Girls' Angle Bulletin

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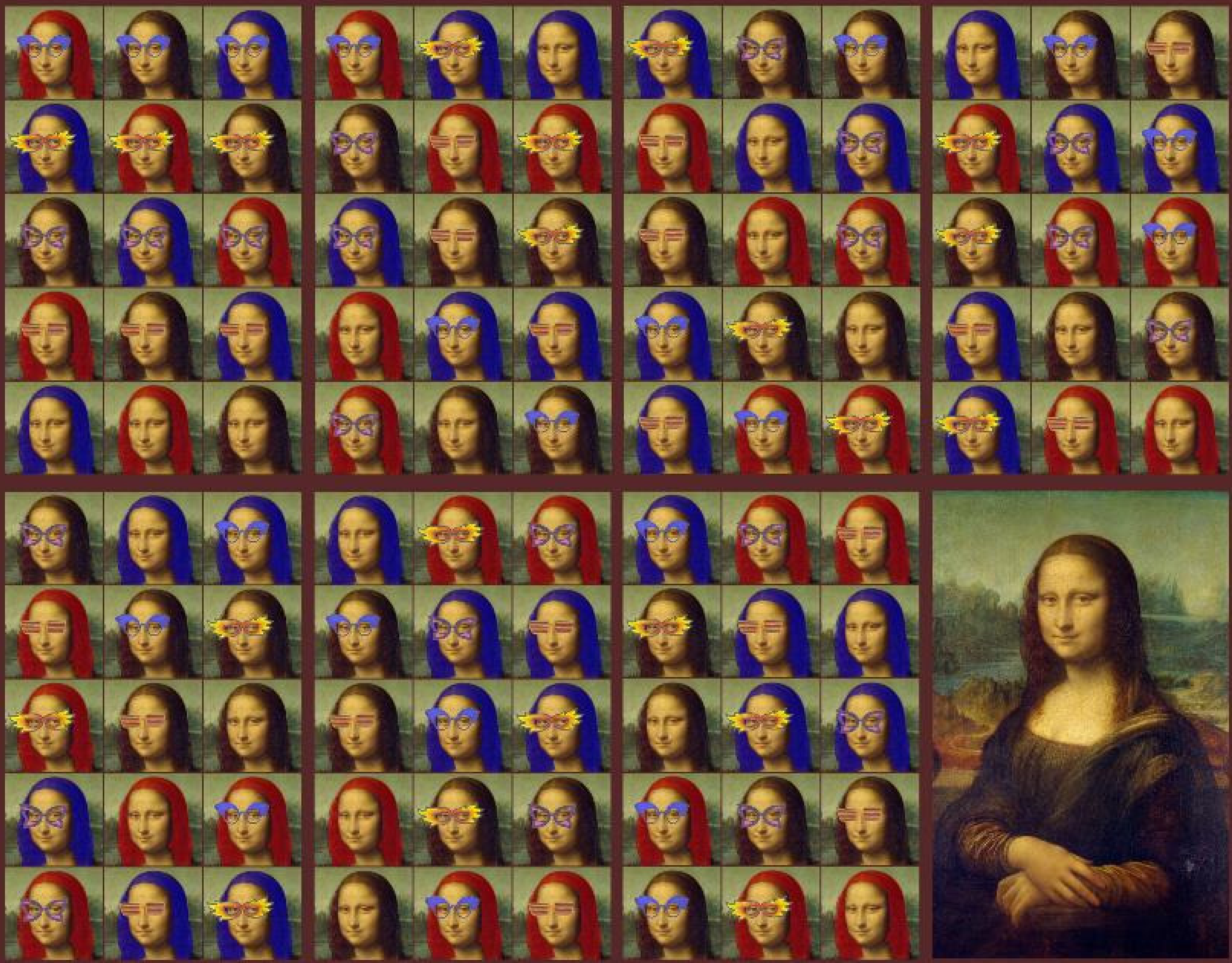
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

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- Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Lisa Meets Thomas*, by Ken Fan and Julia Zimmerman. A solution to Kirkman's schoolgirl problem, see *Unsuspected Geometry* on page 27.

An Interview with Kathryn Mann

It's good to … admit easily when you are wrong (I'm wrong a lot!) and learn from mistakes.

Kathryn Mann is a Morrey Visiting Assistant Professor and National Science Foundational postdoctoral fellow at the University of California, Berkeley. This fall, she will begin an assistant professorship at Brown University. She received her doctoral degree in mathematics from the University of Chicago under the supervision of Benson Farb.

Ken: What motivated you to become a mathematician? What do you like about being a mathematician?

Kathryn: I have always liked figuring out how things work. I was (and still am) interested in all kinds of science. But doing experiments was not my favorite part of science. What I really liked was coming up with ideas for experiments and deducing conclusions from my experience; the steps that required thinking and imagining.

I remember learning about the famous "thought experiment" of Galileo, who dispelled the common perception that heavier objects always fall faster than lighter objects. His thought experiment was something like this: "imagine attaching a brick to a feather with a string. If heavier objects always fall faster, then the brick should fall faster than the feather and the feather will pull on the string and slow the brick down a little. But if heavier objects always fall faster, well, the total mass of the brick and feather and string tied together is heavier than the brick, so it should fall even faster than the brick would fall by itself. Since these are two contradictory outcomes, our old assumption (that speed of falling is proportional to weight) must be wrong."[1]

This example really speaks to me – I love how one can challenge one's understanding of the world just by thinking about things carefully.

That's exactly what I like about mathematics. My job as a mathematician is to think about things that I find curious or interesting and try to figure out (just by thinking!) how they work. It took me a while to discover that math was like that – when I was in grade school, a lot of the math I had to do was just computational and not creative. But once I saw that real mathematics was about challenging assumptions and figuring things out from scratch, and requires a lot of creativity, I was sold.

The other thing I particularly like about mathematics is the way mathematicians (and even my students!) talk to each other. In other disciplines, like politics or philosophy for example, people discuss by arguing: they take different sides and debate. In mathematics, when two people work together they are both looking for an answer and trying to find out what's true – they are both on the same team. I really like this collaborative aspect.

Ken: Wow, I love your response! What was the first mathematical idea that caught your interest?

Kathryn: One idea that sticks in my memory is learning about mathematical proof, and in particular proof by induction.

Here's a nice example of what that means: If you add up all the square numbers from 1^2 to 15^2 say, you get 15(16)(31)/6, i.e.

[1] This thought experiment appears in Galileo's work *De Motu Antiquiora*.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Kathryn Mann and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

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Thank you and best wishes,
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President and Founder
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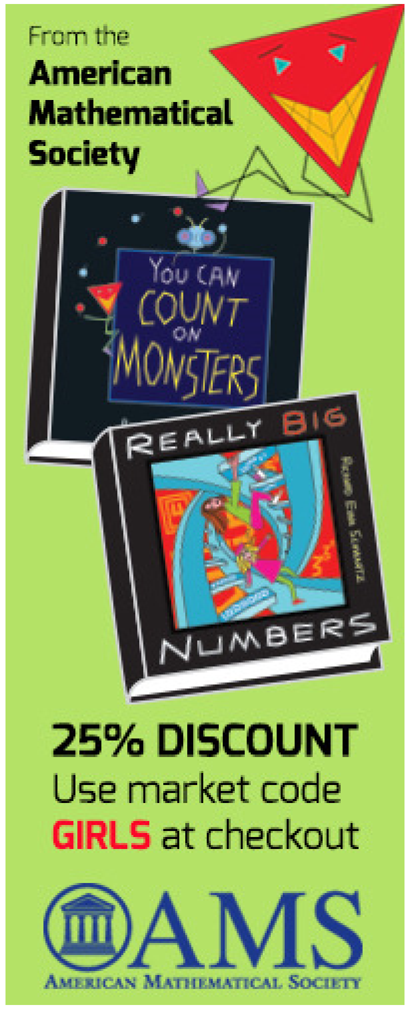

The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

In Search of Nice Triangles, Part 12

by Ken Fan | edited by Jennifer Silva

Jasmine: Are you ready to tackle the even cases?

Emily: Not only am I ready, I'm *dying* to know what $\Phi_n(i)$ is for even n!

Jasmine: Me too! We do know $\Phi_n(i)$ for some even n.

Emily: Right. We know $\Phi_2(i) = i + 1$, $\Phi_4(i) = 0$, and $\Phi_n(i) = 2$ when n is a power of 2 greater than 4.

Jasmine: How should we proceed to compute $\Phi_n(i)$ for other even values of n?

Emily: Maybe we can start with numbers n of the form $2^k p$ where p is an odd prime; if we can do those, then let's consider numbers of the form $2^k m$ where m is a product of two odd primes, and so on.

Jasmine: Or, since we know $\Phi_n(i)$ for all odd n, another approach would be to try to compute $\Phi_n(i)$ for n of the form $2^k m$ for odd m by induction on k.

Emily: Hmm. I like your approach since we'd be keeping the odd part of n fixed, while multiplying by 2 each time for the inductive step. Let's try it!

Jasmine: Okay, let's fix an odd number m. Can we relate $\Phi_{2m}(i)$ to $\Phi_m(i)$?

Emily: We know that $x^{2m} - 1 = \prod_{d|2m} \Phi_d(x)$ and $x^m - 1 = \prod_{d|m} \Phi_d(x)$. Since m is odd, the odd divisors of $2m$ comprise all of the divisors of m.

Jasmine: And the even divisors of $2m$ are twice the divisors of m. So that means that

$$x^{2m} - 1 = (x^m - 1) \prod_{d|m} \Phi_{2d}(x).$$

Emily: If we divide through by $x^m - 1$, we get $x^m + 1 = \prod_{d|m} \Phi_{2d}(x)$.

Emily and Jasmine continue their investigation into nice triangles. They've been using "nice" to denote angles that measure a rational multiple of π radians.

Previously, they decided to embark on a study of the minimum polynomials of the cosines of rational multiples of π. They defined the polynomials $p_d(x)$, for $d > 1$, to be the product of all linear factors of the form $x - \cos(2\pi k/d)$, where $1 \le k \le d/2$ and $(k, d) = 1$. They defined $p_1(x) = x - 1$.

They observed that, for n odd,

$$T_n(x) - 1 = 2^{n-1} p_1(x) \left(\prod_{d|n, d>1} p_d(x) \right)^2,$$

and for n even,

$$T_n(x) - 1 = 2^{n-1} p_1(x) p_2(x) \left(\prod_{d|n, d>2} p_d(x) \right)^2,$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind.

They showed that the minimum polynomial of $\cos(2\pi k/n)$, where k and n are relatively prime, is $p_n(x)$.

They aim to compute the constant terms of $p_n(x)$ in the hopes that doing so will enable them to determine all triangles with 3 nice angles and 2 sides of integer length.

They're also using $\Phi_n(x)$ to denote the nth cyclotomic polynomial, which is the minimum polynomial of a primitive nth root of unity.

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Modeling Rain

by Ken Fan
edited by Jennifer Silva

Imagine placing a sheet of paper on the ground during a light rain and watching as raindrops begin to pitter-patter upon it. Can you predict where on the sheet the next drop that falls on it will land? Probably not; it appears to be random.

At first, it would be hard to know which parts of the paper will stay dry the longest. If you put several sheets out, the initial pattern of water marks will be different for each one. But we can predict with reasonable confidence that after some time, all of the sheets will be pretty much uniformly soaked.

How can we model this phenomenon mathematically? Because of the randomness, it would seem natural to use probability.

Discrete Probability

Typically, our first encounter with probability is through coin flipping, card dealing, or die rolling. In a die roll, there are six ways that the die can come up, and all six ways are equally likely if the die is fair. When all of the outcomes are equally likely, we can determine the probability of certain desirable outcomes by computing the ratio of the number of desired outcomes to the total number of possible outcomes. Thus, the probability that a fair die comes up odd is 3/6 because there are three outcomes that are odd (1, 3, and 5) and six total outcomes.

But when we try to apply this method for computing probabilities to raindrops falling on a sheet of paper, we immediately run into trouble: there are infinitely many places where a raindrop might fall. Computing probability by counting no longer works.

Continuous Probability

We can get some guidance on our counting problem by approximating the situation with a discrete one. Suppose we draw a line down the middle of our sheet of paper so that the paper is split into two halves, a left half and a right half. What is the probability that a raindrop falls in the left half?

Though there are infinitely many locations where a raindrop may fall, we've now grouped the possible outcomes into just two: either the raindrop falls in the left half, or it falls in the right half. In effect, we have created a situation where we can employ counting again. If there are no peculiarities about the environment in which we're conducting our raindrop experiment, the raindrop would have no preference for the left half over the right half, nor for the right half over the left; thus, both outcomes are equally likely.

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Anna's Math Journal

By Anna B.

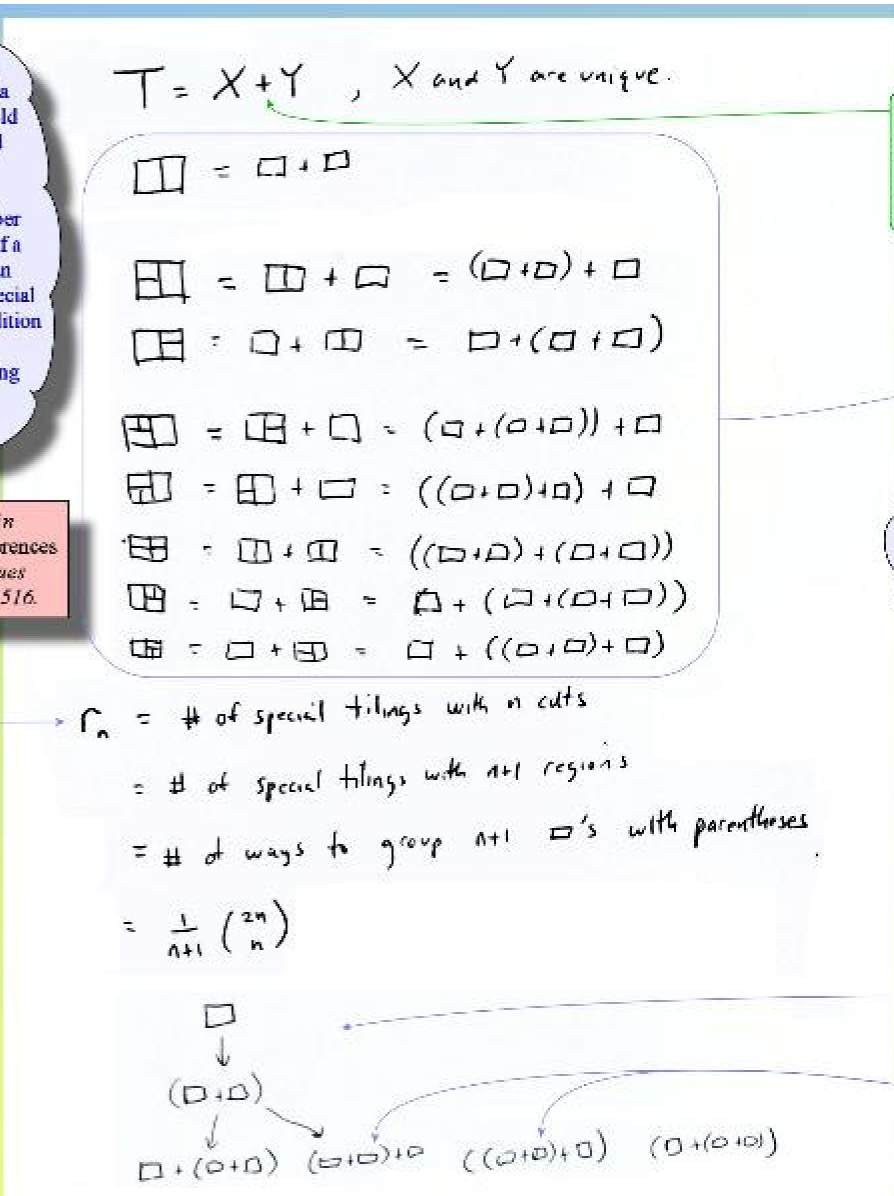
Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna has a new idea for finding the number of special tilings of a 1 by $\sqrt{2}$ rectangle.

I spoke with Ken about the number of special tilings of a 1 by $\sqrt{2}$ rectangle, and he told me that the numbers I found are known as the "Catalan" numbers. He said Catalan originally counted the number of parenthetic expressions of a certain type. That gave me an idea! I can break down a special tiling as a sum using the addition I defined on special tilings. What if I keep going, breaking down a tiling to a sum of plain rectangles?

Catalan's result can be found in Note sur une équation aux différences finies, *in the J. de Mathématiques pures appl., 3 (1838), pp. 508-516.*

So...



Earlier, I defined the sum of two special tilings to mean rotating the special tilings counterclockwise 90 degrees and joining them side by side.

By breaking tilings down, I get different ways to group a sum of plain rectangles, which is apparently exactly what Catalan considered!

If two tilings are the same, then they'll give rise to the same grouping, and every way of grouping yields a special tiling.

I still wish I had a way of showing that the r_n are the Catalan numbers without using generating functions. I wonder if there's a way to see that the number of different ways to group terms is equal to a Catalan number. Maybe I can find an inductive argument by connecting groupings of $n - 1$ rectangles to those of n rectangles.

Hm. It doesn't look promising because adding a new rectangle systematically to a sum can lead to identical groupings.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

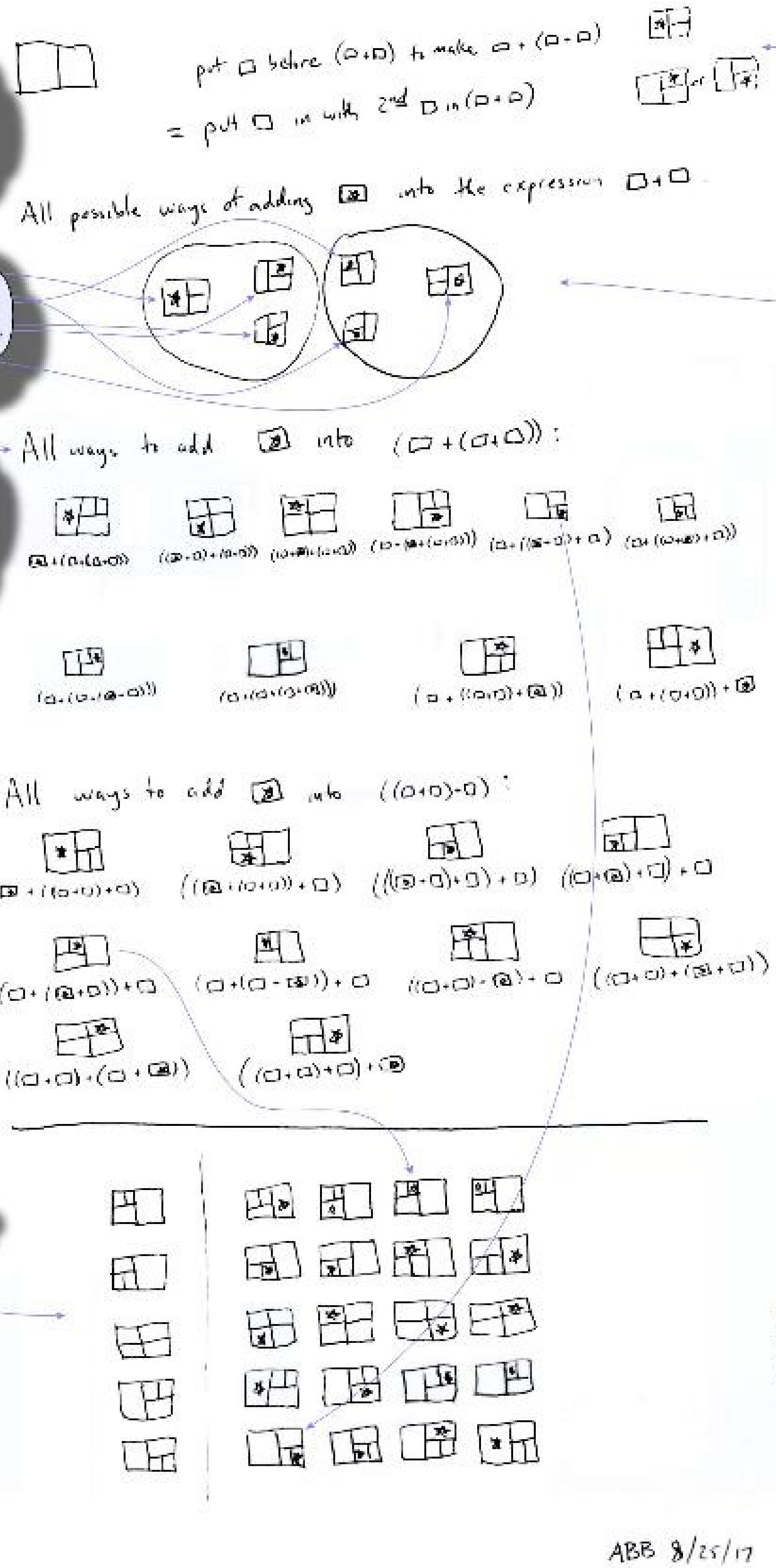
Wait a sec. When I insert a new rectangle into an existing sum, I might get the same tiling pattern, but the new rectangle corresponds to a different region in the tiling.

That makes me think I should perhaps keep track of where the new rectangle ends up by marking it with a star.

Adding a [★] to [ ]+[ ], I get [★]+([ ]+[ ]), ([★]+[ ])+[ ], ([ ]+[★])+[ ], [ ]+([★]+[ ]), [ ]+([ ]+[★]), and ([ ]+[ ])+[★].

In the case where I'm adding a new rectangle to a sum of two rectangles, when I keep track of where the new rectangle ends up in the tiling, I get each tiling pattern 3 times, one for each of the ways I can place the star.

I think I'll work out in detail the case where I add a rectangle to a sum of 3 rectangles. I'll systematically work the new rectangle in from the leftmost position to the right.



I'll group the marked tilings I got by their tiling pattern.

Hey... each tiling pattern occurs 4 times, one for each possible location for the star. That's incredibly regular. If this pattern holds for any number of rectangles, there's got to be a way to work it into a formula for r_n! Hmm...

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Summer Fun!

In the last issue, we presented the 2017 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. When you work on the problem, you will force yourself to think about the ideas associated with the problem. You will gain some familiarity with the related concepts and this will make it easier to read other people's solutions.

With mathematics, don't be passive! Get active!

Move that pencil! Move your mind! You might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Summer Fun!

Cyclotomic Polynomials

by Girls' Angle Staff

1. Since $(e^{2\pi ik/n})^n = e^{2\pi ik} = \cos(2\pi k) + i\sin(2\pi k) = 1$, we see that $e^{2\pi ik/n}$ is a solution to the equation $x^n = 1$. On the other hand, the polynomial $x^n - 1$ has at most n roots (counted with multiplicity). Since the complex numbers $e^{2\pi ik}$ are different for $k = 0, 1, 2, 3, \ldots, n - 1$, they are a complete set of solutions to $x^n = 1$.

2. Since $(-1)^2 = 1$, we know -1 is a square root of unity. Since $(-1)^1 = -1$, we know -1 is, in fact, a primitive square root of unity. We compute that $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^2 = \frac{-1\mp i\sqrt{3}}{2}$ and $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3 = 1$.

These facts show that $\frac{-1\pm i\sqrt{3}}{2}$ are primitive cube roots of unity. The first 4 positive powers of i are i, -1, $-i$, and 1, respectively. This shows that i is a primitive fourth root of unity. Similarly, $-i$ is a primitive fourth root of unity since its first 4 positive powers are $-i$, -1, i, and 1, respectively.

3. We are given that $w^n = 1$. Notice that $(w^k)^n = w^{kn} = (w^n)^k = 1^k = 1$. Hence, w^k is an nth root of unity for all $0 \le k < n$. Suppose that w is a primitive nth root of unity. If $\{ w^k \mid 0 \le k < n \}$ does not constitute a complete set of nth roots of unity, then $w^k = w^j$ for some $0 \le k < j < n$. But then we have $w^{j-k} = 1$, where $j - k > 0$, contradicting the fact that w is primitive. Conversely, suppose $\{ w^k \mid 0 \le k < n \}$ constitutes a complete set of nth roots of unity. Then $w^k \ne 1$ for any $0 < k < n$, for if $w^k = w^0 = 1$ for some $0 < k < n$, then $\{ w^k \mid 0 \le k < n \}$ would have fewer than n elements. But we know that there are n nth roots of unity.

4. The only integers k such that $0 \le k \le n$ and $(e^{2\pi i/n})^k = e^{2\pi ik/n} = 1$ are $k = 0$ and $k = n$. Therefore, $e^{2\pi i/n}$ is a primitive nth root of unity.

5. Let d be the greatest common factor of k and n. Notice that $(w^k)^{n/d} = (w^n)^{k/d} = 1$. Hence, if w^k is a primitive nth root of unity, then $d = 1$, i.e. k and n are relatively prime. Suppose $d = 1$. Let r be the remainder when we divide km by n. Then $km = nq + r$, for some integer q and $0 \le r < n$. If $(w^k)^m = 1$, then $1 = w^{km} = w^{nq+r} = w^{nq}w^r = w^r$. Since w is a primitive nth root of unity, this last equation is only possible if $r = 0$, i.e. n divides km. But since k and m are relatively prime, this implies that n divides m. Hence w^k is a primitive nth root of unity.

6. The nth roots of unity are $e^{2\pi ik/n}$ for $0 \le k < n$. From Problem 5, we know that of these, the primitive ones are precisely the ones where k and n are relatively prime. By definition, the number of positive integers less than or equal to n that are relatively prime to n is given by the Euler totient function $\varphi(n)$.

7. The degree of $\Phi_n(x)$ is $\varphi(n)$.

Summer Fun!

8. We know that $\{ e^{2\pi ik/n} \mid 0 \le k < n \}$ constitutes a complete set of nth roots of unity. Suppose that the fraction k/n becomes the fraction j/d when expressed in lowest terms. Then $e^{2\pi ik/n} = e^{2\pi ij/d}$ and by Problem 6, $e^{2\pi ij/d}$ is a primitive dth root of unity. Therefore, the nth roots of unit are a subset of the union of the P_d for which $d \mid n$. On the other hand, if $d \mid n$ then any primitive dth root of unity is an nth root of unity. Furthermore, if w is in P_d and P_b then $e^{2\pi ik/d} = e^{2\pi ij/b}$, where k is relatively prime to d and j is relatively prime to b. But then $1 = e^{2\pi ik/d - 2\pi ij/b} = e^{2\pi i(kb - jd)/(bd)}$, which means that bd divides evenly into $kb - jd$. Since k is relatively prime to d, this implies that d divides b. But since j is relatively prime to b, we must also have that b divides d. Therefore, if P_d and P_b intersect nontrivially, then $b = d$.

9. The identity $x^n - 1 = \prod_{d|n} \Phi_d(x)$ follows since the polynomials on both sides are monic polynomials with the same roots.

10. From Problem 9, we see that $x^p - 1 = \Phi_1(x)\Phi_p(x)$, since the only divisors of a prime number p are 1 and p. The only "one-th" root of unity is 1 itself, so $\Phi_1(x) = x - 1$. Thus,

$$\Phi_p(x) = (x^p - 1)/(x - 1) = 1 + x + x^2 + x^3 + \ldots + x^{p-1}.$$

11. We organize this answer in a table:

n	$\Phi_p(x)$
1	$x - 1$
2	$x + 1$
3	$x^2 + x + 1$
4	$x^2 + 1$
5	$x^4 + x^3 + x^2 + x + 1$
6	$x^2 - x + 1$
7	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
8	$x^4 + 1$
9	$x^6 + x^3 + 1$
10	$x^4 - x^3 + x^2 - x + 1$
11	$x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
12	$x^4 - x^2 + 1$
13	$x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
14	$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
15	$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$

12. Since $105 = 3\cdot 5\cdot 7$, we have $x^{105} - 1 = \Phi_1(x)\Phi_3(x)\Phi_5(x)\Phi_7(x)\Phi_{15}(x)\Phi_{21}(x)\Phi_{35}(x)\Phi_{105}(x)$. All but the last 3 terms are in the table above and we can use $x^{21} - 1 = \Phi_1(x)\Phi_3(x)\Phi_7(x)\Phi_{21}(x)$ and $x^{35} - 1 = \Phi_1(x)\Phi_5(x)\Phi_7(x)\Phi_{35}(x)$ to find $\Phi_{21}(x)$ and $\Phi_{35}(x)$. We find that

$$\Phi_{105}(x) = x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} - x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x + 1.$$

13. Note that $x^{\varphi(n)}\Phi_n(1/x)$ is a polynomial of degree $\varphi(n)$. If w is a primitive nth root of unity, so is w^{-1}. Therefore, the roots of $x^{\varphi(n)}\Phi_n(1/x)$ are the primitive nth roots of unity. Since $x^{\varphi(n)}\Phi_n(1/x)$ and $\Phi_n(x)$ have the same roots, if we can show that they have the same lead coefficients, then they must be equal. We know that the lead coefficient of $\Phi_n(x)$ is 1. The lead coefficient of $x^{\varphi(n)}\Phi_n(1/x)$ is $\Phi_n(0)$. Actually $\Phi_1(x) = x - 1$, so $\Phi_1(0) = -1$, and, in fact, $x^{\varphi(1)}\Phi_1(1/x) = 1 - x$, which is not equal to $\Phi_1(x) = x - 1$. However, by induction on the number of factors of n, we can see from the identity $x^n - 1 = \prod_{d|n}\Phi_d(x)$, that $\Phi_n(0) = 1$ for all $n > 1$. The coefficient of x^k in the polynomial $x^{\varphi(n)}\Phi_n(1/x)$ is the coefficient of $x^{\varphi(n)-k}$ in $\Phi_n(x)$.

14. The divisors of p^m are p^k for $0 \le k \le m$. Using Problem 9, we find

$$x^{p^m} - 1 = \prod_{k=0}^{m}\Phi_{p^k}(x) = \Phi_{p^m}(x)(x^{p^{m-1}} - 1).$$

Dividing both sides by $x^{p^{m-1}} - 1$, we see that $\Phi_{p^m}(x) = \dfrac{x^{p^m} - 1}{x^{p^{m-1}} - 1} = \Phi_p(x^{p^{m-1}})$, since $\Phi_p(x) = \dfrac{x^p - 1}{x - 1}$.

15. A. Since w is a primitive pn-th root of unity, it is equal to $e^{2\pi ik/(pn)}$ for some $0 < k < pn$ such that k is relatively prime to pn. Then $w^p = (e^{2\pi ik/(pn)})^p = e^{2\pi ik/n}$. If k is relatively prime to pn, then k is relatively prime to n (for if d divides both k and n, then d divides both k and pn). From Problem 5, we see that w^p is a primitive nth root of unity.

B. We can write $w = e^{2\pi ik/n}$ and $v = e^{2\pi ij/n}$ where k and j are relatively prime to n. The pth roots of w are the complex numbers $e^{2\pi i(k + sn)/(pn)}$, $s = 0, 1, 2, \ldots, p - 1$. The *pth* roots of v are the complex numbers $e^{2\pi i(j + tn)/(pn)}$, $t = 0, 1, 2, \ldots, p - 1$. If $e^{2\pi i(k + sn)/(pn)} = e^{2\pi i(j + tn)/(pn)}$ for some s and t, then we must have $k - j + (s - t)n$ divisible by pn. In particular, n divides $k - j$, which means $v = w$.

The numbers $k + sn$, $s = 0, 1, 2, \ldots, p - 1$, give a complete set of residues modulo p since n is relatively prime to p. Note that $k + sn$ is relatively prime to n since if $d > 0$ divides both, then d divides $(k + sn) - sn = k$, which means $d = 1$ since k and n are relatively prime. If $k + sn$ is divisible by p, then $e^{2\pi i(k + sn)/(pn)}$ is an nth root of unity, and since $k + sn$ is relatively prime to n, it is, in fact, a primitive nth root of unity. If $k + sn$ is not divisible by p then it is relatively prime to np and, hence, $e^{2\pi i(k + sn)/(pn)}$ is a primitive pn-th root of unity. There are $\varphi(n)$ primitive nth roots of unity and $\varphi(np) = (p - 1)\varphi(n)$ primitive pn-th roots of unity. Since S contains $p\varphi(n)$ elements, we know that S contains all the primitive pn-th and nth roots of unity.

The roots of $\Phi_n(x^p)$ are the pth roots of the primitive nth roots of unity, i.e., the elements of S. Since the roots of $\Phi_n(x)$ are the primitive nth roots of unity, the roots of $\Phi_n(x^p)/\Phi_n(x)$ are just the primitive pn-th roots of unity. Also, $\Phi_n(x^p)/\Phi_n(x)$ has lead coefficient 1. Therefore, $\Phi_{np}(x) = \Phi_n(x^p)/\Phi_n(x)$.

17. See this issue's installment of *In Search of Nice Triangles*, specifically, page 11.

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Sets

by Girls' Angle Staff

1. The subsets of $\{1, 2, 3\}$ are $\{1, 2, 3\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$, $\{1\}$, $\{2\}$, $\{3\}$, and Ø.

2. To form a subset, we can ask each element if it would like to be in the subset or not. Since each element can answer independently "yes" or "no", there are 2^n subsets of S.

3. The sets S and T are equal.

4. We have

$S \cup T = \{-2, -1, 0, 1, 2, 3, 4, 5\}$ $\quad$ $S \cap T = \{1, 2\}$

$T \cup W = \{$multiples of 3 and 1, 2, 4, and 5$\}$ $\quad$ $T \cap W = \{3\}$

$W \cup S = \{$multiples of 3 and -2, -1, 1, and 2$\}$ $\quad$ $W \cap S = \{0\}$

5. Let A and B be sets. A standard way to show that A and B are the same set is to show that every element of A is in B and every element of B is in A.

For $S \cup T = T \cup S$, suppose x is in $S \cup T$. Then x is in S or x is in T. Thus x is in $T \cup S$. Conversely, if x is in $T \cup S$, then x is in T or x is in S. Hence, x is in $S \cup T$.

A similar argument works to how that $S \cap T = T \cap S$.

Note: When mathematicians say "A or B is true", they mean that at least one of, and possibly both of, A and B are true.

6. We have $S \cap T = S$ precisely when S is a subset of T.

7. We have $S \cup T = S$ precisely when T is a subset of S.

8. This is false. For a counterexample, let T = Ø and let $S = W = \{0\}$.

9. Each number 1, 2, and 3, must be in at least one of the sets S, T, or W, but it cannot be in all of them. For each number, there are 3 ways it can be in exactly one of the sets and 3 ways it can be in exactly 2, for a total of 6 ways to place that number. Since each of the three numbers can be placed into the sets independently of each other, there are $6^3 = 216$ solutions to this problem!

10. Suppose x is in $S \cap (T \cup W)$. Then x is in both S and $T \cup W$. Since x is in $T \cup W$, we know x is in at least one of T or W. Therefore x is in at least one of $S \cap T$ or $S \cap W$, that is x is in $(S \cap T) \cup (S \cap W)$. Now suppose x is in $(S \cap T) \cup (S \cap W)$. Then x is in $S \cap T$ or $S \cap W$. In either case, x is in S. Also, x must be in T or W, i.e. x is in $T \cup W$. Thus, x is in $S \cap (T \cup W)$.

11. This can be solved using an argument quite similar to the solution for Problem 10.

12. Suppose x is in $(S \cup T)^c$. Then x is not in $S \cup T$, which means x is in neither S nor T. This means x is in both S^c and T^c, and hence x is in $S^c \cap T^c$. Now suppose x is in $S^c \cap T^c$. Then x is in both S^c and T^c, i.e. x is neither in S nor T. This means x is not in $S \cup T$, hence x is in $(S \cup T)^c$.

A similar argument can be used to show that $(S \cap T)^c = S^c \cup T^c$.

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13-15. Let a and b be positive integers. If n is in $S_a \cap S_b$, then n is a multiple of both a and b and we can write $n = ak = bj$ for some integers k and j. Let d be the greatest common factor of a and b. Then $n = (a/d)kd = (b/d)jd$, where a/d and b/d are both integers. Dividing by d we get $(a/d)k = (b/d)j$. Since a/d and b/d are relatively prime, b/d must divide k, that is $k = (b/d)l$, for some integer l. Thus, $n = (a/d)bl$. Note that $(a/d)b$ is the least common multiple of a and b. Hence we see that n is a multiple of the least common multiple of a and b. Conversely, if n is a multiple of the least common multiple of a and b, then it is divisible by both a and b (since the least common multiple is). We conclude that $S_a \cap S_b = S_{\mathrm{LCM}(a,\, b)}$.

Since LCM(3, 12) = 12 and LCM(12, 18) = 36, we see $S_3 \cap S_{12} = S_{12}$ and $S_{12} \cap S_{18} = S_{36}$.

16. Let a and b be positive integers. If a is in S_c, then $S_a \subseteq S_c$. Therefore, we seek the largest integer c such that both a and b are in S_c, i.e. such that a and b are divisible by c. By definition, this is the greatest common factor of a and b. Thus, the largest c for which $S_a \cup S_b \subseteq S_c$ is the greatest common factor of a and b.

17. Let $p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$ be a prime factorization of a, where the p_i are distinct primes. Then $P_a = \{p_1, p_2, p_3, \ldots, p_k\}$ and $D_a = \{\, p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_k^{m_k} \mid 0 \le m_i \le n_i,\ i = 1, \ldots, n\}$. The number of subsets of P_a is 2^k. The number of elements in D_a is $(n_1 + 1)(n_2 + 1)(n_3 + 1)\cdots(n_k + 1)$. Since $n_i + 1 \ge 2$, the only way $2^k = (n_1 + 1)(n_2 + 1)(n_3 + 1)\cdots(n_k + 1)$ is if each factor in the right hand side is equal to 2, i.e. $n_i = 1$ for all i. This means that a is a product of distinct primes. Such an integer is called **square-free** because it is not divisible by any perfect square other than 1.

18. If S and T are disjoint, then the number of elements in $S \cup T$ would be $|\,S\,| + |\,T\,|$. However, if $S \cap T$ is not empty, then $|\,S\,| + |\,T\,|$ would double-count the elements in $S \cap T$. Therefore, $|\,S \cup T\,| = |\,S\,| + |\,T\,| - |\,S \cap T\,|$.

19. Let T be a member of C that has the smallest number of elements. Since the empty set is not in C, we know that T is not empty. So let x be in T. We claim that $T = \{x\}$. If not, then we know $\{x\}$ is not a member of C since T was chosen to be a member of C with the smallest number of elements, and if $T \ne \{x\}$, then T has more than 1 element, whereas $\{x\}$ only has 1 element. By property iii, the complement of $\{x\}$ in S is a member of C. Let U be the intersection of the complement of $\{x\}$ (in S) with T. By property ii, U is a member of C. However, U is a strictly smaller subset of T because U does not contain x. This contradicts the choice of T as being a member of C with the least number of elements. Thus, $T = \{x\}$.

If V is any member of C, it must contain x, for if it doesn't, then by property ii, $V \cap T = \varnothing$ would be in C, contradicting property i. Therefore all the members of C contain x. Furthermore, if W is any set that contains x, then the complement of W in S does not contain x and, hence, cannot be a member of C. By property iii, we conclude that W is in C.

Thus, C consists precisely of those subsets of S that contain x.

20. Problem 19 does not hold if the finiteness condition on S is dropped. A collection of subsets of S that satisfy properties i, ii, and iii is known as an **ultrafilter**. Interested readers are urged to read about ultrafilters on the internet or take a look at the book *The Theory of Ultrafilters* by Comfort and Negrepontis.

Summer Fun!

The Fourth Dimension

by Girls' Angle Staff

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Unsuspected Geometry

by Matthew de Courcy-Ireland

1. Please see the problem set in the previous issue.

2. The method described after the statement of Problem 1 works because it takes two pieces of information to find the equation of a line. To find the equation of a parabola instead would take three pieces of information. Given three points on a parabola $y = ax^2 + bx + c$, we have three linear equations in the three unknown coefficients a, b, and c, which can then be solved as long as we initially made sure to give each person their own point with no repeats. Thus any three people can decode the message, but two people would need more information. The same idea works for more people: Using a cubic equation $y = ax^3 + bx^2 + cx + d$ would allow groups of four people to decode, but not any smaller group. A scheme for groups of any fixed size can be made with an equation of degree one less. A different method can be used if the goal is for decryption to be possible only with the cooperation of the entire group. For that, generate a number for each person (ideally at random) and calculate the sum of the numbers. Then give each person one of the numbers. Use the sum to encrypt the message, for example by cycling the alphabet that many letters. If everyone reveals their number, they can add them up to find the sum. But if even one value is missing, the sum could be anything.

3. Please see the problem set in the previous issue.

4. Any line through the origin can be parametrized in the form $(x, y, z) = t(a, b, c)$ where (a, b, c) is a non-zero vector thought of as a "direction" and t varies over all the numbers modulo 7. Several different direction vectors correspond to the same line: If (a, b, c) is one of them, then the rest are the non-zero multiples of (a, b, c). Thus the number of directions is greater than the number of lines by a factor of $7 - 1 = 6$. The number of directions is $7^3 - 1 = 342$ because every vector is allowed except (0, 0, 0). Therefore the number of lines is 57.

5. The planes through the origin are given by $ax + by + cz = 0$, where not all of a, b, and c are 0. As in the case of lines, scaling all of (a, b, c) by the same factor does not change the plane that solves $ax + by + cz = 0$. Accounting for this factor of 6, the number of planes is therefore

$$\frac{7^3 - 1}{7 - 1} = 57,$$

the same as the number of lines.

6. The 57 lines and 57 planes above are more than enough to cover the 55 cards in *Spot It!* Given any two lines with different slopes, there is a unique plane containing both lines. Likewise, two planes intersect in a line (unless they are "parallel"). One can interpret the symbols as lines and the cards as planes. Equally well, one could interpret the symbols as planes and the cards as lines – this is an instance of "duality". Each plane contains 8 lines (each line

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contains the origin and 6 points specific to it, so $1 + 6 \times 8 = 49 = 7^2$ is enough points to fill a plane), and likewise each line can be extended to a plane in 8 different ways. A plane can thus be thought of as a "card" containing 8 "symbols", and there will be one line common to a given pair of planes. This is true for any subset of planes, so removing two cards to have the 55 used in the game preserves the required property.

7. The considerations above apply just as well for any prime number, leading to

$$\frac{p^3 - 1}{p - 1} = p^2 + p + 1$$

planes through the origin or, equally well, lines through the origin. To count spaces of dimension d, we first count the number of ways to choose d independent "directions". The first direction offers $p^d - 1$ options, since it can be any vector except (0, 0, 0). At the next step, we must not choose a multiple of the first vector since we want independent directions. At each step, we cannot choose a linear combination of the previous vectors but are otherwise free. Thus the number of directions is a product $(p^d - 1)(p^d - p)(p^d - p^2)\cdots(p^d - p^{d-1})$. It is possible to change coordinates, so that the same space can be given by many different choices of direction. In the cases above, this was given only by scaling a chosen direction. In the higher-dimensional case, there are more ways to change coordinates.

8. This is not so easy to solve by trial and error. One solution is (see the cover of this issue):

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
01 06 11	01 02 05	02 03 06	05 06 09	03 05 11	05 07 13	11 13 04
02 07 12	03 04 07	04 05 08	07 08 11	04 06 12	06 08 14	12 14 05
03 08 13	08 09 12	09 10 13	12 13 01	07 09 15	09 11 02	15 02 08
04 09 14	10 11 14	11 12 15	14 15 03	08 10 01	10 12 03	01 03 09
05 10 15	13 15 06	14 01 07	02 04 10	13 14 02	15 01 04	06 07 10

See below for conceptual ways to find how to find a solution.

9. The direction vector of a line can be anything other than (0, 0, 0, 0), so there are $2^4 - 1 = 15$ possibilities. Modulo 2, the only numbers are 0 and 1. This means that there is no overcounting due to scaling, as in the modulo 7 case. So the directions match the lines perfectly, and thus 15 is also the number of lines through the origin. In this case where 0 and 1 are the only scalars, a line consists only of the origin and one other point, so it can be thought of simply as that point.

10. We can represent the students as lines through the origin in 4-dimensional space modulo 2, there being 15 such lines. In 4 dimensions, a plane containing the origin must be specified using 2 direction vectors. There are 15×14 ways to choose two independent directions, but this overcounts the number of planes by a factor of 6 because of the freedom to change coordinates within the plane. Thus there are $5 \times 7 = 35$ planes in total. Let $A = (1, 0, 0, 0)$, $B = (0, 1, 0, 0)$, $C = (0, 0, 1, 0)$, and $D = (0, 0, 0, 1)$. The lines of our 4-dimensional space correspond to nonzero subsets of $\{A, B, C, D\}$, where the line consists of the origin plus the point that is the sum of the vectors in the subset. But how do we pick out the

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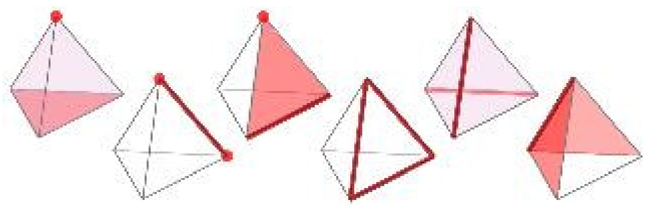
35 planes, and how do we partition these 35 planes into 7 sets of 5 to solve Kirkman's problem? For this, first read the solutions to Problems 11 and 12, then return here. Label the vertices of the tetrahedron in those solutions A, B, C, and D. Then each facet corresponds to the line that contains the origin and the vector which is the sum of the vertices of that facet. For example, the interior is the line $\{0, A + B + C + D\}$. If we make one small change, namely defining the 6^{th} group to be a pair of faces together with the edge connecting the vertices that are not shared by the two faces, then each group corresponds exactly to the nonzero vectors in a plane! We can then use the solution (with the slight modification of the previous sentence) given for Problem 12 to find a way to group the 35 planes into 7 groups of 5 that solve Kirkman's problem.

11. A handy way to picture a tetrahedron is to spread out your thumb, index, and middle fingers. The tips of those three fingers, plus the point on your palm where the fingers meet, are the four vertices of the tetrahedron. Your fingers are three of the edges, and there are three additional edges connecting the tips of each pair of fingers that we have to imagine. There are four faces: a triangle formed by your fingertips, two more triangles as if your fingers were webbed, and a fourth that stretches between thumb and middle finger.

12. It helps to visualize the possibilities (shown left to right in order of the problem statement).



For the first type of group, everything is determined by which of the 4 vertices we use. For the second, there are 6 ways to choose the pair of vertices. For the third kind, we can first choose one of the 4 vertices and then choose one of 3 edges to connect it to, for a total of 12 groups. For the fourth, there are 4 triangular circuits from which to choose. For the fifth, we have only to choose one of 6 edges to start since there is then only one edge not meeting it. However, this counts each group twice since the other order of the two edges does not give a different group. For the last, we must choose two faces, which can be done in 6 ways (as in the second case). Tallying all the different types, we have 4 + 6 + 12 + 4 + 3 + 6, for a total of 35 groups.

The magic of this tetrahedron is that $35 = 5 \times 7$ and each of the 35 groups covers three students, represented as either vertices, edges, faces, or solid interior. Thus we have 7 days worth of 5 rows of 3 students. In each case – vertex-to-vertex, vertex-to-edge, vertex-to-face, vertex-to-solid, edge to edge, edge to face, edge to solid, face to solid – there is only one way to connect two students. Thus, if we can find 7 ways to pick 5 groups whose union contains all 15 facets of the tetrahedron, we will have solved Kirkman's problem, as no two girls will walk together more than once. The 7 groups that include the interior (the 1^{st} and 5^{th}) must be put on different days. Consider a day with the 1^{st} type of group. Such a group has a vertex V and the face F opposite the vertex. Looking at the other groups, the edges of F can only be covered by groups of the 2^{nd} or 4^{th} types. Groups of the 2^{nd} type cannot be exclusively used because then the vertices of F will be selected twice. So we must cover the edges of F with a group of the 4^{th} type. The only way to cover the facets that remain is to take 3 groups of the 3^{rd} type. On a day with a group of the 5^{th} type, there will be 2 groups of the 2^{nd} type and 2 of the 6^{th} type. The details are left to the reader. (Draw pictures!)

Calendar

Session 21: (all dates in 2017)

September	7	Start of the twenty-first session!
	14	
	21	No meet
	28	
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Girls' Angle has hosted over 100 Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors strive to get members to do math through inspiration and not assignment. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where are Girls' Angle meets held? Girls' Angle meets take place near Kendall Square in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: ____________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____