomial $p(x)$ with rational coefficients, lead coefficient 1, and of least degree such that $p(\lambda) = 0$."

They defined $p(x)$ to be the minimum polynomial of $\cos X$ and $q(x)$ to be the minimal polynomial of $\cos Y$. They let d be the degree of $p(x)$.

Emily: Hey! Then in that case, I think we *can* conclude that $q(x) = p(rx)/r^d$. By replacing x with x/r, the equation[1]

$$p(rx)/r^d = s(x)q(x)$$

can be rewritten as $p(x) = r^d\, s(x/r)q(x/r)$. And if we substitute $\cos X$ for x, we find that

$$0 = r^d\, s((\cos X)/r)\, q((\cos X)/r).$$

This tells us that $\cos X$ is a root of either the polynomial $s(x/r)$ or $q(x/r)$. But $p(x)$ is the *minumum* polynomial for $\cos X$. There can't be a polynomial with rational coefficients of lower degree than $p(x)$ that has $\cos X$ as a root. That means that either $s(x/r)$ or $q(x/r)$ must have degree at least as great as that of $p(x)$. And since the degree of a product of polynomials is the sum of the degrees of the factors, it must be that either $s(x/r)$ or $q(x/r)$ has degree exactly equal to the degree of $p(x)$, while the other would have degree 0. Since we know that $q(x)$ is the minimum polynomial for $\cos Y$, its degree cannot be 0. Therefore, $q(x/r)$ has the degree of $p(x)$, and $s(x/r)$ must be a constant. By comparing the lead coefficients in the equation $p(x) = r^d s(x/r)q(x/r)$, we can further conclude that $s(x/r) = 1$. So $p(x) = r^d q(x/r)$, or to put it another way, $q(x) = p(rx)/r^d$!

Jasmine: Super! I think you're saying that the minimum polynomial for a number λ cannot be factored as a product of polynomials of smaller degree. For if it were so, one of the factors would be a polynomial of lower degree with λ as a root, and this would contradict minimality of degree.

Emily: Yes, that really hits the nail on the head!

Jasmine: You know what, Emily?

Emily: What?

Jasmine: The equation $p(x) = r^d q(x/r)$ actually says that the roots of $p(x)$ are *exactly* r times the roots of $q(x)$. That is, to get the roots of $p(x)$ we just scale all the roots of $q(x)$ by the factor r.

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[1] At the end of Part 6 (see Volume 9, Number 5 of this Bulletin), Emily and Jasmine showed that $q(x)$ must divide evenly into $p(rx)/r^d$ and wrote $p(rx)/r^d = s(x)q(x)$.

Emily: Hmm. You're right. That seems awfully restrictive. I'm curious to see this at work in the case of the 30-60-90 triangle.

Jasmine: In that case, X is 90° and Y is 30°.

Emily: I think you mean X is 0° and Y is 60° … because when we moved on to considering cosines of angles instead of sines of angles, we had to switch to looking at the complements of the angles of the triangle.

Jasmine: Oh yeah, I forgot! So X is 0° and Y is 60°.

Emily: In this case, both cos X and cos Y are rational numbers themselves, so their minimum polynomials are both linear.

Jasmine: Then this case is not an interesting illustration of what we've found since the minimum polynomials each have only a single root. The isosceles case isn't interesting in this regard, either, because $X = Y$ and $r = 1$ in that instance.

Emily: Now I'm suspecting that the only examples of triangles with 3 nice angles and 2 sides of integer length are the isosceles triangles and the 30-60-90 triangle. This condition on the roots of the minimum polynomials just seems too rigid.

Jasmine: Maybe. But how do we prove it?

Emily and Jasmine sit quietly, thinking.

Jasmine: I think we need to study the minimum polynomials of cosines of nice angles.

Emily: Yes, that sounds like a good plan. Plus, it seems like a fun project in its own right. I love minimum polynomials!

Jasmine: Okay, let's do it! So our goal is to answer this: What is the minimum polynomial of cos $k\pi/n$, where k and n are integers? We can assume that k/n is in lowest terms so k and n are relatively prime and, from properties of cosine, we can assume that $0 \le k \le n$.

Emily: Yes, because we can get all possible cosines by restricting to angles between 0 and π.

Jasmine: Right.

Emily: Well, at least we know that we can find a polynomial with rational coefficients that has cos $k\pi/n$ as a root by using the Chebyshev polynomials of the first kind. If we substitute $k\pi/n$ for x in the identity $\cos(nx) = T_n(\cos x)$, we find that cos $k\pi/n$ is a root of either $T_n(x) + 1$ or $T_n(x) - 1$, depending, respectively, on whether k is odd or even.

Jasmine: That helps a lot, because then we know that the minimum polynomial of cos $k\pi/n$ *must* be a factor of $T_n(x) - \cos k\pi$.

Emily: Let's compute some minimum polynomials by factoring $T_n(x) + 1$ and $T_n(x) - 1$.

Jasmine: Okay!

Emily: We might as well start with $n = 1$. We know $T_1(x) = x$, so cos 0 is a root of $x - 1$ and cos π is a root of $x + 1$. Both of these are linear, so they're the corresponding minimum polynomials.

Jasmine: When $n = 2$, we have $T_2(x) = 2x^2 - 1$. So cos $\pi/2$ is a root of $(2x^2 - 1) + 1$, or $2x^2$. That's strange. It means that the minimum polynomial of cos $\pi/2$ is just plain old x.

Emily: Jasmine – the cosine of $\pi/2$ *is* zero!

Jasmine laughs.

Jasmine: How silly of me! Talk about a fancy way to see that the minimum polynomial of 0 is x!

Emily: And cos $\pi/3 = 1/2$, so its minimum polynomial has to be $x - 1/2$.

Jasmine: Right. That means that $x - 1/2$ must be a factor of $T_3(x) + 1$. What was $T_3(x)$? I remember we computed the first few Chebyshev polynomials already.

Jasmine searches through her notebook and finds the table of the first few Chebyshev polynomials of the first kind.

Jasmine: Here it is … $T_3(x) = 4x^3 - 3x$, so $T_3(x) + 1 = 4x^3 - 3x + 1$. In fact, 1/2 is a root of this polynomial since $4(1/2)^3 - 3(1/2) + 1 = 0$.

Emily: And the full factorization is $4x^3 - 3x + 1 = (x - 1/2)(4x^2 + 2x - 2) = 4(x - 1/2)^2(x + 1)$.

Jasmine: Let's try to find the minimum polynomial for cos $\pi/4$ by factoring $T_4(x) + 1$ instead of using the fact that cos $\pi/4 = \sqrt{2}/2$.

Emily: Okay.

Jasmine: We previously computed that $T_4(x) = 8x^4 - 8x^2 + 1$, so $T_4(x) + 1 = 8x^4 - 8x^2 + 2$. That's a quadratic in x^2, so we can use the quadratic equation to factor it.

Emily computes.

Emily: It's got a double root because its discriminant is 0. I get $8x^4 - 8x^2 + 2 = (x^2 - 1/2)^2$.

Jasmine: And $x^2 - 1/2$ isn't a product of linear factors with rational coefficients, so $x^2 - 1/2$ is the minimum polynomial of cos $\pi/4$. Indeed, $\sqrt{2}/2$ is a root of $x^2 - 1/2$.

Emily and Jasmine continue computing, finding that the minimum polynomials for both cos $\pi/5$ and cos $3\pi/5$ is $x^2 - x/2 - 1/4$, for both cos $2\pi/5$ and cos $4\pi/5$ is $x^2 + x/2 - 1/4$, for cos $\pi/6$ and cos $5\pi/6$ is $x^2 - 3/4$, for cos $\pi/7$, cos $3\pi/7$, and cos $5\pi/7$ is $x^3 - x^2/2 - x/2 + 1/8$, and for cos $2\pi/7$, cos $4\pi/7$, and cos $6\pi/7$ is $x^3 + x^2/2 - x/2 - 1/8$. Do you agree with their work?

Emily: This all seems so helter skelter.

Jasmine: Maybe we should take a step back and think about just what we're doing.

Emily: Good idea. We're trying to find the minimum polynomial for cos $k\pi/n$ by exploiting the fact that the minimum polynomial must be a factor of $T_n(x) - \cos k\pi$, where $T_n(x)$ is the nth Chebyshev polynomial of the first kind.

Jasmine: I'm trying to imagine how things might go if we did try to continue factoring these polynomials for larger and larger values of n. Factoring polynomials of high degree seems difficult in general. I don't think we'd be able to get very far unless we had some inkling as to what the factors were.

Emily: The funny thing is that we actually know what the roots of these polynomials are.

Jasmine: We do?

Emily: Yes, because the key identity for these Chebyshev polynomials is $\cos nx = T_n(\cos x)$. That identity enables us to determine all the solutions to $T_n(\cos x)$ equal to 1, -1, or $\cos z$ for any angle z.

Jasmine: I see what you're saying. We just have to solve the equation $\cos nx = \cos z$, where z is some fixed angle. This is like an exercise from trig.

Emily: Right. And the solutions are $x = z/n$, $z/n + 2\pi/n$, $z/n + 2(2\pi/n)$, $z/n + 3(2\pi/n)$, and so on, all the way up to $z/n + (n-1)(2\pi/n)$.

Jasmine: So we can write $T_n(x) - \cos z = \prod_{k=0}^{n-1} (x - \cos(z/n + 2\pi k/n))$.

Emily: Yes, except for one thing …

Jasmine: What's that?

Emily: The coefficient of the leading term of $T_n(x) - \cos z$ isn't 1, but the coefficient of the leading term on the right hand side is 1.

Jasmine: Oh yeah, good catch! I remember we figured out the lead coefficient of $T_n(x)$. It was a power of 2. Was it 2^n or 2^{n-1}?

Emily: I think it was 2^{n-1} for $n > 0$, and it is 1 if $n = 0$.

Jasmine: So, actually, $T_n(x) - \cos z = 2^{n-1}\prod_{r=0}^{n-1} (x - \cos(z/n + 2\pi r/n))$. Since we know that the minimum polynomial of $\cos 2\pi k/n$ must be a factor of $T_n(x) - \cos 0 = T_n(x) - 1$, we know that its roots must be some subset of the numbers $\{ \cos 2\pi r/n \}_{0 \le r < n}$.

Emily: Hey, that explains why we were finding so many double roots for $T_n(x) - 1$: in our factorization of $T_n(x) - \cos z$, as z goes to 0, the roots $\cos(z/n + 2\pi k/n)$ and $\cos(z/n + 2\pi(n-k)/n)$ become equal to each other since $\cos x = \cos(2\pi - x)$.

Jasmine: Emily? I just noticed something else.

Emily: What's that?

Jasmine: Suppose d divides n. Then the set of numbers $\{ \cos 2\pi s/d \}_{0 \le s < d}$ is a subset of the numbers $\{ \cos 2\pi r/n \}_{0 \le r < n}$ …

Emily: … and that means that $T_d(x) - 1$ is a factor of $T_n(x) - 1$!

Jasmine: Exactly!

Emily: That's super helpful! We can just factor out all $T_d(n) - 1$ where d divides n. What's left will be … oh, we can't just do that because some of these $T_d(n) - 1$, for various divisors of n, will share roots.

Emily and Jasmine were quick to realize the value of defining the polynomials $p_d(x)$ because they've both read Lola Thompson's article *An Euler φ-For-All* on page 8, Volume 6, Number 5 of this Bulletin.

Emily and Jasmine are both seized with a thought and look at each other wide-eyed with excitement.

Emily and Jasmine: Relatively prime!

Jasmine: Yeah! Let's see. We organize the roots $\{ \cos 2\pi k/n \}_{0 \le k < n}$ according to their denominators when k/n is written in lowest terms. So we define $p_d(x)$ to be the product of linear factors $x - \cos 2\pi k/d$ for values of k that are relatively prime to d and are between 0 and $d/2$.

Emily: Yes, at least for values of d greater than 2. We should define $p_1(x) = x - \cos 0$ and $p_2(x) = x - \cos \pi$. That is, $p_1(x) = x - 1$ and $p_2(x) = x + 1$.

Jasmine: I agree. Then, collectively, the roots of $p_d(x)$ for d dividing n will partition the roots of $T_n(x) - 1$. Since the roots of $p_d(x)$ for $d > 2$ will be double roots of $T_n(x) - 1$, we can say that

$T_n(x) - 1 = 2^{n-1} p_1(x)\left(\prod_{d|n, d>1} p_d(x)\right)^2$ if n is odd, and $T_n(x) - 1 = 2^{n-1} p_1(x) p_2(x)\left(\prod_{d|n, d>2} p_d(x)\right)^2$ if n is even.

Emily: What a beautiful factorization!

Jasmine: Wouldn't it be amazing if the minimum polynomial of $\cos(2\pi k/n)$, when k is relatively prime to n, turns out to be none other than $p_n(x)$?

Emily: It would! We know, by construction, that $\cos(2\pi k/n)$ is a root of $p_n(x)$. But even if $p_n(x)$ is the minimum polynomial of $\cos(2\pi k/n)$, how could we show it?

Jasmine: We'd have to not only show that it has rational coefficients, but also show that it can't be further factored as a product of non-constant polynomials.

Emily and Jasmine think.

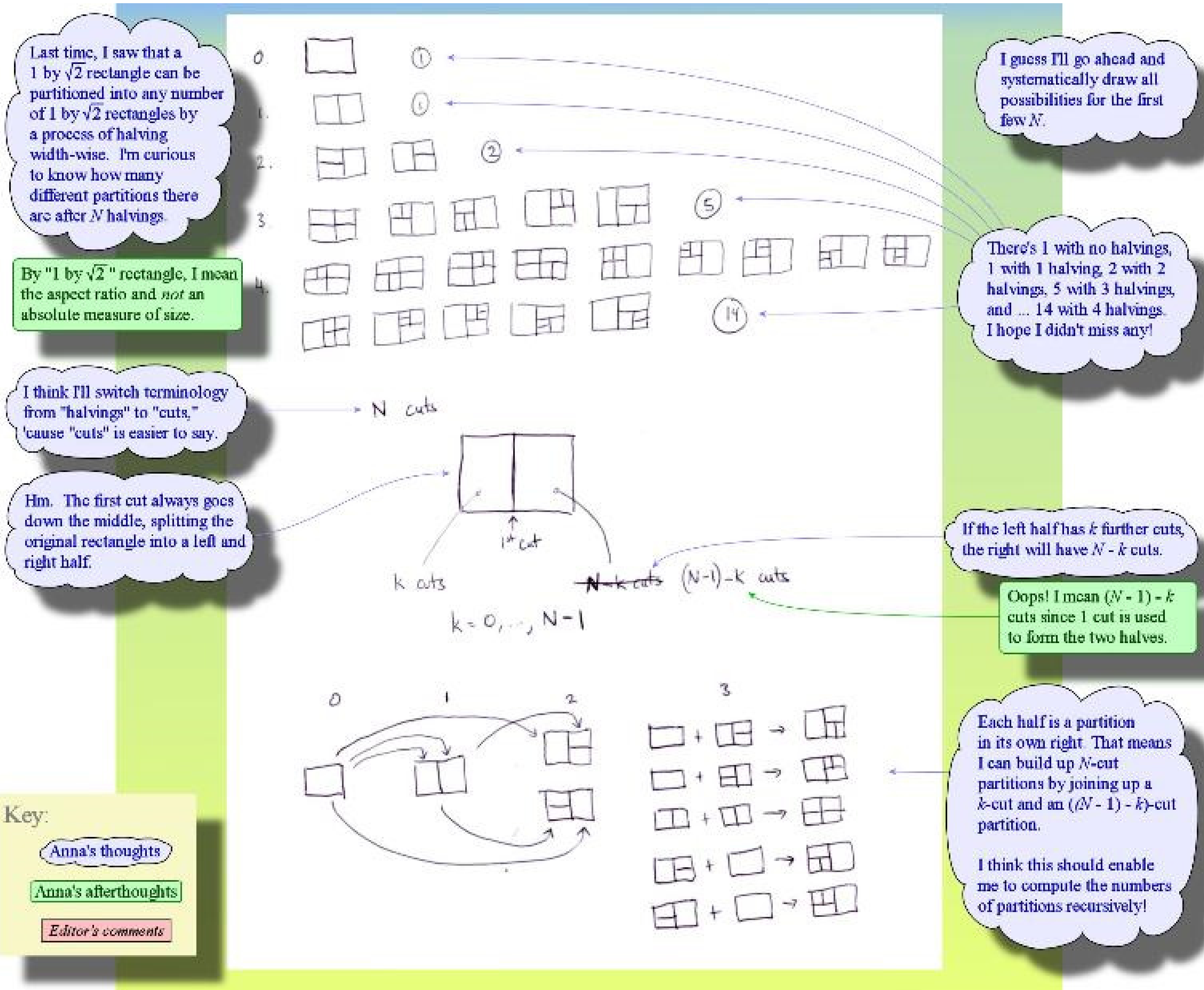
To be continued …

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Extending from last time, Anna counts special N-tilings of a 1 by $\sqrt{2}$ rectangle.



I'm thinking that the N-cut partitions can be organized into bins, where all the partitions in the same bin share the same number of cuts on their left halves and, also, on their right halves.

Let R_n be the set of patterns obtained with n cuts.

Let $R_{n,k}$ be the set of patterns in R_n where the left rectangle (formed by the 1^{st} cut) has k cuts (and the right has $n-k-1$ cuts).

I'll introduce notation to reflect this idea.

The left half of every partition must have *some* number of cuts, from 0 to $n - 1$, so every partition has to be in one of the bins $R_{n,k}$, and can only be in one of these bins, so we do have a disjoint union.

Then $R_n = \coprod_{k=0}^{n-1} R_{n,k}$

The symbol $\sqcup$, stands for "disjoint union" (as opposed to $\cup$ for "union".

Because these maps are inverse to each other, I've got a bijection.

$$R_{n,k} \underset{g}{\overset{f}{\rightleftarrows}} R_k \times R_{n-1-k}$$

The partitions in $R_{n,k}$ can be split in half to form a pair of partitions, one with k cuts, and the other with $n - 1 - k$ cuts, and vice versa. I need to fix orientations to make sure I have a well-defined map, so I'll split, then rotate each half counterclockwise 90 degrees.

Let $r_n = \# R_n$

From the bijection, I know that $r_n = r_k r_{n-1-k}$.

$$r_n = \# R_{n,k} = r_k r_{n-1-k}$$

Because the union of the $R_{n,k}$ is disjoint, the number of partitions is the sum of the numbers in each bin.

$$r_n = \sum_{k=0}^{n-1} r_k r_{n-1-k}$$

There's the recurrence relation!

n		
0		1
1	1·1	1
2	1·1 + 1·1	2
3	1·2 + 1·1 + 2·1	5
4	5·1 + 2·1 + 1·2 + 1·5	14
5	14·1 + 5·1 + 2·2 + 1·5 + 1·14	42
6	42·1 + 14·1 + 5·2 + 2·5 + 1·14 + 1·42	132

I'll use the recurrence relation to compute a few values of r_n. The first five values agree with what I found earlier.

That worked out rather nicely!

ABB 10/29/16

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Crazy Counting

by Addie Summer | edited by Jennifer Silva

Sometimes, I like to try seemingly crazy things in math and see where they lead. It's fun!

Here's an example. Recently, I counted the number of elements in some familiar sets, but not in any straightforward way. In fact, I even knew how big these sets were before I started counting! But I decided to count them in a crazy way, using the **principle of inclusion-exclusion**.

Since the principle of inclusion-exclusion (sometimes referred to as "PIE") is so critical to what I'm about to do, let me take a moment to explain how it works. The principle of inclusion-exclusion tells us how to count all the elements in a collection of sets. If the sets are disjoint, meaning that no element is contained in more than one set, all we have to do to determine the total number of elements in all the sets is to add up the numbers of elements in each set. But if some of the sets overlap, this sum would be an overestimate, because some elements would be counted more than once. The principle of inclusion-exclusion explains how to find the total number of elements even when the sets overlap.

Here's what the principle of inclusion-exclusion says:

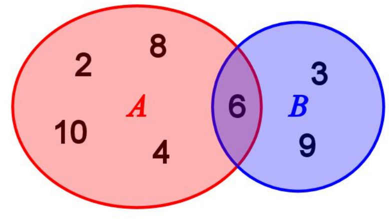
To compute the total number of elements in a collection of sets:

Add up the numbers of elements in each set,
… then subtract the numbers of elements in the intersection of every pair of sets,
… then add the numbers of elements in the intersection of every triple of sets,
… then subtract the numbers of elements in the intersection of every quadruple of sets,
… then add the numbers of elements in the intersection of every quintuple of sets,
… and so on.

To illustrate, let's count how many numbers between 1 and 10 (inclusive) are divisible by either 2 or 3 using the principle of inclusion-exclusion. Let A be the set of numbers between 1 and 10 (inclusive) that are divisible by 2. Let B be the set of numbers between 1 and 10 (inclusive) that are divisible by 3. We want to know the total number of numbers in A and B by applying the principle of inclusion-exclusion.

Here, $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9\}$.

We begin by adding the numbers of elements in each set. There are 5 elements in A and 3 in B, so we compute $5 + 3 = 8$.



A Venn diagram of the situation.

We then subtract the numbers of elements in the intersection of every pair of sets. In this example, there is only one such pair because we only have two sets. The only number in both A and B is 6. So we subtract 1 from 8 to get 7.

We then add the numbers of elements in the intersection of any 3 sets. But there aren't 3 sets, so we can stop the counting procedure here.

Using the principle of inclusion-exclusion, we find 7 numbers that are divisible by 2 or 3.

We can double-check by making a list of the numbers and counting directly. The numbers between 1 and 10 (inclusive) that are divisible by 2 or 3 are: 2, 3, 4, 6, 8, 9, and 10. Indeed, there are 7 such numbers.

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Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 19 - Meet 1 September 15, 2016	Mentors:	Sarah Coleman, Anna Ellison, Neslly Estrada, Katie Fisher, Hannah Larson, Isabel Macenka, Christine Soh, Isabel Vogt, Jane Wang

To break the ice, we played a few rounds of Round and Round, with a twist. To play Round and Round, organize participants into a big circle. Define a sequence of numbers, such as "perfect squares from 1 to 400" or "prime numbers from 2 to 101". Pick someone in the circle to start. That person must state the first term in the sequence. Participants take turns going around the circle in the clockwise direction saying the next term in the sequence. If a participant makes a mistake, she must restart the sequence with its first term. The goal for the participants is to get through the defined sequence without making a mistake.

The twist was that each member sported a number between 0 and the total number of participants, less one. These numbers did not go around the circle in any particular order. The girls had to play Round and Round computing numbers of the sequence modulo the number of participants. Also, instead of going clockwise around the circle, on a member's turn, she would figure out what term of the sequence she had to announce, then find the member showing this number, state that member's name, and then that member would go next. Organizers chose sequences that would contain every possible remainder to ensure that every participant got to go each round.

In our case, we had too many participants for this activity to work well, but members did get to know each other's names. If you'd like to try this, I'd suggest doing it only if you have fewer than 12 or so participants.

Session 19 - Meet 2 September 22, 2016	Mentors:	Bridget Bassi, Sarah Coleman, Anna Ellison, Neslly Estrada, Suzanne O'Meara, Christine Soh, Jane Wang
	Visitor:	Jane Kostick, Woodworker

Jane Kostick creates amazing woodworks that are rich with geometric detail. She has also been a frequent visitor to Girls' Angle. In this, her 5th, visit to Girls' Angle, she led the girls in an exploration of the cross sections of tetrahedrons, cubes, and rhombic dodecahedra using Play-Doh.

Using Play-Doh, she had members chop up rhombic dodecahedra along 3 mutually perpendicular planes to produce 8 congruent pieces that could be put back together to make 2 identical cubes. If you're interested in making a rhombic dodecahedron and trying this, one way to do it is by stuffing the hollow of the Kostick's Tetraxis puzzle with Play-Doh.

For more on Jane and her woodworks, see her website at kosticks.com or check out Volume 2, Number 2 and Volume 7, Number 4 of this *Bulletin*.

Session 19 - Meet 3 September 29, 2016	Mentors:	Bridget Bassi, Anna Ellison, Neslly Estrada, Isabel Macenka, Suzanne O'Meara, Christine Soh, Jane Wang

In mathematical writing, we strive for clarity without sacrificing precision. Some members have been working on developing such clarity and precision by formulating definitions of mathematical objects. They started by defining a square and eventually tried to produce a rigorous definition that captured their intuitive sense of what it means for a sequence of numbers to "tend to zero."

Here's a geometry problem that **Coral** and **GoogolInfinity** solved in an elegant way:

> Figure out a way to cut up a regular hexagon into pieces that can be rearranged to form two congruent equilateral triangles.

For their solution, see the bottom of the next page. This problem is an instance of the Bolyai-Gerwein theorem which asserts that a polygon can be dissected into a finite number of polygons that can be rearranged to form *any* other collection of polygons with the same total area.

Session 19 - Meet 4 October 6, 2016	Mentors:	Bridget Bassi, Sarah Coleman, Neslly Estrada, Katie Fisher, Hannah Larson, Isabel Macenka, Suzanne O'Meara, Christine Soh, Anuhya Vajapeyajula, Isabel Vogt

Some members managed to solve Fibonacci's 30 birds problem (for a statement of this problem, see Matthew de Courcy-Ireland's Summer Fun problem set in Volume 9, Number 5 as well as his detailed solution in Volume 9, Number 6). As a further challenge in this direction, we constructed the following "100 dogs problem":

> You used exactly 100 gold coins to purchase 100 dogs. You bought 4 types of dog: Bulldogs, Beagles, Boxers, and Bloodhounds. The following chart shows how much each type of dog costs.

Dog Type	Cost in Coins
Bulldogs	1/5
Beagles	1/4
Boxers	8
Bloodhounds	20

> You bought at least 1 of each type of dog. Coins cannot be split (so, for instance, you must have bought a multiple of 5 Bulldogs). How many of each type of dog did you get?

Can you solve it? Is the answer unique?

Some members are working on multiplication with negative numbers. We hope that eventually, members will see that it is not that "there is a correct way to multiply negative numbers that one must learn how to do," but, instead, "we create math and we can define how to multiply negative numbers as we wish, but one method was universally adopted because it proved so useful, namely, to define it so that the basic properties of arithmetic remain valid." For more, see Volume 3, Number 3 of this *Bulletin*.

Session 19 - Meet 5 Mentors: Bridget Bassi, Sarah Coleman,
October 13, 2016 Neslly Estrada, Isabel Macenka

Let $C_{n,\,k}$ denote "n choose k" and extend to cases where $n < k$ by declaring that $C_{n,\,k} = 0$ whenever $n < k$. Some members have been exploring properties of these binomial coefficients, such as this nifty identity

$$\sum_{k=0}^{n} C_{n,k} C_{k,d} = 2^{n-d} C_{n,d} .$$

Can you prove it? For a similar type of identity, see *Crazy Counting* on page 22.

Session 19 - Meet 6 Mentors: Bridget Bassi, Sarah Coleman, Anna Ellison,
October 20, 2016 Neslly Estrada, Hannah Larson, Suzanne O'Meara,
Christine Soh, Anuhya Vajapeyajula,
Isabel Vogt, Jane Wang

Here's an interesting activity: Take the product of 4 consecutive whole numbers. Add 1. Compute several examples. What do the resulting numbers have in common? Can you prove it?

Every now and then, Lunga Lee appears at the Girls' Angle club. Lunga loves functions, but she has an uncanny knack for expressing functions in a terribly long winded way. Here's an example:

INPUT: a positive integer N

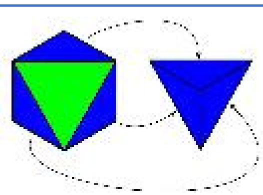
Form a sequence S_k where $S_1 = N$. The number S_{m+1} is a weighted sum of the decimal digits of S_m. The weight assigned to the digit in the 10^p place depends on the remainder you get when you divide p by 6. If the remainder is 0, the weight is 1. If the remainder is 1, the weight is 3. If the remainder is 2, the weight is 2. If the remainder is 3, the weight is 6. If the remainder is 4, the weight is 4. If the remainder is 5, the weight is 5. Got that? Find the smallest n such that S_n is a single digit. If $S_n > 6$, then let $x = S_n - 7$, otherwise let $x = S_n$.

OUTPUT: x

What is the simplest description of this function that you can come up with? For more long-windedness from Lunga Lee, see page 21 of Volume 8, Number 1 of this *Bulletin*.

Session 19 - Meet 7 Mentors: Sarah Coleman, Anna Ellison, Neslly Estrada,
October 27, 2016 Katie Fisher, Hannah Larson, Isabel Macenka,
Suzanne O'Meara, Christine Soh, Jane Wang

Coral and **GoogolInfinity** found this nifty dissection of a regular hexagon that can be used to form 2 congruent equilateral triangles.



A wild game of "Define This" (see pages 8-9 of Volume 4, Number 4 of this *Bulletin*) led to intense discussions about the nature of circles, numbers, and techniques for writing good definitions.

Calendar

Session 19: (all dates in 2016)

September	15	Start of the nineteenth session!
	22	Jane Kostick, woodworker
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	
	8	

Session 20: (all dates in 2017)

January	26	Start of the twentieth session!
February	2	
	9	
	16	
	23	No meet
March	2	
	9	
	16	
	23	
	30	No meet
April	6	
	13	
	20	No meet
	27	
May	4	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________