Girls' Angle Bulletin

October/November 2017 • Volume 11 • Number 1

To Foster and Nurture Girls' Interest in Mathematics

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From the Founder

Math stretches the mind, making it concoct the most fanciful thoughts. For proof, check out this issue's Math Buffet, which features the mathematical scratch work of mathematicians.

- Ken Fan, President and Founder

Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Two mating polynomial Julia sets* by Arnaud Chéritat. Image courtesy of Professor Arnaud Chéritat, CNRS/Institut de Mathématiques de Toulouse.

An Interview with Sarah Koch

Sarah Koch as an Associate Professor of Mathematics at the University of Michigan. She has doctoral degrees in mathematics from l'Université de Provence and Cornell University, both under the supervision of John Hubbard. She was a National Science Foundation (NSF) Postdoctoral Fellow and a Benjamin Peirce Assistant Professor of Mathematics at Harvard. She is the recipient of numerous honors including an NSF CAREER Grant.

Ken: Could you please explain the earliest mathematical notion that captivated your interest that you can recall?

Sarah: The logical structure of proofs really appealed to me when I was learning math in school. I liked knowing that I was sure I had a problem figured out, and I especially liked being able to understand both a problem, and a solution inside and out. This is what first attracted me to mathematics. I didn't have the same level of certainty in my work for other classes in school growing up.

Ken: What led you to decide to become a mathematician?

Sarah: I grew up in Concord, NH and went to public school. I was extremely fortunate to have had some wonderful teachers, especially in my math classes. This was another big draw for me toward mathematics; my math teachers not only encouraged me to succeed, but they also made math fun and exciting. I really enjoyed learning math in school to the point that it was hard for me to imagine not thinking about it or studying it.

Ken: Was becoming a mathematician a straightforward process or was it a roller coaster ride?

I really like mysteries, and being a mathematician is sort of like being a detective.

Sarah: A big part of becoming a mathematician is learning to do research, and research can definitely be a roller coaster ride. In graduate school, I was working on a particular project for my dissertation. I had discovered a really neat phenomenon in a space of complex **Hénon maps**; these are dynamical systems that depended on two complex variables. This space has 4 real dimensions, so it is hard to visualize. However, there are computer programs that help us to see it. As I was exploring this mysterious space with the computer, I noticed something strange about how the pictures were changing as I clicked in different places. As I was struggling to put this phenomenon into words, my thesis advisor, John Hubbard at Cornell, started reading an exciting new paper about a completely different problem in complex dynamics.

This paper, by mathematicians Bartholdi and Nekrashevych, contained a solution to something called the "Twisted Rabbit Problem". When Hubbard presented this work in a seminar, he asked the audience if there was anyone who wanted to try a little experiment. There was a really interesting construction in the paper that used the fact that a particular set contained just four elements. Hubbard asked if anyone wanted to try to make the construction work by putting five elements in the set. Nobody seemed interested in doing that. So I volunteered, thinking I would spend a few hours over the weekend thinking about it, and then I would return to trying to understand my computer pictures.

In hindsight, I was extremely fortunate to have thought about putting five elements into that set. This turned out to be a really exciting thing to do, and that work

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Thank you and best wishes,
Ken Fan
President and Founder
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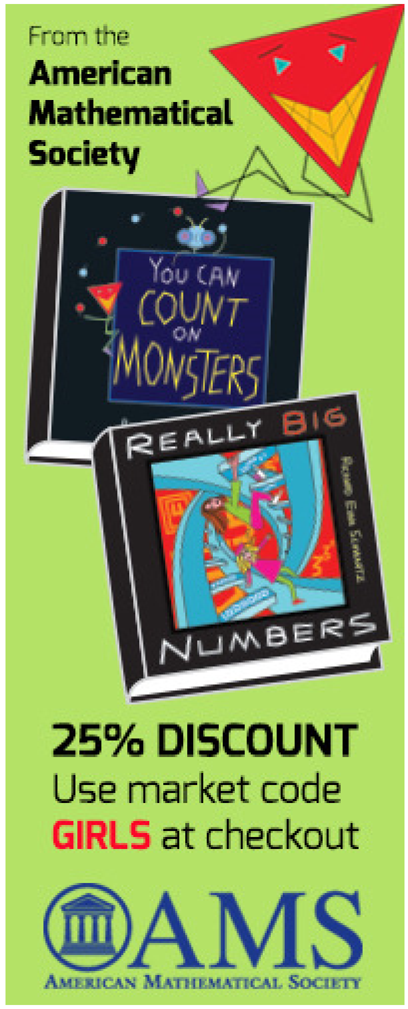
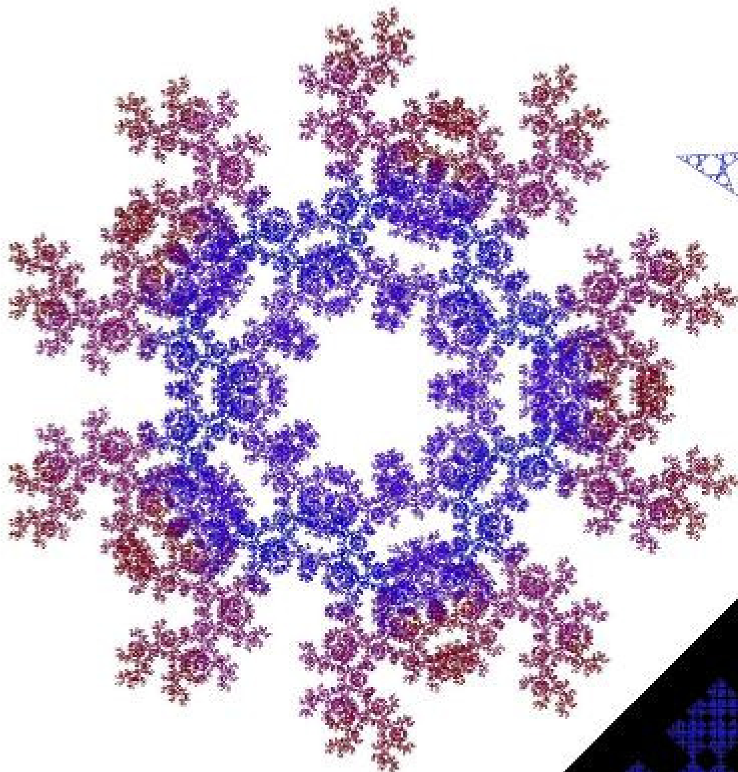
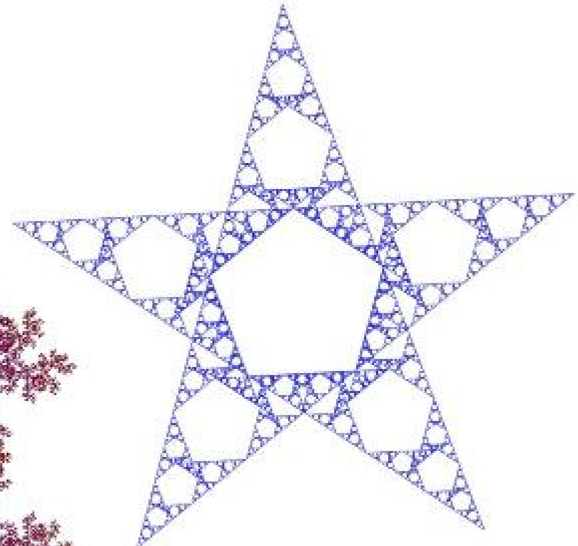


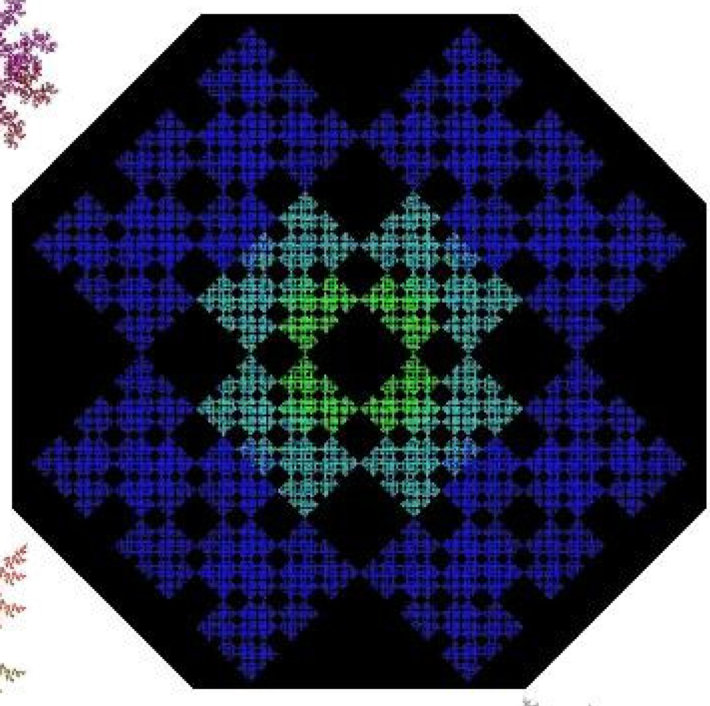
The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

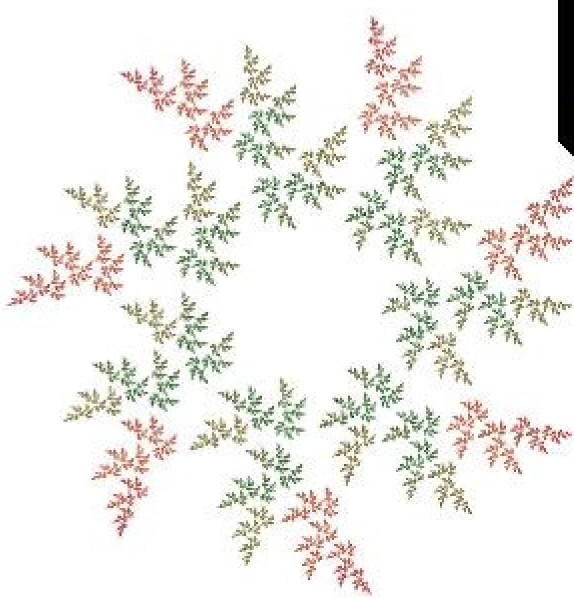
More Chaos

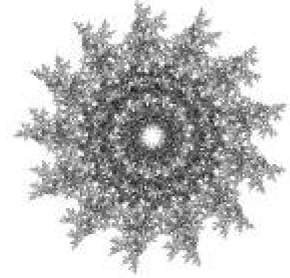
Here are more results of playing the Chaos Game described by Sarah Koch on the previous page. Each was generated by a computer executing millions of iterations.











Pac-Man Meets Euler

by **π** and Ken Fan | edited by Jennifer Silva

The center of mass of a uniform circular disc, by symmetry, is located at its center. But where is the center of mass of a semicircle?

That's the question Girls' Angle member **π** posed to give herself practice evaluating integrals. One question led to another, and before long, we had a possibly new way to think about beautiful formulas of Euler and Vieta. Here, we'll retrace our journey.

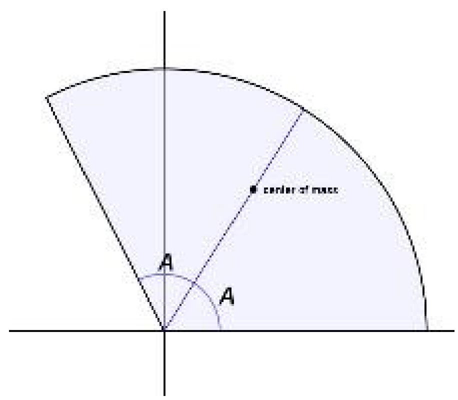
The Center of Mass of a Uniform Circular Sector

To begin, what is the center of mass of a uniform circular sector with a central angle $2A$? The reason for the factor of 2 will be explained later.

For definiteness, let's center a circle of radius 1 over the origin of a Cartesian xy-plane and measure $2A$ counterclockwise from the positive x-axis. Let the coordinates of the center of mass be (x_c, y_c). We can find x_c by computing the average value of x over the sector.

In polar coordinates, $x = r\cos\theta$, and since the area of the sector is $\frac{2A}{2\pi}\pi = A$, we have

$$\begin{aligned}
x_c &= \frac{1}{A}\int_0^{2A}\int_0^1 r\cos\theta \cdot r\,dr\,d\theta \\
&= \frac{1}{A}\int_0^{2A}\cos\theta\int_0^1 r^2\,dr\,d\theta \\
&= \frac{1}{A}\int_0^{2A}\cos\theta\left(\frac{1}{3}r^3\right)\Big|_{r=0}^{1}d\theta \\
&= \frac{1}{A}\int_0^{2A}\cos\theta\cdot\frac{1}{3}\,d\theta \\
&= \frac{1}{3A}(\sin\theta)\Big|_{\theta=0}^{2A} \\
&= \frac{1}{3A}\sin(2A).
\end{aligned}$$



By symmetry, the center of mass must sit along the bisector of the central angle. Therefore, if we define r_c to be the distance of the center of mass from the vertex of the sector, then

$$r_c\cos A = \frac{1}{3A}\sin(2A).$$

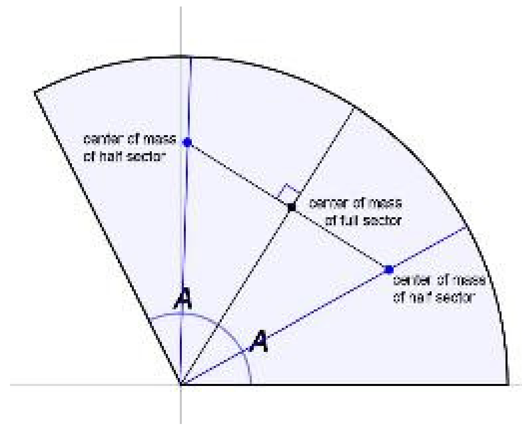
Solving for r_c, we find $r_c = \frac{1}{3A}\frac{\sin(2A)}{\cos A} = \frac{2\sin A\cos A}{3A\cos A} = \frac{2}{3}\frac{\sin A}{A}$.

The location of the center of mass of a uniform circular sector is well known. For instance, Gearhart and Shultz derive it in their article about the function $\sin x / x$ [1], which is also known as the "sinc function," sinc x. Thus, $r_c = 2/3$ sinc A. As Gearhart and Shultz note, the limit of r_c as A tends to 0 is 2/3, which is how far the center of mass of a uniform isosceles triangle is from its apex along its axis of symmetry.

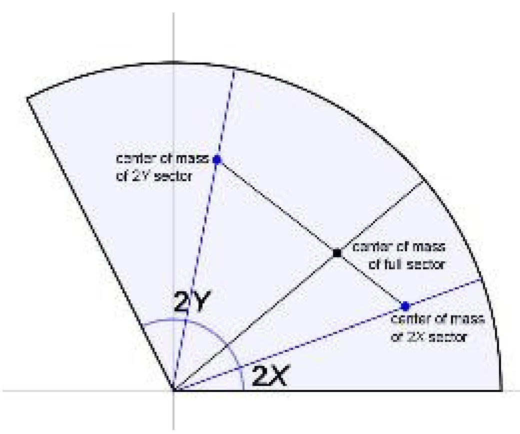
We shall write $r_c(A)$ to denote the distance of the center of mass of a uniform circular sector with central angle $2A$ from its apex to emphasize the dependence on A.

Exploiting Regularity and the Piecemeal Property of the Center of Mass

A circular sector is a union of circular sectors. This enables us to apply the piecemeal property for computing center of mass to get identities involving the sinc function. The piecemeal property for computing center of mass says that to compute the center of mass of an object, you can split it into pieces, place the entire mass of each piece at a point mass located at that piece's center of mass, then compute the center of mass of all of the point masses. For more details on this property, see "Center of Mass" in Volume 7, Number 3 of this *Bulletin*.



For example, let's bisect a sector with central angle $2A$, viewing it as a union of two congruent sectors with central angle A, then apply the piecemeal property of center of mass. Since the smaller sectors are congruent, they have the same mass. This means that the center of mass of the large sector must be located at the midpoint between the centers of mass of the two smaller sectors. In other words, the line segment connecting the centers of mass of the two smaller sectors intersects the bisecting radial line exactly at the center of mass of the larger sector. By symmetry, the bisecting radial line perpendicularly bisects the line segment connecting the center of masses of the smaller sectors, creating two identical right triangles. By looking at the measurements of one of these right triangles, we find that $r_c(A) = r_c(A/2)\cos(A/2)$. This identity can be derived directly by using the double-angle identity $\sin 2x = 2 \sin x \cos x$.



What happens if instead of bisecting our sector, we split it into pieces with central angles $2X$ and $2Y$, so that $2A = 2X + 2Y$? (See figure at left.) Because we are assuming uniform mass density, the masses of our circular sectors are proportional to their central angles. So when we split our sector in this way, the smaller sectors will contain a fraction of X/A and Y/A of the total mass.

The center of mass of the sector with central angle $2X$ has coordinates $(r_c(X)\cos X, r_c(X)\sin X)$. If the sector with central angle $2Y$ were positioned to lie against the positive x-axis, then its center of mass would be $(r_c(Y)\cos Y, r_c(Y)\sin Y)$ – but it's rotated from that position by an angle of $2X$. Consequently, its center of mass is actually located at the point

$$(r_c(Y)\cos(Y + 2X), r_c(Y)\sin(Y + 2X)).$$

The piecemeal property of center of mass gives us two identities, one for each coordinate:

$$r_c(X + Y) \cos(X + Y) = (X/A)\, r_c(X) \cos X + (Y/A)\, r_c(Y) \cos(Y + 2X)$$

and

$$r_c(X + Y) \sin(X + Y) = (X/A)\, r_c(X) \sin X + (Y/A)\, r_c(Y) \sin(Y + 2X).$$

Using our formula $r_c(A) = (2/3) \sin A / A$, these identities become:

$$\sin(X + Y)\cos(X + Y) = \sin X \cos X + \sin Y \cos(Y + 2X)$$

and

$$\sin^2 (X + Y) = \sin^2 X + \sin Y \sin(Y + 2X).$$

More generally, let $A_1, A_2, A_3, \ldots, A_n$ be n positive angle measures. By applying the piecemeal property to a sector with central angle $2(A_1 + A_2 + A_3 + \ldots + A_n)$, we get the identities:

$$\sin(\sum_{k=1}^{n} A_k)\cos(\sum_{k=1}^{n} A_k) = \sum_{k=1}^{n} \sin A_k \cos(A_k + 2\sum_{j=1}^{k-1} A_j)$$

and

$$\sin^2(\sum_{k=1}^{n} A_k) = \sum_{k=1}^{n} \sin A_k \sin(A_k + 2\sum_{j=1}^{k-1} A_j).$$

Set $A_k = x$ in these identities and get nifty formulas for

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n - 1)x$$

and

$$\sin x + \sin 3x + \sin 5x + \ldots + \sin(2n - 1)x.$$

Can you prove these trigonometric identities directly?
If you recognize these trigonometric identities from elsewhere, please let us know.

The Spiral

Note that our formula for $r_c(A)$ is not restricted to values of A between 0 and π. (A full circular disc corresponds to $A = \pi$.) When $A > \pi$, the object whose center of mass we're finding consists of a number of full circular discs plus a sector. The fact that $r_c(A) < 0$ for some values of A means that the center of mass is in the opposite direction from the origin of the radial line that makes an angle A with the positive x-axis.

Since the center of mass of a stack of circular discs is at their center, each time we add another circular disc to a sector, the center of mass is pulled closer toward the origin. This fact is reflected in the identity

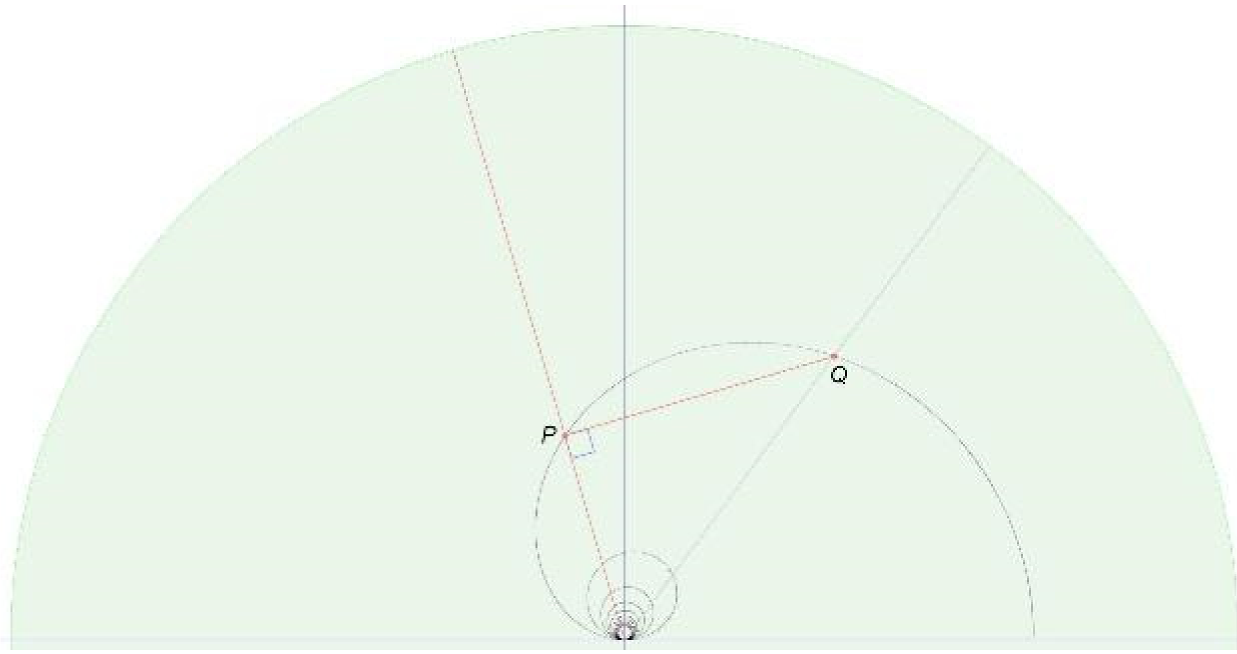
$$r_c(A + \pi k) = (-1)^k \frac{A}{A + k\pi} r_c(A).$$

All of these facts describe properties of the curve given in polar coordinates (r, θ) by

$$r = r_c(\theta).$$

This is why we chose to have $r_c(A)$ pertain to the center of mass of a sector with a central angle of $2A$ instead of A.

We've plotted the graph of $r = r_c(\theta)$ on the next page. Notice that the spiral never ventures below the horizontal axis. Can you explain why?



Consider the point P with polar coordinates $(r_c(A), A)$. It is the center of mass of a sector with central angle $2A$. If you draw a line through P perpendicular to its radial line, it must intersect the spiral at a point Q along the radial line with angle $A/2$.

What happens if we iterate this process? We should obtain a sequence of points P_k that converge to the point (2/3, 0). Let (r_k, A_k) be the polar coordinates of P_k. From P_k, we find P_{k+1} by traveling from P_k in a direction perpendicular to the radial line $\theta = A_k$ until we hit the radial line $\theta = A_k/2$. That is, $r_{k+1}\cos(A_k/2) = r_k$ and $A_{k+1} = A_k/2$.

Because the angles halve with each iteration, we have $A_k = A_0/2^k$. Thus,

$$r_{k+1} = \frac{1}{\cos(A_0/2^k)}\frac{1}{\cos(A_0/2^{k-1})}\frac{1}{\cos(A_0/2^{k-2})}\cdots\frac{1}{\cos(A_0/2^1)}r_0 .$$

Taking the limit, as k tends to infinity, we recover, after rearranging terms, a formula of Euler:

$$\frac{\sin A_0}{A_0} = \prod_{k=1}^{\infty}\cos(A_0/2^k) .$$

If we take $A_0 = \pi/2$, we recover Vieta's formula: $\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$.

Two beautiful formulas fall into one's lap just by playing around with the center of masses of circular sectors!

References

[1] W. B. Gearhart and H. S. Shultz, The function sin *x* / *x*, *The College Mathematics Journal*, Vol. 21, No. 2 (1990), 90-99.

The Quadratic Formula, Revisited

by Lightning Factorial | edited by Jennifer Silva

Last year, Addie Summer wrote an article about how she discovered the quadratic formula (see "Discovering the Quadratic Formula" on page 21 of Volume 10, Number 2 of this *Bulletin*). Soon after, Lightning Factorial mentioned that she had also found the quadratic formula, but in a different way than what Addie described. I asked Lightning to explain her method because it's interesting and because it shows that in math, there are often many ways to get from point A to point B. - *Editor*

This is the story of how I found the solutions to the equation $ax^2 + bx + c = 0$.

One day some time ago, a teacher challenged me with the following problem:

> I'm thinking of two numbers. The sum of the numbers is 100 and the product of the numbers is 2139. What are the two numbers*?*

At first, I thought, *maybe the two numbers are both 50*, but I computed $50 \times 50 = 2500$ and saw that that was wrong. Then I thought, *maybe the two numbers are 49 and 51*, which seemed promising because I could see that the units digit of 49×51 would be 9, which is what the product is supposed to end in. But then I computed that $49 \times 51 = 2499$, so that was wrong, too.

I figured that I could systematically continue checking pairs of numbers that add up to 100 by next considering 48 and 52, then 47 and 53, then 46 and 54, and so on, computing their products until I found the one whose product was 2139. But that seemed awfully tedious. *Besides*, I thought, *what if the numbers aren't integers?*

Then I noticed something. Two numbers that add up to 100 are balanced about 50. That is, if one of the numbers is $50 + x$, then the other number would have to be $50 - x$. To find the pair, I would just have to find the value of x that satisfies the equation

$$(50 + x)(50 - x) = 2139.$$

Expanding the left-hand side, I got $2500 - x^2 = 2139$, or $x^2 = 2500 - 2139 = 361$. Taking square roots, I found that $x = \pm 19$. Thus, the two numbers my teacher was thinking of were 69 and 31.

After I showed my teacher that I'd found the answer, he said, "I didn't know you knew the quadratic formula!"

But at the time, I *didn't* know the quadratic formula. "The what?" I replied.

My teacher said, "Didn't you use the quadratic formula to solve a quadratic equation?"

I thought about what I'd done. I recalled that I did, indeed, solve a quadratic equation, but that I hadn't used the quadratic formula. So I responded, "Well, I did solve a quadratic equation, but I didn't use any special formula."

My teacher replied, "Really? You solved the quadratic equation $x^2 - 100x + 2139 = 0$ without using the quadratic formula?"

$x^2 - 100x + 2139 = 0$? I thought to myself. "That's not the quadratic equation I solved!" I said.

My teacher said, "But you just did. The numbers I had in mind, 69 and 31, *are* the solutions to the quadratic equation $x^2 - 100x + 2139 = 0$. In general, if two numbers have a sum of S and a product of P, then they are the solutions to the quadratic equation $x^2 - Sx + P = 0$."

Really? I wondered in silence. *If a and b are the solutions to $x^2 - Sx + P = 0$, then we must have $x^2 - Sx + P = (x - a)(x - b)$, but $(x - a)(x - b) = x^2 - ax - bx + ab = x^2 - (a + b)x + ab$. I see! By comparing coefficients, $a + b = S$ and $ab = P$.*

"Neat!" I said. "I see what you mean."

Later that day, I tried to generalize the way I found the answer to my teacher's question about finding the solutions to the quadratic equation $x^2 - Sx + P = 0$.

As I saw, this equation is equivalent to finding two numbers that add up to S and whose product is P. Knowing that the solutions add up to S, I could say that if $S/2 + d$ was one of the solutions, then $S/2 - d$ would have to be the other, just as I did in the specific case $s = 100$. Since the product of the two numbers is P, I'd have the equation

$$(S/2 + d)(S/2 - d) = P.$$

Expanding and rearranging terms, this is equivalent to $d^2 = S^2/4 - P$. Hence the two solutions to the quadratic equation $x^2 - Sx + P = 0$ are

$$S/2 + \sqrt{S^2/4 - P} \text{ and } S/2 - \sqrt{S^2/4 - P}.$$

Then it dawned on me that the sum S and the product P could be anything, so this method enabled me to find a formula for the solutions of the general quadratic equation $ax^2 + bx + c = 0$. First, I divided by a to get $x^2 + (b/a)x + c/a = 0$. (If $a = 0$, then the equation isn't quadratic.) By comparing this last quadratic to the quadratic $x^2 - Sx + P$, I could see that its solutions would be two numbers that sum to $-b/a$ and whose product is c/a. Substituting $-b/a$ for S and c/a for P into the formulas above, I found that the solutions to $ax^2 + bx + c = 0$ are

$$-\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \text{ and } -\frac{b}{2a} - \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}.$$

These simplify to the traditional form of the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hard as I tried, though, I could not figure out a way to extend this kind of approach to one that could be used to solve the general cubic equation $ax^3 + bx^2 + cx + d = 0$. Can you?

Anna's Math Journal

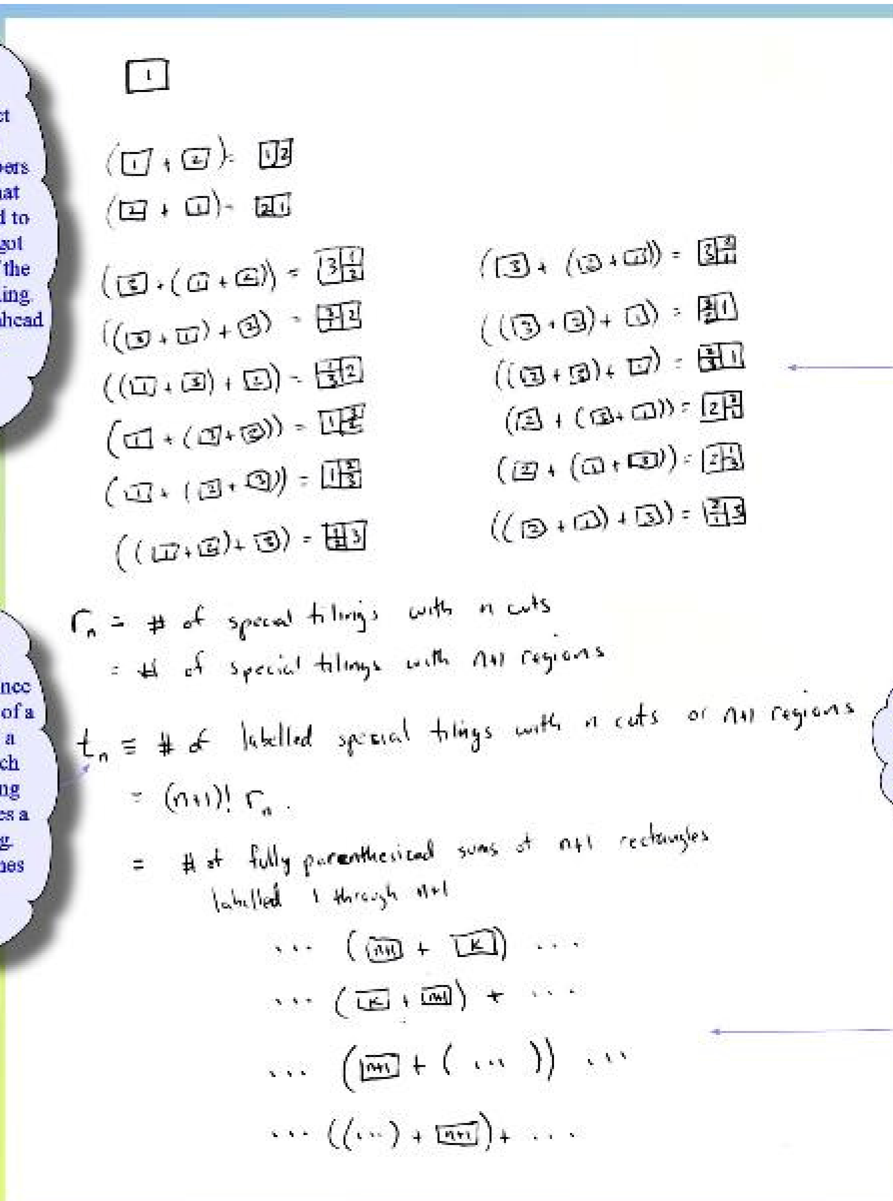
By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna finds a way to get the formula for the Catalan numbers without a generating function.

Last time, I broke down special tilings as sums of rectangles in order to connect special tilings to the original way Catalan found the numbers named after him. I noticed that by marking a rectangle added to such sums of n rectangles, I got every way of marking one of the rectangles in every special tiling. It occurred to me to just go ahead and uniquely mark *all* of the rectangles in a sum with the numbers 1 through n.

Every special tiling occurs with every possible way of numbering the tiles!



If I let t_n be the number of "labeled" special tilings with n cuts, I will get $(n-1)!r_n$, since a given parenthetic grouping of a sum of $n+1$ rectangles gives a specific special tiling, and each of the $(n+1)!$ ways of labeling the rectangles in the sum gives a different labeled special tiling. Also, every special tiling comes from a different parenthetic grouping.

If I can somehow find another way to count t_n, then I might be able to get the formula for r_n, without using a generating function.

If I have a sum of $n+1$ labeled rectangles, how can the rectangle labeled $n+1$ occur in the expression? As the sum is evaluated, when the $n-1$ rectangle is added in, it is either added to another rectangle, either on the left or right, or it is added to a parenthetic subexpression, either to the left or the right.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

So maybe induction on n will work. I just have to keep track of what I get when a rectangle labeled $n + 1$ is systematically added to an expression that is the sum of n labeled rectangles.

Build t_n from t_{n-1}?

Given a fully parenthesized sum of n rectangles labelled 1 through n, can form such an expression with $n+1$ rectangles by adding $\boxed{n+1}$

– before a left parenthesis: $\cdots(\cdots)\cdots \to \cdots(\boxed{n+1}+(\cdots))\cdots$

– before a rectangle: $\cdots(\boxed{k}+\cdots)\cdots \to \cdots((\boxed{n+1}+\boxed{k})+\cdots)\cdots$

or $\cdots(\cdots+\boxed{k})\cdots \to \cdots(\cdots(\boxed{n+1}+\boxed{k}))\cdots$

– after a rectangle: $\cdots(\boxed{k}+\cdots)\cdots \to \cdots((\boxed{k}+\boxed{n+1})+\cdots)\cdots$

or $\cdots(\cdots+\boxed{k})\cdots \to \cdots(\cdots+(\boxed{k}+\boxed{n+1}))\cdots$

– after a right parenthesis: $\cdots(\cdots)\cdots \to \cdots((\cdots)+\boxed{n+1})\cdots$

Maybe I should examine carefully how many expressions I get when I add a new rectangle to an existing sum of n rectangles. As I go through the expression left to right, I get a new expression by putting the new rectangle before a left parentheses, before a rectangle, after a rectangle, or after a right parentheses.

It seems all I have to do is count the number of rectangles and parentheses. For each labeled sum of $n + 1$ rectangles, I get one new labeled sum of $n + 2$ rectangles for each parenthesis, and two new labeled sums of $n + 2$ rectangles for each rectangle.

In a fully parenthesized sum with n rectangles, there are $2n-2$ parentheses, $n-1$ left and $n-1$ right.

I switched my indexing here. Above, I was thinking n rectangles to $n + 1$ rectangles. Below, I'm thinking $n + 1$ rectangles to $n + 2$ rectangles.

$$t_{n+1} = t_n \times (\#\text{ of parentheses in fully parenthesized sum of } n+1 \text{ rectangles} + 2\cdot\#\text{ of rectangles})$$

$$= t_n \times (2n + 2(n+1))$$

$$= t_n \cdot (4n+2)$$

$$= t_{n-1}(4n-2)(4n+2) = \cdots = t_0(2\cdot 6\cdot 10\cdots(4n-2)(4n+2))$$

$$= 2^{n+1}\cdot(1\cdot 3\cdot 5\cdots(2n-1)(2n+1))$$

$$= 2^{n+1}\cdot\frac{(2n+1)!}{2^n\cdot n!} = 2\,\frac{(2n+1)!}{n!}$$

$$r_n = \frac{1}{(n+1)!}t_n = \frac{1}{(n+1)!}\,\frac{2(2n-1)!}{(n-1)!}$$

$$= \frac{1}{n+1}\cdot\frac{1}{n!}\cdot\frac{2n\cdot(2n-1)!}{n\cdot(n-1)!}$$

$$= \frac{1}{n+1}\,\frac{(2n)!}{n!\,n!} = \frac{1}{n+1}\binom{2n}{n}$$

It worked! Counting special tilings does give a way to derive the formula for the Catalan numbers without using a generating function. The trick is to count labeled special tilings instead of plain ones. Neat!

ABB 10/24/17

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Mathematical Buffet

Scratch Work | Layout by Ashley Wang

If you're having trouble seeing how to solve a problem, putting your pencil to paper might be just what you need to do to get unstuck. But you don't have to take my word for it. Here we present a rare window onto mathematicians' personal process of discovery: a gallery of their scratch work, the refinery where crude ideas are forged into polished theories. *–Editor*

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Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 21 - Meet 1 September 7, 2017	Mentors:	Sarah Coleman, Karia Dibert, Anna Ellison, Neslly Estrada, Alexandra Fehnel, Suzanne O'Meara, Kate Pearce, Jacqueline Shen, Jane Wang

Mathematical identities combine lots of ideas into a compact package. It can take some effort to unpack these facts and realize the wealth of ideas contained within.

Consider the following problem: Pulled noodles is a Chinese noodle-making technique where a chef repeatedly stretches and doubles over a lengthening string of dough to make many strands of noodles quickly. Starting with a wad of dough, the chef pulls to make a rather thick noodle about a meter long. The chef folds the dough over and stretches again to make what looks like two strands of noodle, though it's really just one strand doubled over. Again, the chef folds this over to make what now looks like 4 strands, then 8 strands, then 16, each time doubling the apparent number of strands. When satisfied, the chef cuts off the ends to free the strands and cooks them in a pot of boiling water. Using this technique, a master chef can make just over a thousand strands of noodles in about a minute. Even though that's a lot of noodles, it doesn't require that much work because 2^{10} is 1,024, so only 10 doublings are needed. Now for the question: If it takes a chef a minute to make just over a thousand strands, how much longer will it take for the chef to make one million strands?

To answer this question, a naive approach would be to simply continue doubling, starting from $2^{10} = 1{,}024$, until we arrive at the first number that exceeds one million. We keep track of the number of doublings required to get there, and then compute the time required to do that based on the rate of 10 doublings per minute.

However, there's a faster way. We are looking for the exponent we must raise 2 to in order to first exceed one million, that is, what is the smallest whole number p such that $2^p > 10^6$? If we take the logarithm, base 2, of both sides, we get $p > \log_2 10^6$. When we study logarithms, we usually learn the identity

$$\log_b x^y = y \log_b x.$$

Perhaps you're adept at using it to solve any pertinent homework problems. *But do you see what is says about pulled noodles?*

It says that if you want to square the number of strands of noodles, then you double the time required to make them, and if you cube the number of strands, you triple the time. More generally, if you raise the number of strands to a power p, the time required to make them *multiplies* by p. One million is one thousand squared, so it'll take about two minutes to make a million strands. Hence, the answer to the question is that the chef can do it in another minute – provide the noodles don't break! This idea is packed into that logarithm identity. Do you see?

Here's another example. We often learn about similarity in high school geometry class. We learn that when two triangles are similar to each other, ratios of the lengths of corresponding sides are equal to the scaling factor. But do we realize that it is this principle that we are using all the time when we use Google Maps to estimate how far some city is? Do we realize that a submarine and a scale model of it are geometrically similar in the very sense of similarity defined in math class? All theorems about similarity apply directly to the submarine and its model.

Some of you might find everything I just wrote about "obvious". But for others, there can be a disconnect between what we learn in math class and the rest of our lives. I hope this helps you to see that there really isn't a barrier between math and the rest of the world.

Session 21 - Meet 2 September 14, 2017	Mentors:	Sarah Coleman, Karia Dibert, Anna Ellison, Neslly Estrada, Alexandra Fehnel, Kate Pearce, Jacqueline Shen, Sarah Tammen, Isabel Vogt, Jane Wang, Josephine Yu

At the club, we've been hosting a station whose goal is to get participants involved in a math investigation where they can start to "play with math" effectively. A number of interesting questions have come out of this station, and when an investigation gathers steam, that investigation is split off into its own station.

Another group began working on writing their own self-referential test.

Session 21 - Meet 3 September 28, 2017	Mentors:	Rachel Burns, Sarah Coleman, Anna Ellison, Neslly Estrada, Molly Humphreys, Suzanne O'Meara, Jacqueline Shen, Isabel Vogt, Josephine Yu

Some members worked on deciphering a secret code.

Others thought about equal area division problems, such as, how can you divide a pizza into seven equal parts? One might respond: Just cut it into 7 equal sectors. But, with what tools would you do this and how would you find the center of the pizza? Even if we assume that the pizza is a perfect circle, one would not necessarily want to stick one end of a compass into a piece of pepperoni and drag the other end through the cheese.

Some members worked on making perspective drawings of circles in various orientations. In a perspective drawing, circles are drawn either as circles, ellipses, or line segments. Can you describe the circumstances under which these three cases occur? Can you develop a reliable technique for sketching the elliptical case?

Session 21 - Meet 4 October 5, 2017	Mentors:	Rachel Burns, Sarah Coleman, Karia Dibert, Neslly Estrada, Danielle Fang, Molly Humphreys, Suzanne O'Meara, Isabel Vogt, Ashley Wang, Jane Wang, Josephine Yu

Some members posed to themselves the problem of counting the number of 4 by 4 matrices whose entries are all either 0 or 1 and whose row and column sums are all equal to 2. They found 90. Do you agree?

More generally, can you find a formula for the number of n by n matrices whose entries are all either 0 or 1 and whose row and column sums are all equal to 2? Note that if the row and column sums are all equal to 1, then the answer is $n!$ and the matrices are the set of permutation matrices.

One member figured out how to construct an equilateral triangle and a square using compass and straightedge, and she is now working on constructing a regular pentagon. This quest has led to an understanding of all the lengths and angles in a regular pentagon. Can you show that in a regular pentagon with side length 1 unit, the distance between any pair of non-connected vertices is equal to the golden mean?

Session 21 - Meet 5 October 12, 2017	Mentors:	Rachel Burns, Sarah Coleman, Neslly Estrada, Danielle Fang, Alexandra Fehnel, Kate Pearce, Jacqueline Shen, Sarah Tammen, Isabel Vogt, Jane Wang, Josephine Yu

Our youngest member **goey** succeeded in building a donut out of index cards that requires no tape or glue to hold together. She also found a clever way to fold a perfect equilateral triangle from a single index card.

Imagine making a perspective drawing of a person who is 6 feet tall. Let d be the distance that the person is from the drawing canvas, and let D be the ideal distance that an observer of the drawing is supposed to stand from the canvas when admiring it. Let r be d/D. How tall will the depiction of the person in the drawing be as a function of r? Note that when r is 0, the drawing of the person should be exactly 6 feet tall.

Session 21 - Meet 6 October 19, 2017	Mentors:	Rachel Burns, Anna Ellison, Neslly Estrada, Daniel Fang, Alexandra Fehnel, Kate Pearce, Jacqueline Shen, Sarah Tammen, Josephine Yu

Some members explored the relationships between points, lines, and planes in both 2 and 3-dimensional Euclidean space. Here are some sample questions:

1. For any two planes in space, show that there exists a plane perpendicular to both.

2. For any two lines in space, show that there exists a line that intersects both at a right angle.

3. If the faces of a tetrahedron are extended to complete planes, how many regions is space chopped up into?

Other members thought about the concept of parity (or evenness and oddness). Here's a sample question that involves parity: Consider an infinite piece of graph paper. In every square, place a coin, heads up. Your goal is to flip all the coins so that they are showing tails. If you touch a coin, it, and its 4 nearest neighbors are flipped. Which coins can you touch so that all the coins end up showing tails?

Session 21 - Meet 7 October 26, 2017	Mentors:	Rachel Burns, Danielle Fang, Alexandra Fehnel, Molly Humphreys, Suzanne O'Meara, Kate Pearce, Jacqueline Shen, Isabel Vogt, Ashley Wang, Josephine Yu

A number of members explore applications of similarity. Can you answer the following question entirely in your head?

Let $f(x)$ be a quadratic with roots 0 and 1 and a maximum at $x = 1/2$. Let $g(x)$ be a quadratic with roots -2 and 1. Furthermore, assume that the graphs of $y = f(x)$ and $y = g(x)$ are tangent to each other at $x = 1$. If the area captured below $y = f(x)$ and above the x-axis is 9 square units, what is the area captured below $y = g(x)$ and above the x-axis?

Calendar

Session 21: (all dates in 2017)

September	7	Start of the twenty-first session!
	14	
	21	No meet
	28	
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Session 22: (all dates in 2018)

February	1	Start of the twenty-second session!
	8	
	14	
	22	No meet
March	1	
	8	
	15	
	22	
	29	No meet
April	5	
	12	
	19	No meet
	26	
May	3	
	10	

Get ready for a Brand New Math Adventure! SUMIT 2018 registration opens at 2 pm on November 19, 2017. For more info, please visit www.girlsangle.org/page/SUMIT/SUMIT.html.

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________   Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton Univeresity
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, The Dartmouth Institute
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, assistant professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, associate professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ____________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____