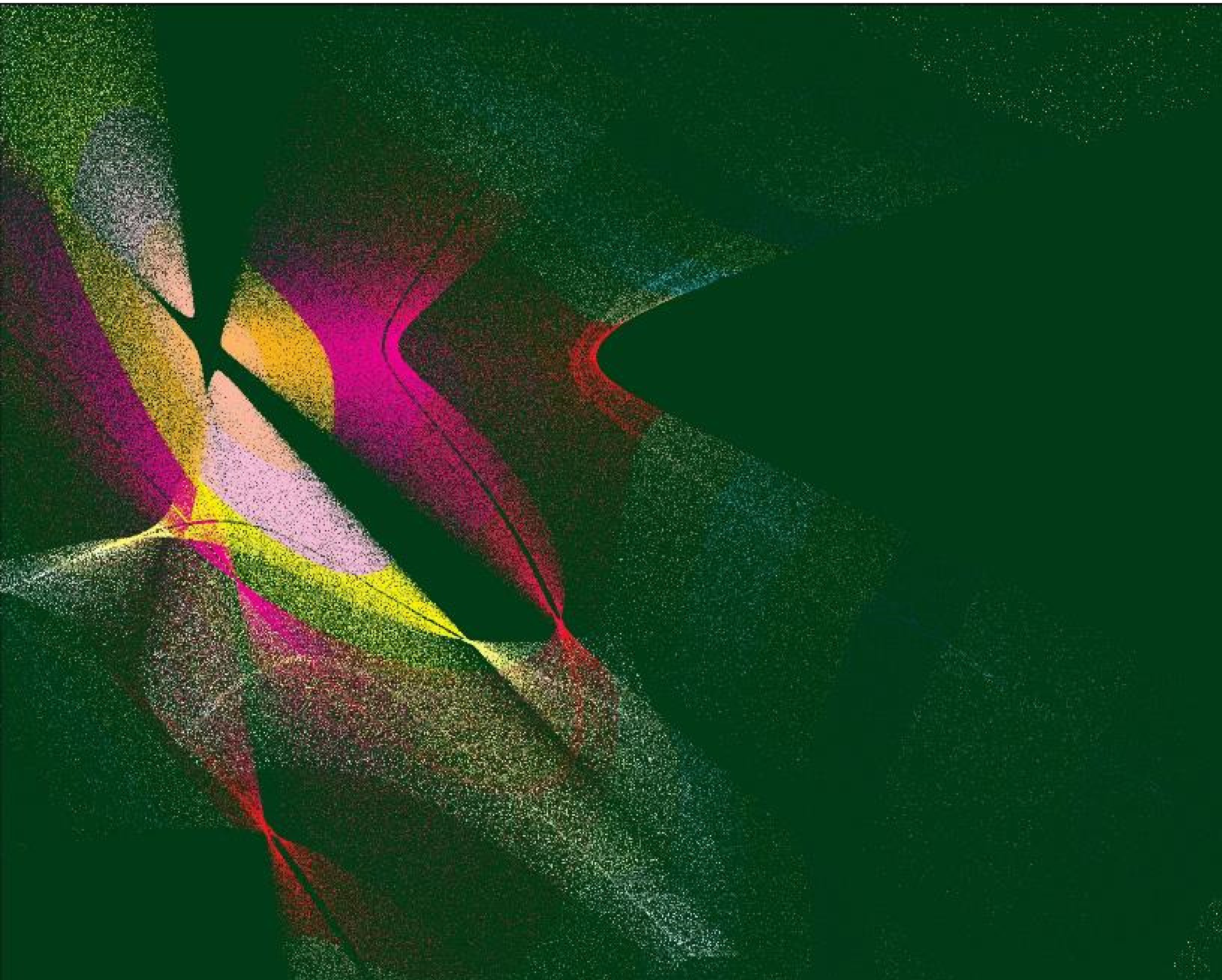
# Girls' Angle Bulletin

April/May 2017 • Volume 10 • Number 4

*To Foster and Nurture Girls' Interest in Mathematics*



# From the Founder

We are grateful to the Broad Institute for giving us a home for most of this academic year, and to MIT for providing us with space when we weren't at the Broad. Both institutions are abuzz with the excitement of discovery. They are magnificent shores for seeking prettier seashells.

- Ken Fan, President and Founder

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## Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Executive Editor: C. Kenneth Fan


## Girls' Angle:
### A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*



On the cover: Cover image produced by Dr. C.-M. Viallet (CNRS / Sorbonne Universités) depicting iterates of an integrable third-order difference equation arising from joint work with Professor Nalini Joshi (Sydney).

# An Interview with Nalini Joshi

Nalini Joshi is a professor in the School of Mathematics and Statistics at the University of Sydney where she is a Georgina Sweet Australian Laureate Fellow and the Chair of Applied Mathematics. She was born and raised in Burma (now Myanmar) and received her doctoral degree in mathematics from Princeton University.

**Ken**: How did you first become aware of mathematics?

**Nalini**: In Burma, I remember being fascinated by games that involved counting and movement, which had many, many different variations and patterns. I could play them all day long and resented being called indoors for meals. Later at school, I remember loving the process of solving mathematics problems, mostly simple ones like working out perimeters of rice paddies. I remember hearing that we would learn theorems (mainly to do with Euclidean geometry) in higher-level classes at school. I thought it was unfair that we didn't get to know what they were immediately. It was like a treasured secret being kept for later in life.

**Ken**: I was moved by Trixie Barretto's video of you (which can be seen on Vimeo). In it, you liken math to music, saying that both are emotional. Thinking back, what is the earliest recollection you have of feeling this emotion with respect to mathematics? What mathematics were you contemplating? What do you find emotional about it?

**Nalini**: Too many things happened in my early life that were unpredictable. But in math classes, the shapes of rice paddies would change and the simple formula for a rectangular perimeter would no longer apply, but you could always work out the length of the perimeter anyway! Mathematics was a landscape I could walk around in, inside which I could always find an answer. I didn't seem to belong in any of the ethnic groups where ever I ended up, but in mathematics, I always felt I belonged. So the earliest emotion I felt was a sense of security.

**Ken**: In the video, you recall how, as a youth, you would make models of the solar system and try to find answers to astronomical questions, such as, "Why does the same face of the moon always face toward us?" So you had great interest in physics. Ultimately, you chose math. What is the difference between math and physics that led you to choose math in the end?

**Nalini**: In first year physics classes at university, I found out that I had no physical intuition whatsoever and I had to guess answers, which often turned out to be wrong! My experiments either didn't work or elicited scepticism from the tutors, because I asked so many questions. In contrast, I felt like mistakes were good in mathematics because they led me to understand more things. I realised that I could take my time, try alternative beginnings, do one step after another, and get to glimpse all kinds of mathematical possibilities along the way. As I went on to higher year mathematics, I found there were problems that appeared to have more than one solution, but when I realized which solution was wrong, it was very, very satisfying. So I knew after a while that maths was right for me.

**Ken**: In the video, you said "[math is] something that you only need a little bit of work to get in through the door, and once

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Nalini Joshi and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
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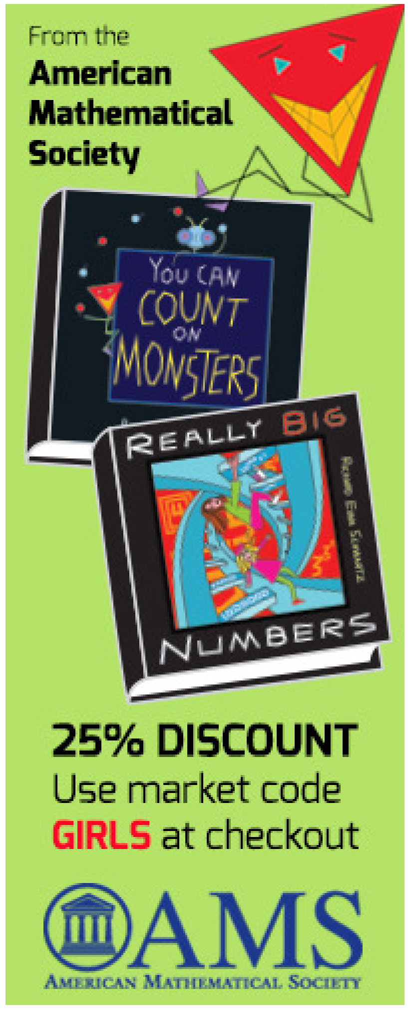



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

# In Search of Nice Triangles, Part 10

by Ken Fan | edited by Jennifer Silva

Emily and Jasmine continue their investigation into nice triangles. They've been using "nice" to denote angles that measure a rational multiple of $\pi$ radians.

They are now looking for triangles with 3 nice angles and 2 sides of integer length.

Previously, they decided to embark on a study of the minimum polynomials of the cosines of rational multiples of $\pi$. They defined the polynomials $p_d(x)$, for $d > 1$, to be the product of all linear factors of the form $x - \cos(2\pi k/d)$, where $1 \le k \le d/2$ and $(k, d) = 1$. They defined $p_1(x) = x - 1$.

They observed that, for $n$ odd,

$$T_n(x) - 1 = 2^{n-1} p_1(x)\left(\prod_{d|n, d>1} p_d(x)\right)^2,$$

and for $n$ even,

$$T_n(x) - 1 = 2^{n-1} p_1(x) p_2(x)\left(\prod_{d|n, d>2} p_d(x)\right)^2,$$

where $T_n(x)$ is the $n$th Chebyshev polynomial of the first kind.

They conjectured that the minimum polynomial of $\cos(2\pi k/n)$, where $k$ and $n$ are relatively prime, is $p_n(x)$.

As Professor Miller leaves Cake Country, several patrons follow her out and the bakery becomes a quiet temple ideal for doing math.

Emily: Professor Miller told us so much, yet I don't immediately see how it helps us with our triangle classification problem.

Jasmine: Yes, and I'm sure I didn't understand everything she said.

Emily: Same here.

Jasmine: But what I gathered is that somehow, the irreducibility of our polynomials $p_n(x)$ follows from the irreducibility of the … the …

Emily: … the "cyclotomic polynomials" …

Jasmine: Right, the cyclotomic polynomials, which are the minimum polynomials of primitive roots of unity.

Emily: Yes. Specifically, $\Phi_n(x) = \prod (x - e^{2\pi ik/n})$, where the product is taken over values of $k$ from 1 to $n$ that are relatively prime to $n$.

Jasmine: I still need to think about the proof Professor Miller showed us that these polynomials are irreducible, but I can do that later. I'm willing to accept it for now, because I really want to understand how this relates to our polynomials $p_n(x)$.

Emily: Okay, I'm with you.

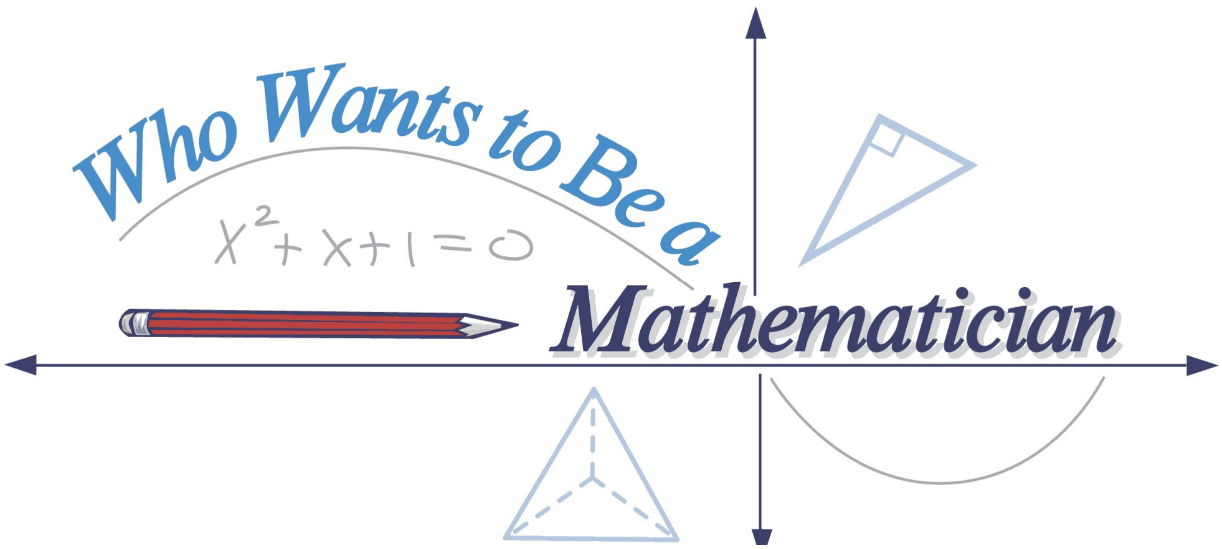
Jasmine: At least we know that $e^{2\pi ik/n} = \cos(2\pi k/n) + i \sin(2\pi k/n)$, so the roots of $\Phi_n(x)$ have some relation to the roots of $p_n(x)$.

Emily: I'm getting tired of writing $e^{2\pi ik/n}$ over and over. Can we agree to let $w = e^{2\pi i/n}$? That way, we can write $w^k$ instead of $e^{2\pi ik/n}$.

Jasmine: Sure!

Emily: We can also write $\cos(2\pi k/n) = (w^k + w^{-k})/2$, because

$$w^{-k} = \cos(-2\pi k/n) + i \sin(-2\pi k/n) = \cos(2\pi k/n) - i \sin(2\pi k/n).$$


Who Wants to Be a
$x^2+x+1=0$
Mathematician

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# Anna's Math Journal

By Anna B.

*Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.*

Anna continues counting special $N$-tilings of a 1 by $\sqrt{2}$ rectangle.

I've been trying to show that the number of special tilings of a 1 by $\sqrt{2}$ rectangle that can be obtained with $n$ cuts is one over $n + 1$ times $2n$ choose $n$ using some kind of counting argument.

*If you're unfamiliar with the concept of a generating function, check out Lauren Williams' Girls' Angle WIM video, Generating Fibonacci Numbers, which can be viewed on the Girls' Angle website.*

$r_n$ = # of special tilings of a 1 by $\sqrt{2}$ rectangle that can be obtained with $n$ cuts.

$$r_0 = 1 \qquad r_n = \sum_{k=0}^{n-1} r_k r_{n-1-k}$$

But I haven't been able to find anything. So now I'll try using generating functions.

Define $r(x) = r_0 + r_1 x + r_2 x^2 + \cdots$

$$r(x)^2 = r_0^2 + (r_0 r_1 + r_1 r_0)x + (r_0 r_2 + r_1 r_1 + r_2 r_0)x^2$$

$$+ \cdots + \left(\sum_{k=0}^{n-1} r_k r_{n-1-k}\right) x^{n-1} + \cdots$$

The recursion relation is just like the formula for the coefficients of the square of a generating function.

$$= r_1 + r_2 x + r_3 x^2 + \cdots + r_n x^{n-1} + \cdots$$

$$= \frac{r(x) - r_0}{x} = \frac{r(x) - 1}{x}$$

$$x\,r(x)^2 - r(x) + 1 = 0$$

So $r(x)$ satisfies this nice quadratic equation. I can solve for $r(x)$ using the quadratic formula.

$$r(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$$

Hm. But is it + or - root $1 - 4x$?

Key:

Anna's thoughts

Anna's afterthoughts

*Editor's comments*

I can decide on the sign by finding an expansion for the square root of 1 - $4x$. Actually, I'll expand the square root of 1 - $x$, then substitute $4x$ for $x$ in the result.

I know that Newton gave an expansion for $(x + y)^r$, where $r$ is any real number, but I couldn't recall it, so I decided to go ahead and find a Taylor expansion for this special case of Newton's "binomial theorem".

$$f(x) = \sqrt{1-x}$$

I'll compute the Taylor series. So I need to compute derivatives. Here goes!

$$f'(x) = -\frac{1}{2}(1-x)^{-1/2}$$

$$f''(x) = -\frac{1}{2^2}(1-x)^{-3/2}$$

$$f^{(3)}(x) = -\frac{3}{2}\frac{1}{2^2}(1-x)^{-5/2}$$

$$f^{(4)}(x) = -\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2^2}(1-x)^{-7/2}$$

This looks like the $k$th derivative. I can verify by differentiating and checking if I get the expression for $k + 1$.

$$f^{(k)}(x) = -\frac{(2k-3)(2k-5)\cdots 5\cdot 3\cdot 1}{2^k}(1-x)^{-\left(\frac{2k-1}{2}\right)}$$

It works!

$$\frac{d}{dx}f^{(k)}(x) = -\frac{(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}{2^{k+1}}(1-x)^{-\left(\frac{2k+1}{2}\right)} \quad \checkmark$$

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 - \frac{1}{3!}\frac{3}{2}\frac{1}{2^2}x^3 - \cdots - \frac{1}{k!}\frac{(2k-3)(2k-5)\cdots 5\cdot 3\cdot 1}{2^k}x^k - \cdots$$

Because the coefficients of $r(x)$ are positive, I have to take the minus sign in the quadratic formula.

$$r(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$= \frac{1}{4}\cdot 4 + \frac{1}{2!}\frac{1}{2^2}4^2x + \frac{1}{3!}\frac{3}{2}\frac{1}{2^2}4^3x^2 + \cdots + \frac{1}{k!}\frac{(2k-3)(2k-5)\cdots 5\cdot 3\cdot 1}{2^{k+1}}4^kx^{k-1} + \cdots$$

$$r_n = \frac{1}{(n+1)!}\frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^{n+2}}4^{n+1}$$

$$= \frac{1}{(n+1)!}\left(\frac{(2n)!}{2^n n!}\right)\frac{4^{n+1}}{2^{n+2}}$$

$$= \frac{1}{(n+1)!}\frac{(2n)!}{n!}$$

$$= \frac{1}{n+1}\cdot\frac{(2n)!}{n!\,n!} = \frac{1}{n+1}\binom{2n}{n} \quad \square$$

How about that! The conjectured formula turns out to be true!

*These numbers are known as "Catalan" numbers.*

ABB 4/25/17

Key:

Anna's thoughts

Anna's afterthoughts

*Editor's comments*

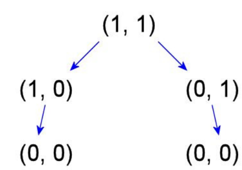
# Counting NIM, Part 1

by Milena Harned and Miriam Rittenberg

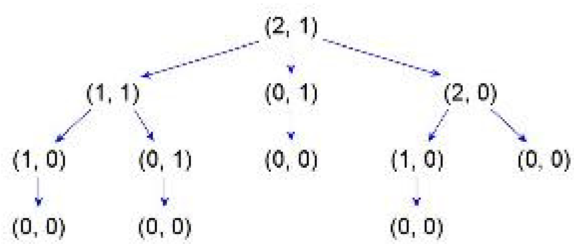
**Introduction**

We were thinking about the game of NIM. In NIM, 2 players start with any nonnegative integer number of piles of any nonnegative integer number of counters. Then, players alternate taking any positive integer number of counters from one of the piles (but they cannot take counters from more than one pile at a time). The goal is to be the person who removes the last counter. We decided to analyze the case of two piles, one with $n$ counters and one with $m$ counters.

Our first goal was to determine whether the first or second player would win if both players played optimally. But we soon switched to a different question: How many different ways to play a game of NIM are there? For example, if we start with 1 counter in each pile, we can take 1 from either pile, and then must take 1 from the other, and then we are done. This is illustrated in the diagram at right. We use the notation $(x, y)$ to indicate that there are $x$ counters in the first pile and $y$ counters in the second. So there are 2 ways to finish this game. We write this as $f(1, 1) = 2$. In general, we will use $f(n, m)$ to represent the number of ways to finish a game of NIM that starts with $n$ counters in the first pile and $m$ counters in the second pile (the piles are treated as distinct: if we have two piles with the same number of counters, taking counters from the first pile is different from taking counters from the second pile).




Here is another example: What is $f(2, 1)$?




From the diagram above, we see that $f(2, 1) = 5$.

We have found formulas for $f(n, m)$ where $n$ and $m$ are nonnegative integers. We used three different strategies. One was to find a recursive formula that connects $f(n + 1, m)$ to $f(n, m)$ and then turn it into a closed form formula for $f(n, m)$ for small values of $m$. The second was to think of the problem as putting dividers between counters. The third is similar to the second, but produces a formula valid for all nonnegative $n$ and $m$.

**First Strategy**

First, we want to find a formula for $f(n, 0)$. Starting from $(n, 0)$, for our first move, our only choice is to take away some $k$ counters from the first pile, leaving us with $(n – k, 0)$. Any value $0 < k \le n$ is an option, so we have

$$f(n, 0) = \sum_{k=1}^{n} f(n-k, 0). \qquad (1)$$

We compute

$$
\begin{aligned}
f(n, 0) &= \sum_{k=1}^{n} f(n-k,0) \\
&= \sum_{k=2}^{n} f(n-k,0) + f(n-1, 0) \\
&= \sum_{k=1}^{n-1} f(n-1-k,0) + f(n-1, 0) \\
&= f(n-1, 0) + f(n-1, 0) \\
&= 2f(n-1, 0). \qquad (2)
\end{aligned}
$$

From this and the fact that $f(1, 0) = 1$, we deduce that $f(n, 0) = 2^{n-1}$.

Now think of $m$ as a fixed positive integer. From $(n, m)$, we can start by either taking away $k$ counters from the pile of $n$, where $0 < k \le n$, leaving us with $(n - k, m)$, or we can take away $k$ counters from the pile of $m$, where $0 < k \le m$, leaving us with $(n, m - k)$. So

$$f(n, m) = \sum_{k=1}^{n} f(n-k,m) + \sum_{k=1}^{m} f(n,m-k).$$

We can now do something similar to what we did in the $m = 0$ case:

$$
\begin{aligned}
f(n, m) &= \sum_{k=1}^{n} f(n-k,m) + \sum_{k=1}^{m} f(n,m-k) \\
&= f(n-1,m) + \sum_{k=2}^{n} f(n-k,m) + \sum_{k=1}^{m} f(n,m-k) \\
&= f(n-1,m) + \sum_{k=1}^{n-1} f(n-1-k,m) + \sum_{k=1}^{m} f(n,m-k) \\
&= f(n-1,m) + f(n-1,m) - \sum_{k=1}^{m} f(n-1,m-k) + \sum_{k=1}^{m} f(n,m-k) \\
&= 2f(n-1,m) - \sum_{k=1}^{m} f(n-1,m-k) + \sum_{k=1}^{m} f(n,m-k) \qquad (3)
\end{aligned}
$$

Now we'll use (3) to find a formula for $f(n, 1)$.

$$
\begin{aligned}
f(n, 1) &= 2f(n-1,1) - \sum_{k=1}^{1} f(n-1,1-k) + \sum_{k=1}^{1} f(n,1-k) \\
&= 2f(n-1, 1) - 2^{n-2} + 2^{n-1}.
\end{aligned}
$$

Using standard techniques, we find that $f(n, 1) = (n + 3)\cdot 2^{n-2}$.

A similar computation enables us to find that

$$f(n, 2) = (n^2 + 13n + 26)\cdot 2^{n-4}$$

and, after an even longer computation,

$$f(n, 3) = (n^3/3 + 10n^2 + 215n/3 + 110)\cdot 2^{n-5}.$$

In principle, this technique can be applied sequentially to find a formula for $f(n, m)$ for any fixed $m$. In practice, the algebra gets longer quickly as $m$ rises.

**Second Strategy**

Each player in NIM will pick any positive amount of pieces from any pile on their turn. We can make a record of each game by using 3 symbols ●, ○, and |, in the following way. We use ● to stand for a token in the first pile and ○ to stand for a token in the second pile. Each time a player removes tokens, we write down a string of the appropriate token. If the next player draws from the same pile, we will place a | symbol so that we can tell that tokens from the same pile were not drawn in the same move.

For example, suppose that we have 4 tokens in the first heap and 1 token in the second heap. Then, the record ●|●○●● stands for the following sequence of moves:

1. One piece is taken from pile 1.
2. One piece is taken from pile 1.
3. One piece is taken from pile 2.
4. Two pieces are taken from pile 1.

Any string of such symbols represents a NIM game provided that a | only appears between two symbols representing tokens from the same pile. Thus, to compute $f(n, m)$ we count the number of such "record strings" that have $n$ ● symbols and $m$ ○ symbols.

**Computation of $f(n, 0)$**

Any record string will contain $n$ ● symbols and no ○ symbols. A | may be placed between any two adjacent ● symbols. There are $n - 1$ possible places for a |, and each place can contain a | or not independently of the other places, so there are $2^{n-1}$ possible ways to place | symbols into the string of $n$ ● symbols to get a record string. Thus, $f(n, 0) = 2^{n-1}$.

**Computation of $f(n, 1)$**

This time, we have a string of $n + 1$ tokens, of which $n$ are ● and one is ○. We will split into two cases depending on whether ○ is at an end of the string of token symbols or not.

Case 1: The symbol is at one end of the string.

The is either at the beginning or the end of the string. There are $n - 1$ places where we can put a |, and each place is independent of the others. Hence there are $2 \cdot 2^{n-1}$ or $2^n$ such game records.

Case 2: The symbol is not at one end of the string.

There are $n - 1$ places between adjacent ’s to place a , and for each placement, there are $n - 2$ possible places for a |. Therefore, the total number of such records is $(n - 1)\cdot 2^{n-2}$.

Adding up the results of both cases yields $f(n, 1) = 2^n + (n - 1)\cdot 2^{n-2} = (n + 3)\cdot 2^{n-2}$.

**Computation of $f(n, 2)$**

This time, we have a string of $n + 2$ tokens, of which $n$ are and two are . We will split into two cases depending on whether the two ’s appear adjacent to each other or not in the record string.

Case 1: The two ’s appear adjacent to each other.

In this case, the number of strings is $2f(n, 1)$ because when the two ’s are adjacent, they act as a unit except that we have the additional option of placing a | between them.

Case 2: The two ’s are separated by at least one .

If neither appears at the end of the record string, there are $(n - 1)(n - 2)/2$ possible ways to place the two ’s among the $n$ ’s. For each such placement, there are $2^{n-3}$ ways to place the |’s. This gives us $(n - 1)(n - 2)\cdot 2^{n-4}$ record strings.

If exactly one of the symbols is at an end of the record string, there are $n - 1$ ways to place the other so that it isn’t at the end of the record string. For each arrangement of ’s and ’s, there are $2^{n-2}$ ways to place the |’s. This gives us $2 \cdot 2^{n-2} \cdot (n - 1) = (n - 1)\cdot 2^{n-1}$ record strings.

Finally, there’s only one way to put both of the symbols at the ends of the record string, and there are $2^{n-1}$ ways to add | symbols giving us an additional $2^{n-1}$ record strings.

Thus, $f(n, 2) = 2f(n, 1) + (n - 1)(n - 2)\cdot 2^{n-4} + (n - 1)\cdot 2^{n-1} + 2^{n-1} = (n^2 + 13n + 26)\cdot 2^{n-4}$.

**Computation of $f(n, 3)$**

This time, we have a string of $n + 3$ tokens, of which $n$ are and three are . We again split into cases, as organized in the following table.

| Configuration of | Location of | Number of Ways to Place \|'s | Number of Record Strings |
|---|---|---|---|
| 3-in-a-row | At one end of the record string | $2^{n+1}$ | $2^{n+2}$ |
| | None at the ends of the record string | $2^n$ | $(n-1)\cdot 2^n$ |
| 2 together, 1 apart | Two at the ends of the record string | $2^n$ | $2^{n+1}$ |
| | One at the end of the record string | $2^{n-1}$ | $(n-1)\cdot 2^{n+1}$ |
| | None at the ends of the record string | $2^{n-2}$ | $(n-1)(n-2)\cdot 2^{n-2}$ |
| All 3 separated by at least one | Two at the ends of the record string | $2^{n-2}$ | $(n-1)\cdot 2^{n-2}$ |
| | One at the end of the record string | $2^{n-3}$ | $(n-1)(n-2)\cdot 2^{n-3}$ |
| | None at the ends of the record string | $2^{n-4}$ | $(n-1)(n-2)(n-3)\cdot 2^{n-5}/3$ |

Adding these together, we find $f(n, 3) = (n^3/3 + 10n^2 + 215n/3 + 110)\cdot 2^{n-5}$.

In principle one could use this technique to compute $f(n, m)$ for any fixed positive integer $m$, but the cases grow rapidly with $m$.

**Looking ahead**

In the Part 2, we'll present a general formula for $f(n, m)$ based on a way to organize the counting similar to our second strategy and use the formula to determine the asymptotic behavior of $f(n, m)$ for fixed $m$.

# Art and the Harmonic Mean

by Ken Fan
edited by Jennifer Silva

Logo Design by Hana Kitasei

You find yourself at the end of a long, pleasant, palm tree-lined avenue. You're inspired to make a drawing. On paper, the palm trees need to be drawn smaller and smaller as they recede into the distance. But how, exactly, should the trees' heights shrink from one to the next? And how should they be spaced?

Should the trees be drawn evenly spaced, as they are in real life? Should each successive tree be drawn half the size of the previous tree? In this installment of "Math In Your World," we'll determine the truth.

To gain a foothold into the problem, we'll analyze an idealized scenario: palm trees that are equally spaced and all of the same height. In fact, we'll even think of the palm trees as vertical sticks. Simplifying in this way is often a good strategy for developing understanding. After we establish a basic understanding, we can tweak it to fit real world variations in spacing and palm tree height.

**Perspective Drawing**

If you've studied perspective drawing, you know that drawings of parallel lines look like radial lines all emanating from a single point called a **vanishing point** – unless the parallel lines happen to be parallel to the drawing paper, in which case they'll be rendered as parallel lines in the drawing, too. You can find the exact location of the vanishing point as follows: place your eye where you'd ideally like the viewer to place hers when she looks at your finished drawing, stare in the direction of the parallel lines you're about to draw, and note what point in the plane of the drawing paper you are looking directly upon. In other words, the vanishing point of parallel lines is the point where the unique line that is parallel to these lines and passes through the viewer's eye intersects the plane of your drawing paper. (Note that if the parallel lines are also parallel to the drawing paper, there won't be such an intersection point. This is consistent with the fact that the drawing of such lines would also be parallel in the picture.) Fairly often, the vanishing point will be far off to one side of the drawing paper, but that's okay. Don't force vanishing points into the bounds of your paper if they don't belong there, unless you truly want people to admire your drawing from a nose's length away.

If you haven't studied perspective drawing, check out the "Summer Fun!" problem set on page 21 of Volume 6, Number 5, and the one on page 25 of Volume 3, Number 5 of this *Bulletin*. Also, take a look at **Rowena'**s perspective drawing on the cover of Volume 3, Number 4, along with her article on page 29 of the same issue.

There is a line that passes through the tops of our idealized stick trees. There's also a line that passes through their feet, and these lines are parallel. Let's sketch these lines into our

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# A Fibonacci Related Sequence

by Milena Harned

This paper is inspired by Problem 8 on the 2002 AIME I math contest which went as follows:

> Find the smallest integer $k$ for which the conditions
>
> (1) $a_1, a_2, a_3, \ldots$ is a nondecreasing sequence of positive integers
>
> (2) $a_n = a_{n-1} + a_{n-2}$ for all $n > 2$
>
> (3) $a_9 = k$
>
> are satisfied by more than one sequence.

In this paper, we shall extend this to any nondecreasing sequence of positive integers $F_n$ that satisfies a recurrence relation of the form

$$F_{n+1} = aF_n + bF_{n-1},$$

for $n > 1$ and where $a$ and $b$ are relatively prime positive integers. Let's call such sequences "Fibonacci-like". We will also determine the answer for any index, not just the index 9.

Consider a Fibonacci-like sequence $F_n$ with recurrence relation $F_{n+1} = aF_n + bF_{n-1}$ that begins $F_1 = x$ and $F_2 = y$. (Note that $x \le y$ are positive integers by our definition of "Fibonacci-like".) Let us write out the first few values of this sequence:

$$x,\ y,\ ay + bx,\ (a^2 + b)y + abx,\ (a^3 + 2ab)y + (a^2b + b^2)x,\ \ldots$$

Notice that each term is a linear combination of $x$ and $y$. By the recurrence relation, we can see that all terms will be a linear combination of $x$ and $y$, so we define $x_n$ and $y_n$ to be the coefficients in these linear combinations, that is, $F_n = x_nx + y_ny$. By the recurrence relation, we have

$$\begin{aligned} x_{n+1} &= ax_n + bx_{n-1} \\ y_{n+1} &= ay_n + by_{n-1} \end{aligned}$$

for $n > 1$. Also note that $x_n$ begins $1, 0, b, \ldots$, whereas $y_n$ begins $0, 1, \ldots$. Since both satisfy the same linear recurrence relation, we see that

$$(*) \qquad x_n = by_{n-1}, \text{ for } n > 1.$$

Because of (*), we focus attention on $y_n$. The first few terms of $y_n$ are

$$0,\ 1,\ a,\ a^2 + b,\ a^3 + 2ab,\ a^4 + 3a^2b + b^2,\ \ldots$$

Each term is a polynomial in $a$ and $b$. Let's write down the coefficients of these polynomials in order of decreasing degree:

| $y_n$ | **Coefficients** | | | | |
|---|---|---|---|---|---|
| 0 | 0 | | | | |
| 1 | 1 | | | | |
| $a$ | 1 | | | | |
| $a^2 + b$ | 1 | 1 | | | |
| $a^3 + 2ab$ | 1 | 2 | | | |
| $a^4 + 3a^2b + b^2$ | 1 | 3 | 1 | | |
| $a^5 + 4a^3b + 3ab^2$ | 1 | 4 | 3 | | |
| $a^6 + 5a^4b + 6a^2b^2 + b^3$ | 1 | 5 | 6 | 1 | |
| $a^7 + 6a^5b + 10a^3b^2 + 4ab^3$ | 1 | 6 | 10 | 4 | |
| $a^8 + 7a^6b + 15a^4b^2 + 10a^2b^3 + b^4$ | 1 | 7 | 15 | 10 | 1 |

We can see that this is a misshapen version of Pascal's triangle, which would suggest the following formula for $y_n$:

(**)
$$y_n = \sum_{r=1}^{[n/2]} \binom{n-1-r}{r-1} a^{n-2r} b^{r-1}\,.$$

If true, by (*), we would have $x_{n+1} = \sum_{r=1}^{[n/2]} \binom{n-1-r}{r-1} a^{n-2r} b^{r}$ for all $n > 0$ and $x_1 = 1$.

We will use induction on $n$ to prove (**).

First, note that the formula (**) is true for $n = 1$ and 2, thereby establishing the base cases.

Now assume that (**) holds for $y_{n-1}$ and $y_n$. We must show that

$$ay_n + by_{n-1} = \sum_{r=1}^{[(n+1)/2]} \binom{n-r}{r-1} a^{n+1-2r} b^{r-1}\,,$$

or, substituting the formulas for $y_n$ and $y_{n-1}$,

$$\sum_{r=1}^{[n/2]} \binom{n-1-r}{r-1} a^{n-2r+1} b^{r-1} + \sum_{r=1}^{[(n-1)/2]} \binom{n-2-r}{r-1} a^{n-1-2r} b^{r} = \sum_{r=1}^{[(n+1)/2]} \binom{n-r}{r-1} a^{n+1-2r} b^{r-1}\,.$$

We shall check that the coefficients of like terms are the same on both sides of this equation.

The highest degree term on both sides is $a^{n-1}$. For lower degree terms, we split into cases depending on whether $n$ is even or odd.

Case 1. Suppose $n$ is even.

Pick $r > 1$. The coefficient of $a^{n+1-2r}b^{r-1}$ is $\binom{n-1-r}{r-1}+\binom{n-1-r}{r-2}$ on the left hand side and $\binom{n-r}{r-1}$ on the right hand side. The fact that $\binom{n-1-r}{r-1}+\binom{n-1-r}{r-2} = \binom{n-r}{r-1}$ is the well-known Pascal's triangle recursion formula.

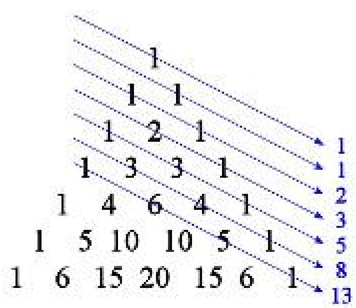
Case 2. Suppose $n$ is odd.

If $r = [(n + 1)/2]$, the coefficient of $b^{r-1}$ is $\binom{n-1-r}{r-2}$ on the left hand side and $\binom{n-r}{r-1}$ on the right hand side. To see these are equal, let $n = 2m + 1$. Then $r = m + 1$ and we can see that $\binom{n-1-r}{r-2} = \binom{m-1}{m-1} = 1$ and $\binom{n-r}{r-1} = \binom{m}{m} = 1$.

Now pick $1 < r \le [n/2]$. The coefficient of $a^{n+1-2r}b^{r-1}$ is $\binom{n-1-r}{r-1}+\binom{n-1-r}{r-2}$ on the left hand side and $\binom{n-r}{r-1}$ on the right hand side, which are equal by the Pascal's triangle recursion formula.

These computations prove the inductive step and (**) follows.

Notice that if $a = b = 1$, then $y_n$ yields the Fibonacci numbers and the formula (**) recovers the famous way to get the Fibonacci numbers from Pascal's triangle illustrated below.




We now turn our attention to generalizing the AIME problem stated at the beginning. Fix two positive integers $a$ and $b$ and assume $a$ and $b$ are relatively prime. (The reason why we insist that $a$ and $b$ be relatively prime will become clear shortly.) Let $m_{n,a,b}$ be the smallest integer for which there exist two different Fibonacci-like sequences $\{ F_i \}_{i=1,2,3,\ldots}$ and $\{ G_i \}_{i=1,2,3,\ldots}$ that satisfy the same recurrence relations $F_{n+1} = aF_n + bF_{n-1}$ and $G_{n+1} = aG_n + bG_{n-1}$, for $n > 1$, and $m_{n,a,b} = F_n = G_n$.

We shall find a formula for $m_{n,\,a,\,b}$, but first, we show that $(x_n, y_n) = 1$ for all $n > 0$ by induction on $n$. The first two terms of $x_n$ and $y_n$ are 1, 0 and 0, 1, respectively, and the only common divisor of 0 and 1 is 1. This establishes our base case.

Now assume that $(x_n, y_n) = 1$ for some $n > 1$. We must show that $(x_{n+1}, y_{n+1}) = 1$. Suppose $p$ is a prime that divides into both $x_{n+1}$ and $y_{n+1}$. Since $x_{n+1} = by_n$, we must have that $p$ divides $b$ or $p$ divides $y_n$. If $p$ divides $b$, then from the recurrence relation $y_{n+1} = ay_n + by_{n-1}$, we deduce that $p$ must also divide $ay_n$. Since $(a, b) = 1$, it must be that $p$ divides $y_n$. Since $x_n = by_{n-1}$ and $p$ divides $b$, we also see that $p$ divides $x_n$. This contradicts the inductive hypothesis. Thus $p$ cannot divide $b$ and we must have that $p$ divides $y_n$. From the recurrence relation $y_{n+1} = ay_n + by_{n-1}$, we deduce that $p$ must then divide $by_{n-1}$. Since $p$ does not divide $b$, it must be that $p$ divides $y_{n-1}$. However, $x_n = by_{n-1}$, so if $p$ divides $y_{n-1}$, it must divide $x_n$, and, once again, we find that $x_n$ and $y_n$ are both divisible by $p$, a contradiction to the inductive hypothesis. We conclude that no prime divides both $x_{n+1}$ and $y_{n+1}$, and, hence, $(x_{n+1}, y_{n+1}) = 1$. This establishes the inductive step.

Thus, $(x_n, y_n) = 1$ for all $n > 0$ by induction.

Because $x_n$ and $y_n$ are relatively prime, all integer solutions to an equation of the form

$$x_n s + y_n t = c,$$

where $c$ is a constant, will have the form $s = s_0 + my_n$ and $t = t_0 - mx_n$, where $s_0$ and $t_0$ are integers that satisfy $x_n s_0 + y_n t_0 = c$ and $m$ is an integer.

By definition of $m_{n,\,a,\,b}$, there exists positive integers $x \le y$ and $u \le v$ such that

$$m_{n,\,a,\,b} = x_n x + y_n y = x_n u + y_n v,$$

where $(x, y) \ne (u, v)$ (as ordered pairs), and no smaller integer has this property. Without loss of generality, assume $x < u$. From the previous paragraph, it must be that $u = x + my_n$ for some positive integer $m$ and $v = y - mx_n$. Since we only insist that $m_{n,\,a,\,b}$ belong to two different Fibonacci-like sequences, there is no harm in assuming that $m = 1$.

Since $u \le v$, we must have $x + y_n \le y - x_n$. That is, $x + x_n + y_n \le y$.

Because both $x_n$ and $y_n$ are positive, to minimize $x_n x + y_n y$ subject to the conditions that

$$0 < x \text{ and } x + x_n + y_n \le y,$$

we must take $x = 1$ and $y = 1 + x_n + y_n$.

Therefore, $m_{n,\,a,\,b} = x_n(1) + y_n(1 + x_n + y_n) = (x_n + y_n)(1 + y_n)$.

In the case $a = b = 1$, we find $m_{n,\,1,\,1} = F_{n+1}(1 + F_n)$, where $F_n$ is the Fibonacci sequence that begins $F_1 = 0$, $F_2 = 1$. The first few terms of $m_{n,\,1,\,1}$ are: 1, 2, 4, 9, 20, 48, 117, 294, 748, ….

# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

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| | | |
|---|---|---|
| Session 20 - Meet 5<br>March 2, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Jessica Dyer,<br>Anna Ellison, Kate Fisher, Isabel Macenka,<br>Jennifer Matthews, Suzanne O'Meara,<br>Christine Soh, Anuhya Vajapeyjula, Jane Wang |

Many members finished solving the self-referential true/false quiz on page 24 of the previous issue. Some members complete the self-referential multiple choice quiz created by Jim Propp, and some of these members went on to create their own self-referential tests.

Other members studied how graphs of different quadratic functions relate to one another and how these differences can be seen in the algebra. As it turns out, just as all circles are similar to each other, so are all parabolas. (See, for instance, the Girls' Angle blog post here: girlsangle.wordpress.com/2011/08/24/circles-squares-and-parabolas/.)

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| | | |
|---|---|---|
| Session 20 - Meet 6<br>March 9, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Jessica Dyer,<br>Neslly Estrada, Katie Fisher, Suzanne O'Meara,<br>Christine Soh, Jane Wang |

We made a deal with members: If they learned about partially ordered sets and Hasse diagrams, we would apply these concepts to chocolate preferences by hosting a chocolate tasting. Several members began by making Hasse diagrams of the partially ordered set of positive integers ordered by divisibility and the set of subsets of a finite set ordered by inclusion.

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| | | |
|---|---|---|
| Session 20 - Meet 7<br>March 16, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Anna Ellison,<br>Neslly Estrada, Isabel Macenka, Jennifer Matthews,<br>Suzanne O'Meara, Christine Soh |

Exploration of the Hasse diagram of the set of positive integers ordered by divisibility morphed into an exploration of prime and composite numbers. For instance: If the Hasse diagram is restricted to the set of positive integers less than or equal to $n > 1$, is the maximum size of an antichain equal to the number of prime numbers less than or equal to $n$? If not, what is? (An antichain is a subset of elements in the partially ordered set such that no two elements are comparable.)

Some members decided to list all the primes less than 100, and after completing the list, took note of the sizes of the gaps between primes. This led to the question: Can you find 9 consecutive composite numbers? More generally, for any $n$, does there exists $n$ consecutive composite numbers? (Note that a string of $2m$ consecutive positive integers, where $m$ is a positive integer, will always be part of a string of $2m + 1$ consecutive positive integers. Why?)

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| | | |
|---|---|---|
| Session 20 - Meet 8<br>March 23, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Jessica Dyer,<br>Anna Ellison, Jennifer Matthews, Suzanne O'Meara,<br>Christine Soh, Jane Wang |

Some members have been exploring properties of Pascal's triangle and various interpretations of the numbers in it. However, all this time, they had been generating Pascal's triangle recursively, starting at the top of the triangle. So for the Pascal's triangle explorers, the question of the day was: What is an efficient way to compute the $n$th row of Pascal's triangle? Is there a way to do it quickly without having to compute all the entries above the desired row?

Other members worked on finding a formula for the number of ways $2n$ people can form themselves into $n$ tennis doubles teams.

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| | | |
|---|---|---|
| Session 20 - Meet 9<br>April 6, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Isabel Macenka,<br>Jennifer Matthews, Suzanne O'Meara, Jane Wang |

A number of members studied perspective drawing and applied their knowledge to the construction of a perspective drawing of a checkerboard, like the one seen in the floor of Vermeer's *The Art of Painting*. In a two-point perspective drawing of a checker board, how does one decide where the two vanishing points should be located? The answer depends on from where you desire your drawing to viewed.

Two members produced the numbers 114, 115, 116, …, 126 as a sequence of 13 consecutive composite numbers. This sequence is, in fact, the earliest occurrence of 13 consecutive (positive) composite numbers. The first time more than 13 consecutive positive numbers occurs is at 524, according to sequence A100964 at the Online Encyclopedia of Integer Sequences.

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| | | |
|---|---|---|
| Session 20 - Meet 10<br>April 13, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Neslly Estrada,<br>Isabel Macenka, Christine Soh, Jane Wang |

As members perfected their perspective drawings of checkerboards, the question arose: How do the sizes of identical objects appear to shrink as they recede into the distance if they are equally spaced, like telephone poles? One member suggested that each successive drawn copy should be halved in size. Can you figure out the correct answer? If not, see this issue's *Math In Your World* on page 19.

Some members continued their focus on prime numbers solving problems such as:

> The product of 4 distinct whole numbers is equal to 729. What are the numbers?
>
> How many triples of positive integers $a \le b \le c$ are there such that $abc = 100$? In general, how can you count the number of triples of positive integers $a \le b \le c$ such that their product $abc$ is equal to a given number?

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| | | |
|---|---|---|
| Session 20 - Meet 11<br>April 27, 2017 | Mentors: | Bridget Bassi, Sarah Coleman, Jessica Dyer,<br>Anna Ellison, Neslly Estrada, Katie Fisher,<br>Isabel Macenka, Suzanne O'Meara, Christine Soh,<br>Jane Wang, Anuhya Vajapeyajula |

We applied the concept of partially ordered sets and Hasse diagrams to a chocolate tasting. Members sampled a number of different chocolates in a semi-blind taste test. Members could see the chocolate, but, for the most part, could not identify the brand or model. They created Hasse diagrams of their chocolate preferences.

 

# Calendar

Session 20: (all dates in 2017)

| | | |
|---|---|---|
| January | 26 | Start of the twentieth session! |
| February | 2 | |
| | 9 | |
| | 16 | |
| | 23 | No meet |
| March | 2 | |
| | 9 | |
| | 16 | |
| | 23 | |
| | 30 | No meet |
| April | 6 | |
| | 13 | |
| | 20 | No meet |
| | 27 | |
| May | 4 | |

Session 21: (all dates in 2017)

| | | |
|---|---|---|
| September | 7 | Start of the twenty-first session! |
| | 14 | |
| | 28 | |
| October | 5 | |
| | 12 | |
| | 19 | |
| | 26 | |
| November | 2 | |
| | 9 | |
| | 16 | |
| | 24 | Thanksgiving - No meet |
| | 30 | |
| December | 7 | |

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

# Girls' Angle: A Math Club for Girls

## Membership Application

**Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.**

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: _____________________________________________________________________

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

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The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

**A Math Club for Girls**

# Girls' Angle
# Club Enrollment

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors strive to get members to do math through inspiration and not assignment. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

**How do I enroll?** You can enroll by filling out and returning the Club Enrollment form.

**How do I pay?** The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

**Where are Girls' Angle meets held?** Girls' Angle meets take place near Kendall Square in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, Founder and Director, The Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton Univeresity
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, The Dartmouth Institute
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, assistant professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, associate professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

 

# Girls’ Angle: Club Enrollment Form

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ______________________________________________________________________

Address: ____________________________________________________________ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

**Emergency contact name and number**: ______________________________________________________________________

**Pick Up Info**: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

______________________________________________________________________________________________

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you’d like us to know about?

______________________________________________________________________________________________

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter’s name in any way. Do we have permission to use your daughter’s image for these purposes? **Yes** **No**

**Eligibility:** Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls’ Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

**Personal Statement (optional, but strongly encouraged!):** We encourage the participant to fill out the optional personal statement on the next page.

**Permission:** I give my daughter permission to participate in Girls’ Angle. I have read and understand everything on this registration form and the attached information sheets.

___________________________________________________ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ___________________________________________________________________

| Members: Please choose one. | Nonmembers: Please choose one. |
|---|---|
| ☐ Enclosed is $216 for one session (12 meets) | ☐ I will pay on a per meet basis at $30/meet. |
| ☐ I will pay on a per meet basis at $20/meet. | ☐ I’m including $50 to become a member, and I have selected an item from the left. |

☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

**Personal Statement (optional, but strongly encouraged!):** This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

## Girls' Angle: A Math Club for Girls

**Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

______________________________________________________________________________,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____________________________________________Date: ________________

Print name of applicant/parent: ____________________________________________

Print name(s) of child(ren) in program: ____________________________________