Girls' Angle Bulletin

June/July 2017 • Volume 10 • Number 5

To Foster and Nurture Girls' Interest in Mathematics

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From the Founder

More than about math, Girls' Angle is about *doing* math. The Bulletin aims to inspire you to do math, show you how math is done, and provide you with opportunities to do math, such as in this summer's batch of Summer Fun problem sets. As Courtney Gibbons writes in *Errorbusters!*, "The way to understanding math is easy to describe ... *Do the math*!"

- Ken Fan, President and Founder

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Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Homage to* Φ_{24} by C. Kenneth Fan. (Φ_{24} is the 24th cyclotomic polynomial. To learn about cyclotomic polynomials, see the Summer Fun problem set on page 20.)

An Interview with Ruth Charney

Ruth Charney is the Theodore and Evelyn Berenson Professor of Mathematics at Brandeis University. She received her doctoral degree in mathematics from Princeton University and has held numerous leadership positions in mathematics including President of the Association for Women in Mathematics and serving on the Board of Trustees of the American Mathematical Society and the Mathematical Sciences Research Institute. She is a Fellow of the American Mathematical Society.

Ken: Please explain to us something in mathematics that caught your interest when you were school-aged.

Ruth: I was always fascinated by infinity. My parents tell the story of me at a very young age sitting in the backseat of the car on a road trip. At some point, I asked my Dad what the largest number was. He told me that there was no largest number. I thought about that for a while, then said, “OK, then what’s the next to largest number?”

Eventually, of course, I got my mind around the concept of infinity. Then in my freshman year of college, I learned that there could be different “sizes” of infinity. In fact, there is no largest infinity! That blew my mind. I had always liked math, but I think that’s when I realized that I loved it.

Ken: What is geometric group theory? What fascinates you about the subject? Would you please explain some of the fundamental questions that geometric group theory addresses?

… you don’t need to be perfect, you don’t need to be the “best” or “fastest” in your class, you just need to love solving problems and have the perseverance to tackle the really challenging ones.

Ruth: Math can be used for practical purposes, like modelling real world phenomena. But it can also be used to stimulate our imagination. That’s the part of math that has always intrigued me most. It’s like art … it makes you see things in a new way.

In high school geometry we mostly study the geometry of a flat plane. But the earth and the universe are not flat. Studying the geometry of more varied shapes (we call them metric spaces) is both interesting and useful for many applications. Let’s say you live in some strange shaped universe and you need to get from point A to point B. How do you go about figuring out the most efficient way to get there? Are there multiple “best routes” or just one? What do triangles in this universe look like? Do their angles always sum to 180 degrees? Do two straight paths that start off in the same direction stay parallel?

That’s the geometric part of “geometric group theory” but the subject also involves algebraic objects called groups. A group consists of a set with an operation such as addition or multiplication or composition. Some groups can be described as symmetries of a geometric object. For example, the symmetry group of a hexagon is the set of rotations and flips that preserve the shape of the hexagon. However, many other groups arise very abstractly and are

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Ruth Charney and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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President and Founder
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Counting NIM, Part 2

by Milena Harned and Miriam Rittenberg

In the first half, we introduced the function $f(n, m)$ which gives the total number of ways that a NIM game with two piles, one containing n counters and the other containing m counters, can be played out. We gave explicit formulas for $f(n, m)$ for fixed, small values of m (see Table 1 below for a summary). In this concluding half, we give a general formula for $f(n, m)$ when $n \geq 1$ and examine its form.

Assume throughout that $n \geq 1$.

General Formula

m	$f(n, m)$
0	2^{n-1}
1	$(n + 3)\cdot 2^{n-2}$
2	$(n^2 + 13n + 26)\cdot 2^{n-4}$
3	$(n^3/3 + 10n^2 + 215n/3 + 110)\cdot 2^{n-5}$

Table 1. Explicit formulas for $f(n, m)$ for fixed $m = 0, 1, 2$, and 3.

If we have n pieces in one of the piles, there are $\binom{n-1}{a-1}$ ways to split the n pieces into a groups. We can think of this as putting $a - 1$ dividers into $n - 1$ possible spots between the n pieces. Similarly, there are $\binom{m-1}{b-1}$ ways to split a pile of m pieces into b groups.

Now we have one pile split into a groups and one pile split into b groups and we would like to know how many orders we can put them in. To count this, we can imagine that there are $a + b$ possible spots, and we need to choose a of these spots to place the a groups from the first pile. The b groups from the second pile will fill the remaining b spots. So there are $\binom{a+b}{a}$ ways to do this.

We want to sum over all the possible numbers (a and b) of groups that the two piles can be divided into. The pile of n pieces can be divided into anywhere from 1 to n groups and the pile of m pieces can be divided into anywhere from 1 to m groups.

Putting this all together, our general formula is

$$f(n, m) = \sum_{a=1}^{n}\sum_{b=1}^{m}\binom{n-1}{a-1}\binom{m-1}{b-1}\binom{a+b}{a}.$$

Note that while this is a general formula that works for any n and m, our previous formulas were easier to calculate with, especially for large n. This is because our previous formulas were in polynomial form and had a fixed number of terms, while this general formula has a number of terms that increases as n and m increase.

Analysis of the General Formula

We wondered if, given a fixed m, the function $f(n, m)$ took a consistent form. Observing our formulas for $f(n, 0)$, $f(n, 1)$, $f(n, 2)$, and $f(n, 3)$ (see Table 1 above), we noticed that for a fixed m, the function $f(n, m)$ looked like a polynomial in n multiplied by 2 raised to a power that depended on n and m.

More specifically, we're going to prove the following proposition.

Proposition. Let $f(n, m)$ denote the total number of ways a NIM game can be played out if it consists of two piles of n and m counters, respectively. Then

$$f(n, m) = \frac{2^{n-m-1}}{m!} \cdot P(n), \tag{1}$$

where $P(n)$ is a polynomial of degree m with lead coefficient 1 and integer coefficients.

We begin by proving two helpful facts.

Helpful Fact 1

Our first helpful fact is the following identity:

$$\binom{N}{M} \cdot 2^{N-M} = \sum_{i=0}^{N} \binom{N}{i}\binom{i}{M}. \tag{2}$$

where N and M are fixed nonnegative integers. This is a known formula, but we provide a proof here anyway.

Notice that from the binomial formula, we have that

$$(1 + x)^N = \sum_{i=0}^{N} \binom{N}{i} \cdot x^i .$$

Differentiating M times with respect to x, we get

$$N \cdot (N-1) \cdot \cdot \cdot (N - M + 1) \cdot (1 + x)^{N-M} = \sum_{i=0}^{N} \binom{N}{i} \cdot (i) \cdot (i-1) \cdots (i - M + 1) \cdot x^{i-M} .$$

Dividing both sides by $M!$, we have that

$$\binom{N}{M} \cdot (1+x)^{N-M} = \sum_{i=0}^{N} \binom{N}{i} \cdot \binom{i}{M} \cdot x^{i-M} .$$

Plugging in 1 for x, we arrive at the desired identity (2).

Later, we will apply this first helpful fact in the form

$$\binom{N-1}{M} \cdot 2^{N-M-1} = \sum_{i=1}^{N} \binom{N-1}{i-1} \cdot \binom{i-1}{M}. \tag{3}$$

which is obtained from the helpful fact by substituting in $N - 1$ for N and reindexing the summation.

Helpful Fact 2

Now, we move on to our second helpful fact, which is that if $P(a)$ is a polynomial with integer coefficients in the variable a of degree m, then we can write

$$P(a) = c_0\binom{a-1}{0}+c_1\binom{a-1}{1}+\ldots+c_m\binom{a-1}{m}, \tag{4}$$

where the c_i are integers. Furthermore, c_i is divisible by $i!$.

We prove this fact using induction on m.

Editor's note: Milena and Miriam's second helpful fact is related to a special case of a more general fact we believe was first stated and proven by George Pólya and Alexander Ostrowski in separate 1919 papers both entitled "Über ganzwertige Polynome in algebraischen Zahlkörpern". One consequence of Pólya and Ostrowski's result is that any polynomial that maps integers to integers can be written as an integer linear combination of binomial coefficients. Note that the set of polynomials that map integers to integers is larger than the set of polynomials with integer coefficients.

For our base case, suppose that $P(a)$ is a polynomial of degree 0. Then $P(a)$ is a constant polynomial. Say $P(a) = c_0$. Then

$$P(a) = c_0\binom{a-1}{0}.$$

For the inductive step, assume the result for polynomials of degree less than m. Let

$$P(a) = k_m a^m + k_{m-1}a^{m-1} + \ldots + k_1 a + k_0,$$

where the k_i are integers. We know that $\binom{a-1}{m} = \frac{a^m + q(a)}{m!}$, where $q(a)$ is a polynomial in a of degree $m-1$ and has integer coefficients. Multiplying by $c_m = k_m \cdot m!$ (which is an integer divisible by $m!$), we have

$$c_m \cdot \binom{a-1}{m} = k_m \cdot m! \cdot \binom{a-1}{m} = k_m a^m + k_m q(a).$$

We can subtract this from $P(a)$ to get

$$P(a) - c_m \cdot \binom{a-1}{m} = k'_{m-1}a^{m-1} + \ldots + k'_1 a + k'_0,$$

for some integers k'_i. By induction, we can write

$$k'_{m-1}a^{m-1} + \ldots + k'_1 a + k'_0 = c_0\binom{a-1}{0}+c_1\binom{a-1}{1}+\ldots+c_{m-1}\binom{a-1}{m-1},$$

where the c_i are integers such that c_i is divisible by $i!$. Thus,

$$P(a) = c_0\binom{a-1}{0}+c_1\binom{a-1}{1}+\ldots+c_{m-1}\binom{a-1}{m-1}+c_m\binom{a-1}{m},$$

where c_i is an integer divisible by $i!$, as desired.

Proof of the Proposition

We now move on to the proof of our proposition. Fixing a value of m, we have

$$f(n, m) = \sum_{a=1}^{n}\binom{n-1}{a-1}\sum_{b=1}^{m}\binom{m-1}{b-1}\binom{a+b}{a}.$$

We notice that for fixed a,

$$\sum_{b=1}^{m}\binom{m-1}{b-1}\binom{a+b}{a}$$

is a polynomial in a of degree m. This is because for each b, $\binom{m-1}{b-1}$ is a constant and the degree of $\binom{a+b}{a}$ is b as a polynomial in a. So the leading term of the whole sum comes from the highest value of b, which is m, and the whole sum has degree m. Furthermore, when we multiply this polynomial by $m!$, the coefficients are all integers. This is true because $\binom{m-1}{b-1}$ is an integer and

$$\binom{a+b}{a} = \binom{a+b}{b} = \frac{(a+b)(a+b-1)\cdots(a+1)}{b!}.$$

The numerator has integer coefficients and the denominator always divides evenly into $m!$ since $b \leq m$. Furthermore, the leading coefficient in the numerator is equal to 1.

So we have that

$$f(n, m) = \frac{1}{m!}\sum_{a=1}^{n}\binom{n-1}{a-1}\cdot \text{(degree } m \text{ polynomial in } a \text{ with integer coefficients, lead coefficient 1).}$$

Using our second helpful fact (4), we have that

$$f(n, m) = \frac{1}{m!}\sum_{a=1}^{n}\binom{n-1}{a-1}\left(c_0\binom{a-1}{0} + c_1\binom{a-1}{1} + \ldots + c_m\binom{a-1}{m}\right)$$

where c_i is an integer divisible by $i!$. Furthermore, in this case, by comparing lead coefficients, we see that $c_m = m!$. We rewrite this as

$$\begin{aligned} f(n, m) &= \frac{1}{m!}\sum_{a=1}^{n}\binom{n-1}{a-1}\sum_{i=0}^{m}c_i\binom{a-1}{i} \\ &= \frac{1}{m!}\sum_{i=0}^{m}c_i\sum_{a=1}^{n}\binom{n-1}{a-1}\binom{a-1}{i}. \end{aligned}$$

By our first helpful fact in the form of identity (3), we have

$$f(n, m) = \frac{1}{m!}\sum_{i=0}^{m} c_i \binom{n-1}{i} \cdot 2^{n-i-1}.$$

Since c_i is divisible by $i!$, we can replace the c_i by the integers $c'_i \equiv c_i/i!$. Note that $c'_m = 1$. Then,

$$\begin{aligned} f(n, m) &= \frac{1}{m!}\sum_{i=0}^{m} c'_i \cdot i! \cdot \binom{n-1}{i} \cdot 2^{n-i-1} \\ &= \frac{2^{n-m-1}}{m!}\sum_{i=0}^{m} c'_i \cdot i! \cdot \binom{n-1}{i} \cdot 2^{m-i}. \end{aligned}$$

The expression $\binom{n-1}{i}$ has the form $\frac{n^i + r(n)}{i!}$, where $r(n)$ is a polynomial in n of degree $i - 1$ and has integer coefficients. Thus, $c'_i \cdot i! \cdot \binom{n-1}{i} \cdot 2^{m-i}$ is a polynomial in n with integer coefficients, and, hence, so is $P(n) \equiv \sum_{i=0}^{m} c'_i \cdot i! \cdot \binom{n-1}{i} \cdot 2^{m-i}$. The highest power of n that we have in the summation is m, coming from the $i = m$ term. Therefore, $f(n, m)$ is a polynomial in n of degree m with lead coefficient $\frac{1}{m!} \cdot 2^{n-m-1}$. We conclude that

$$f(n, m) = \frac{2^{n-m-1}}{m!} P(n),$$

where $P(n)$ is a polynomial in n with integer coefficients, degree m, and lead coefficient 1.

Conclusion

This is a good illustration of how thinking about a problem in many different ways can yield various insights and solutions. Our initial analyses produces formulas $f(n, m)$ for m small that were easy to compute but hard to generalize for higher m. Thinking about the problem in a different way, we found a general formula for $f(n, m)$ that works for all m but was difficult to compute with. However, we were able to find nice patterns in the formulas for fixed m.

m	$2^{m+1-n} \cdot m! \cdot f(n, m)$
0	1
1	$3 + n$
2	$26 + 13n + n^2$
3	$330 + 215n + 30n^2 + n^3$
4	$5496 + 4350n + 851n^2 + 54n^3 + n^4$
5	$113160 + 104314n + 25835n^2 + 2365n^3 + 85n^4 + n^5$

Errorbusters!

by Courtney Gibbons / edited by Jennifer Silva

One of the most tempting – and pernicious – mathematical errors isn't really a math issue so much as an attitude issue. When I'm in a rush to solve a problem, the last thing I want to see is something that is difficult to simplify. Wouldn't life be easier if we just substituted $\sqrt{a}+\sqrt{b}$ for $\sqrt{a+b}$? Or $\log a + \log b$ for $\log(a + b)$?

Life might feel easier in the moment, but incorrect substitutions like this actually make life harder, and, in some cases, impossible! I think of the error of wantonly substituting $f(a) + f(b)$ for $f(a + b)$ as an **error of apathy**. It's not that you really *believe* that $(x + y)^2$ is equivalent to $x^2 + y^2$. It's just easier to write down something that sounds good instead of spending time thinking through how to properly expand $(x + y)^2$. In my own experience, this error happens when I don't really understand the mathematics I'm doing, or when I don't care about it. It's worse than being careless; it's being mentally lazy about really understanding something.

To avoid errors of apathy, strive for understanding!

The way to understanding math is easy to describe, but requires effort to practice: *Do the math*! The effort is worth it, and you'll often be rewarded by a vision of mathematical beauty.

Let me illustrate with the logarithm, which, I'll admit, took me a long time to understand or care about. When I find myself looking at an expression of the form $\log(a + b)$ and wishing that I could simply replace it with $\log a + \log b$, I try to stop myself and think about what I'm doing.

Yes, it's syntactically tempting to assert (incorrectly) that $\log(a + b) = \log a + \log b$. It's especially tempting if you don't have an understanding of what logs are. So let's stop and think about this. The mathematical expression $\log a = n$, where we understand the base of the logarithm to be 10, means $10^n = a$. That's by definition. So "$\log a = n$" is a mathematically concise way of saying "n is the power to which you need to raise 10 to get a." Because of this relationship, logarithms are stand-ins for exponents. If you are comfortable with properties of exponents – particularly the identity $10^x \cdot 10^y = 10^{x+y}$ – then you can use that comfort to appreciate how logarithms work. The identity $10^x \cdot 10^y = 10^{x+y}$ tells us that the power I must raise 10 to in order to get the product $10^x \cdot 10^y$ is $x + y$ (which makes sense because the total number of 10's being multiplied together in the product $10^x \cdot 10^y$ is $x + y$). If I set $a = 10^x$ and $b = 10^y$, we can see that this is the same as saying that to get ab, I need to raise 10 to the $x + y$ power. But $x = \log a$ and $y = \log b$, by definition, so $x + y = \log a + \log b$. In other words, the power to which I need to raise 10 to get ab, which, by definition, is $\log(ab)$, is also $\log a + \log b$. In other words, the true "sum of logs" identity is

$$\log(ab) = \log a + \log b.$$

The relationship $\log a + \log b = \log(ab)$ is interesting and useful because it captures a property of exponentiation. Exploring the log laws from the point of view of the exponential relationships they represent can really help guard against logarithmic errors of apathy!

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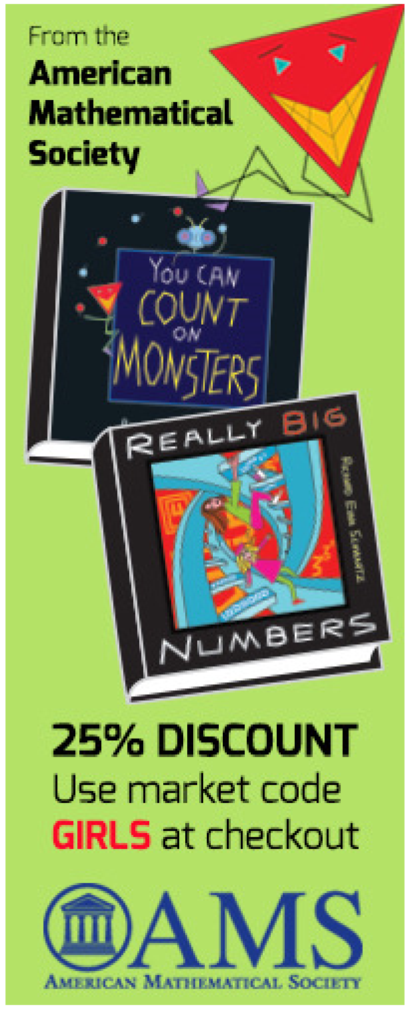

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In Search of Nice Triangles, Part 11

by Ken Fan | edited by Jennifer Silva

Emily and Jasmine continue their investigation into nice triangles. They've been using "nice" to denote angles that measure a rational multiple of π radians.

Previously, they decided to embark on a study of the minimum polynomials of the cosines of rational multiples of π. They defined the polynomials $p_d(x)$, for $d > 1$, to be the product of all linear factors of the form $x - \cos(2\pi k/d)$, where $1 \le k \le d/2$ and $(k, d) = 1$. They defined $p_1(x) = x - 1$.

They observed that, for n odd,

$$T_n(x) - 1 = 2^{n-1}\, p_1(x)\left(\prod_{d|n, d>1} p_d(x)\right)^2,$$

and for n even,

$$T_n(x) - 1 = 2^{n-1}\, p_1(x)\, p_2(x)\left(\prod_{d|n, d>2} p_d(x)\right)^2,$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind.

They showed that the minimum polynomial of $\cos(2\pi k/n)$, where k and n are relatively prime, is $p_n(x)$.

They aim to compute the constant terms of $p_n(x)$ in the hopes that doing so will enable them to determine all triangles with 3 nice angles and 2 sides of integer length.

They're also using $\Phi_n(x)$ to denote the nth cyclotomic polynomial, which is the minimum polynomial of a primitive nth root of unity.

Emily: Hopefully, working with cyclotomic polynomials will prove to be more manageable than the work we did with the minimum polynomials of cosines of nice angles.

Jasmine: We're after the constant terms of $p_n(x)$, that is, we want to find $p_n(0)$ for all n. We know that

$$\Phi_n(x) = (2x)^{\varphi(n)/2} p_n((x + x^{-1})/2),$$

for $n > 2$. So finding $p_n(0)$, at least for $n > 2$, is equivalent to evaluating $\Phi_n(x)$ at a solution to the equation $(x + x^{-1})/2 = 0$.

Emily and Jasmine separately solve $(x + x^{-1})/2 = 0$.

Jasmine: I got $x = \pm i$.

Emily: I agree. So $\Phi_n(i) = (2i)^{\varphi(n)/2} p_n(0)$, for $n > 2$. Instead of finding $p_n(0)$, we can try to find $\Phi_n(i)$.

Jasmine: Let's get to work! We'll need the identity $x^n - 1 = \prod_{d|n} \Phi_d(x)$ a lot. Since the product is over divisors of n, it would seem sensible to compute $\Phi_n(i)$ for n with few divisors, starting with the prime numbers and then gradually adding more and more prime factors.

Emily: That seems prudent, and we might as well start with $n = 1$! Since $\Phi_1(x) = x - 1$, we have $\Phi_1(i) = i - 1$. Now let's compute $\Phi_n(i)$ for prime numbers n. The first prime number is 2 and $\Phi_2(x) = x + 1$, so $\Phi_2(i) = i + 1$.

Jasmine: From our main identity, $x^3 - 1 = \Phi_3(x)\Phi_1(x)$, so $\Phi_3(x) = (x^3 - 1)/(x - 1)$. When we substitute i for x, we get $\Phi_3(i) = (i^3 - 1)/(i - 1) = (-i - 1)/(i - 1) = i$. In general, for primes p, we have $x^p - 1 = \Phi_p(x)\Phi_1(x)$, so $\Phi_p(x) = (x^p - 1)/(x - 1)$.

Emily: Since $i^4 = 1$, we know that i^p only depends on the remainder of p after dividing by 4. For odd primes, there are only two cases: either $p = 1 \pmod 4$ or $p = 3 \pmod 4$. If $p = 1 \pmod 4$, then $i^p = i$ and $\Phi_p(i) = (i - 1)/(i - 1) = 1$. If $p = 3 \pmod 4$, then $\Phi_p(i) = \Phi_3(i) = i$.

Jasmine: So far, so good! Let's try to compute $\Phi_n(i)$ when n is a product of two primes.

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Summer Fun!

The best way to learn math is to do math, so here are the 2017 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems on your own.

Some problems are quite a challenge and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can feel frustrating to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Cyclotomic Polynomials

by Girls' Angle Staff

This Summer Fun problem set is designed to introduce cyclotomic polynomials, the polynomials that Emily and Jasmine have been studying in recent installments of *In Search of Nice Triangles*. We assume familiarity with complex numbers and know Euler's formula $e^{ix} = \cos x + i \sin x$, where $i = \sqrt{-1}$.

A **root of unity** is a complex number w that satisfies $w^n = 1$ for some positive integer n. If $w^n = 1$, then we say that w is an ***n*th root of unity**.

1. Show that the solutions to the equation $x^n = 1$ are the complex numbers $e^{2\pi ik/n}$, where k is an integer that satisfies $0 \le k < n$.

A **primitive *n*th root of unity** is a complex number w that satisfies $w^n = 1$, but does not satisfy $w^m = 1$ for any positive integer m less than n.

2. Show that -1 is the unique primitive square root of unity. Show that the primitive cube roots of unity are $\frac{-1 \pm i\sqrt{3}}{2}$. Show that the primitive 4^{th} roots of unity are i and $-i$.

3. Suppose w is an nth root of unity. Show that w is a primitive nth root of unity if and only if the set $\{ w^k \mid 0 \le k < n \}$ constitutes a complete set of nth roots of unity.

4. Show that $e^{2\pi i/n}$ is a primitive nth root of unity.

5. Suppose w is a primitive nth root of unity. Show that w^k is a primitive nth root of unity if and only if k and n are relatively prime.

6. Show that the number of primitive nth roots of unity is equal to the number of positive integers less than n that are relatively prime to n. In other words, the number of primitive nth roots of unity is equal to $\varphi(n)$, where $\varphi(x)$ is the Euler φ-function.

For more on the Euler φ-function, check out Lola Thompson's article, *An Euler φ-For-All*, on page 8 of Volume 6, Number 5 of this Bulletin.

The ***n*th cyclotomic polynomial $\Phi_n(x)$** is the unique monic polynomial whose roots are the primitive nth roots of unity. (A polynomial is **monic** if its lead coefficient is 1.) That is,

$$\Phi_n(x) = \prod_{\substack{0 \le k < n \\ (k,n)=1}} (x - e^{2\pi ik/n}).$$

7. What is the degree of $\Phi_n(x)$?

Summer Fun!

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Sets

by Debbie Seidell

A **set** is a collection of distinct objects. The objects of a set are called its **elements**. For example, the numbers 2, 5, and 11 are objects, and they can be collected together to form a set, which is written {2, 5, 11}. It doesn't matter what order we list the elements of a set when we write it down, so {2, 5, 11} and {11, 2, 5} represent the same set. In general, two sets are considered the same if every element of each set is also an element of the other.

> The notation "$x \in S$" means that x is an element of the set S.

The empty collection of objects, that is, a collection consisting of no objects at all, is considered a set. It is called the **empty set** and is denoted Ø.

Sometimes when we present a set, we simply list all of its elements between curly braces. Other times, we use this nifty shorthand:

$$\{ 2n \mid n \text{ is an integer } \}.$$

This is read, "the set of all numbers of the form $2n$ such that n is an integer." The elements of this set happen to be the even integers. The vertical bar, "|", means "such that".

Let S and T be sets. We say that T is a **subset** of S if and only if every element of T is an element of S. This is denoted $S \subseteq T$. Notice that Ø is a subset of every set.

1. List all the subsets of {1, 2, 3}. (Hint: there are 8 subsets.)

2. Let n be a positive integer. Let $S = \{ k \mid k$ is an integer that satisfies $1 \le k \le n \}$. How many subsets does S have?

3. Let $S = \{ x \mid x$ is a real number$\}$. Let $T = \{ 2x \mid x$ is a real number$\}$. How do S and T relate?

Let S and T be sets. The **union** of S and T is the set consisting of all elements that are in S or T and is denoted $S \cup T$. The **intersection** of S and T is the set consisting of all elements that are in both S and T and is denoted $S \cap T$.

4. Let $S = \{ -2, -1, 0, 1, 2 \}$. Let $T = \{1, 2, 3, 4, 5\}$. Let $W = \{ 3n \mid n$ is an integer $\}$. For every pair of these 3 sets, determine their union and their intersection.

In problems 5-12, the variables S, T, and W represent sets.

5. Show that $S \cup T = T \cup S$. Show that $S \cap T = T \cap S$.

6. When does $S \cap T = S$?

7. When does $S \cup T = S$?

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8. Suppose that $S \cap T = \emptyset$ and $T \cap W = \emptyset$. Is it true that $S \cap W = \emptyset$? If it is, explain why. If not, produce a counterexample.

9. Determine all triples of sets S, T, and W such that $S \cap T \cap W = \emptyset$ and $S \cup T \cup W = \{1, 2, 3\}$.

10. Show that $S \cap (T \cup W) = (S \cap T) \cup (S \cap W)$.

11. Show that $S \cup (T \cap W) = (S \cup T) \cap (S \cup W)$.

12. Suppose that S and T are both subsets of W. For any subset X of W, we define the complement of X in W to be the set of elements that are in W but not in X and denote it X^c. Show that $(S \cup T)^c = S^c \cap T^c$ and $(S \cap T)^c = S^c \cup T^c$.

In problems 13-16, the set S_a stands for the set consisting of all multiples of a. That is,

$$S_a = \{ \, an \mid n \text{ is an integer} \, \}.$$

13. What is $S_3 \cap S_{12}$?

14. What is $S_{12} \cap S_{18}$?

15. Let a and b be positive integers. What is $S_a \cap S_b$? Notice that $S_a \cap S_b = S_c$ for some c. How does c related to a and b?

16. Let a and b be positive integers. What is the largest integer c such that $S_a \cup S_b \subseteq S_c$?

17. Let a be a positive integer. Let $P_a = \{ \, p \mid p \text{ is a (positive) prime factor of } a\}$ and let $D_a = \{ \, d \mid d \text{ is a (positive) factor of } a\}$. For what a is the number of subsets of P_a equal to the number of elements in D_a?

18. Given a finite set S, let $| \, S \, |$ denote the number of elements in S. Let S and T be finite sets. Show that $| \, S \cup T \, | = | \, S \, | + | \, T \, | - | \, S \cap T \, |$.

19. Let S be a finite set. Suppose C is a collection of subsets of S with the following properties:

i. The empty set is not in C.
ii. The intersection of any two subsets in C is also in C.
iii. For any subset of S, either it or its complement (in S) is in C.

Show that there exists an element x in S such that C consists precisely of the subsets of S that contain x.

20. Do you think the conclusion of Problem 19 is still true when S is not a finite set?

Summer Fun!

The Fourth Dimension

by Girls' Angle Staff

Modern mathematicians routinely work with high-dimensional spaces. We are accustomed to think in 3D because we inhabit an apparently 3D world, with 3 mutually perpendicular directions: forward-backward, left-right, and up-down. In four dimensions, there are four mutually perpendicular directions and phenomena that challenge our intuition.

This Summer Fun problem set is designed to help you develop 4D intuition. In fact, there are many different 4D spaces that mathematicians work with. Here, we will be concerned with **Euclidean 4D-space**, which is a generalization to 4 dimensions of the space in which the geometry of the ancient Greeks was done. We will use the security of coordinates to orient ourselves.

Euclidean 2D-Space

The Euclidean plane is perhaps the most widely acknowledged geometric space. We can specify a point in the plane by giving an ordered pair of real numbers (x, y) and declare that the distance between two points (x_1, y_1) and (x_2, y_2) is given by the Pythagorean-inspired Euclidean distance:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

In this plane, the sets $\{\ (x, 0) \mid x \text{ is a real number}\ \}$ and $\{\ (0, y) \mid y \text{ is a real number}\ \}$ are called axes and they are perpendicular to each other.

For Problems 1-2, pick 3 real numbers a, b, and c with a and b not both equal to zero. Let L be the set of points (x, y) that satisfy the equation $ax + by = c$.

1. Convince yourself that L is a straight line in the Euclidean plane. If you are having trouble, try this problem for specific values of a, b, and c, such as $a = 1$, $b = 0$, and $c = 1$.

2. Show that L splits the Euclidean plane into two halves H_1 and H_2, where H_1 consists of the points (x, y) that satisfy $ax + by > c$ and H_2 consists of the points (x, y) that satisfy $ax + by < c$. Show that any line that passes through a point in H_1 and a point in H_2 must intersect L.

3. Show that the points (1, 0), (0, 1), (-1, 0), and (0, -1) are the vertices of a square.

Euclidean 3D-Space

To get Euclidean 3D-space, we consider ordered triples of real numbers (x, y, z) and define the Euclidean distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) to be

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\ .$$

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For Problems 3-4, pick 4 real numbers *a*, *b*, *c*, and *d* with *a*, *b*, and *c* not all equal to zero. Let *P* be the set of points (x, y, z) that satisfy the equation $ax + by + cz = d$.

3. Convince yourself that *P* is a plane in Euclidean 3D-space.

4. Show that *P* splits Euclidean space into two halves and that any line that passes through a point in one half and a point in the other half must intersect *P*.

5. The *z*-axis consists of the points $(0, 0, z)$ and is a line in Euclidean space. Unlike in the plane, a line does not split space into two halves. Convince yourself that any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) not on the *z*-axis can be connected by a path that avoids the *z*-axis.

6. Show that the points (±1, 0, 0), (0, ±1, 0), and (0, 0, ±1) are vertices of a regular octahedron, which is a polyhedron with 8 equilateral triangular faces and 12 edges all of the same length.

Euclidean 4D-Hyperspace

Euclidean 4D-hyperspace consists of all ordered 4-tuples of real numbers (x, y, z, w) together with the Euclidean distance, which gives the distance between the two points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2}$.

For Problems 7-8 pick 5 real numbers *a*, *b*, *c*, *d* and *e* with *a*, *b*, *c*, and *d* not all equal to zero. Let *H* be the set of points (x, y, z, w) that satisfy the equation $ax + by + cz + dw = e$.

7. Convince yourself that *H* is a Euclidean 3D-space. It is called a **hyperplane**.

8. Show that *H* splits Euclidean 4D hyperspace into two halves and that any line that passes through a point in one half and a point in the other half must intersect *P*.

9. The set of points $P = \{ (0, 0, z, w) \mid z \text{ and } w \text{ are real numbers} \}$ is a Euclidean plane inside Euclidean hyperspace. A plane will split a 3D space into two disconnected halves, but this is no longer the case in hyperspace. Let (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) be two points not on *P*. Show that you can go from (x_1, y_1, z_1, w_1) to (x_2, y_2, z_2, w_2) along a path that avoids *P*.

10. Consider the 8 points (±1, 0, 0, 0), (0, ±1, 0, 0), (0, 0, ±1, 0), and (0, 0, 0, ±1) in Euclidean hyperspace. Show that these 8 points are the vertices of a hypersolid with 16 regular tetrahedrons as "hyperfaces", 32 equilateral triangular faces, and 24 edges all of the same length. This hypersolid has various names, including **4D cross-polytope**, **16-cell**, and **hexadecachoron**.

11. The **hypercube** or **tesseract** is a generalization of the cube to four dimensions. The 16 points (±1, ±1, ±1, ±1) can be taken as its vertices (all possible choices of sign are used). How many edges and 3D faces does a tesseract have?

12. Generic cross-sections (intersections with hyperplanes) of a tesseract are 3D solids. What solids occur as cross-sections of a tesseract?

Summer Fun!

Unsuspected Geometry

by Matthew de Courcy-Ireland

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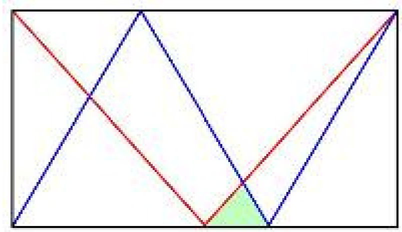
Notes from the Club

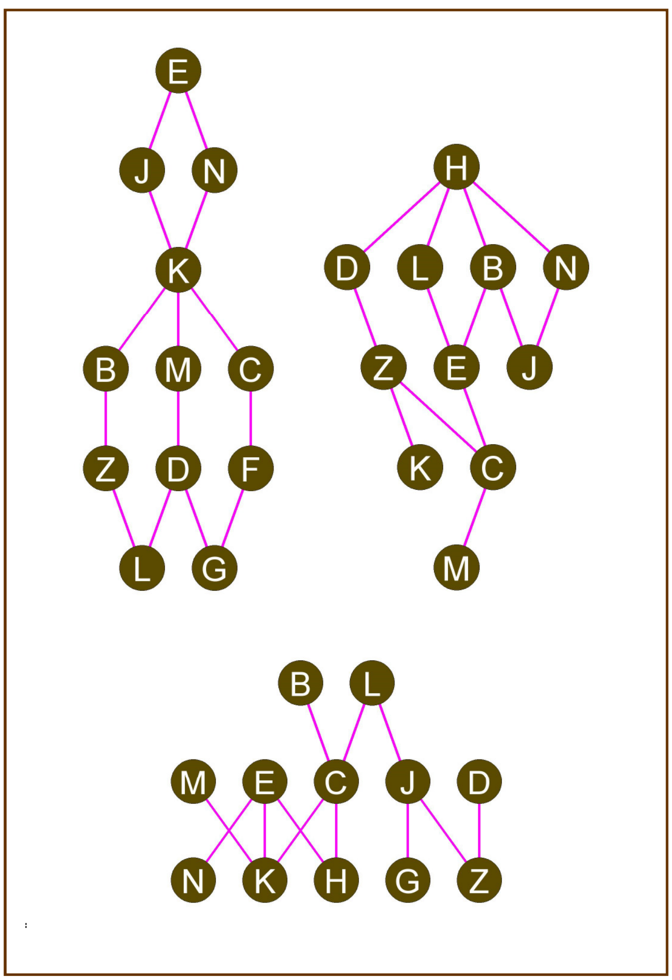
These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 20 - Meet 12 May 4, 2017	Mentors:	Bridget Bassi, Anna Ellison, Neslly Estrada, Jennifer Matthews, Suzanne O'Meara, Christine Soh, Jane Wang

We held our traditional end-of-session Math Collaboration! Here are some problems from the event. Can you solve them?

Two zig-zags are drawn across a rectangle as shown. The red one goes from the upper left corner, to the midpoint of the lower edge, to the upper right corner. The blue one goes from the lower left corner, to a point one-third of the way along the upper edge, to a point two-thirds of the way along the bottom edge, to the upper right corner. The green region has an area of 1.25 square units. What is the area of the rectangle?





Let S be the convex hull of the points (0, 0, 0), (5, 0, 0), (0, 3, 4), (5, 3, 4). Describe what S is in the most concise way that you can.

Make a perspective drawing of a tiled floor, where the tiling represents a tessellation by regular hexagons. The perspective cannot be one of looking straight down upon the floor or edge-on to the floor and must include at least 7 complete tiles.

Consider the partially ordered set consisting of the numbers from 1 to 10, inclusive, ordered by divisibility. What is the size of a maximal antichain?

How many ways are there to write the number 100 as a product of 3 positive integers?

Shown at left are three of the Hasse diagrams of chocolate preferences from the chocolate tasting held on April 27.

Calendar

Session 20: (all dates in 2017)

January	26	Start of the twentieth session!
February	2	
	9	
	16	
	23	No meet
March	2	
	9	
	16	
	23	
	30	No meet
April	6	
	13	
	20	No meet
	27	
May	4	

Session 21: (all dates in 2017)

September	7	Start of the twenty-first session!
	14	
	28	
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	24	Thanksgiving - No meet
	30	
December	7	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________   Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors strive to get members to do math through inspiration and not assignment. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where are Girls' Angle meets held? Girls' Angle meets take place near Kendall Square in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, Founder and Director, The Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, The Dartmouth Institute
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, assistant professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, associate professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: __________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____