Girls' Angle Bulletin

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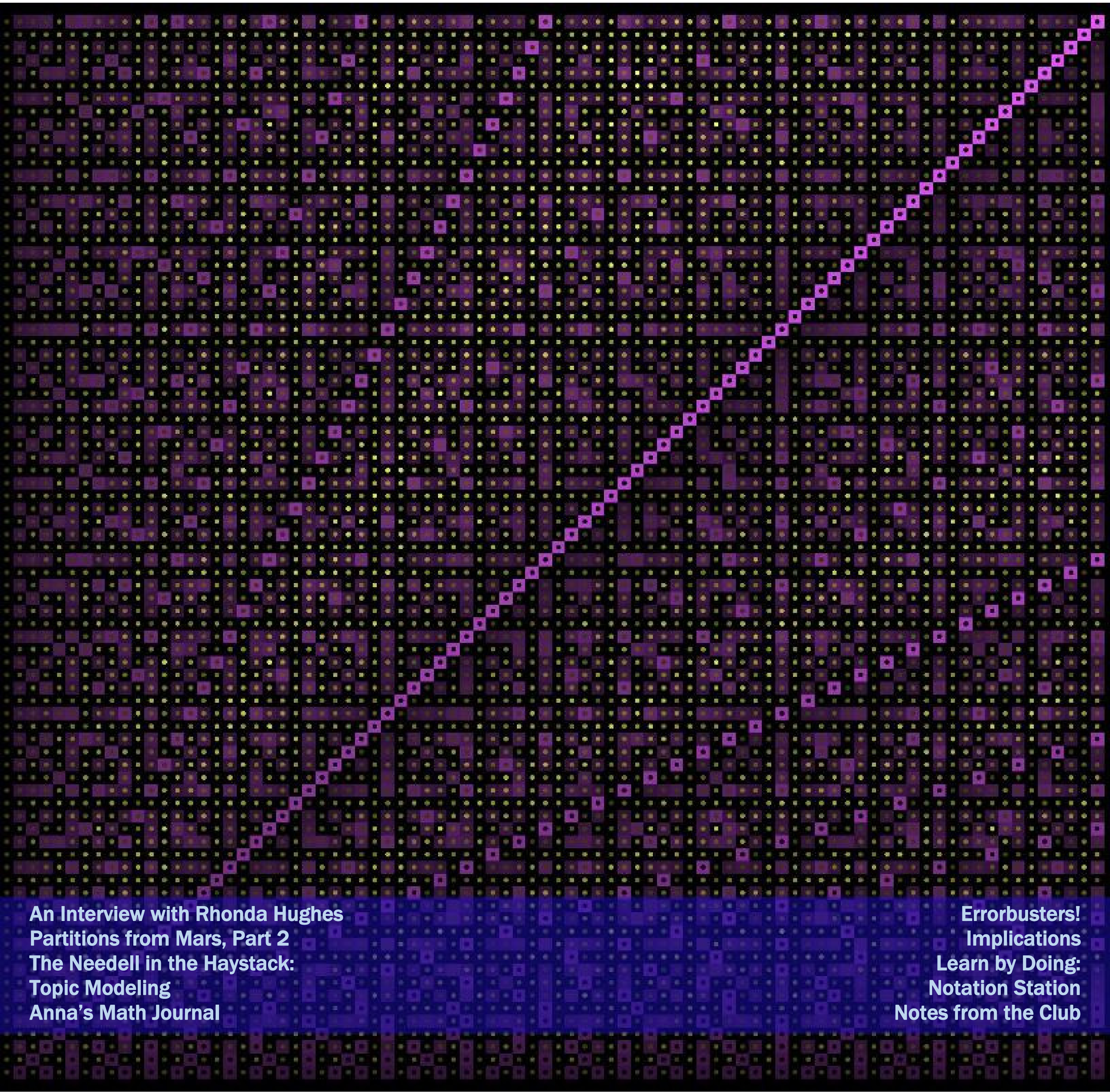
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Understanding that one thing is logically implied by another is key to grasping math. If you don't get this, start with simple connections and ruminate on them until you understand why they are connected, then expand from there. - Ken Fan, President and Founder

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Euclid and the GCF* by C. Kenneth Fan.

An Interview with Rhonda Hughes

Rhonda Hughes is the Helen Hermann Professor Emeritus of Mathematics at Bryn Mawr College. She grew up in Chicago where she was a cheerleader and valedictorian at Gage Park High School. She earned her doctoral degree in mathematics from the University of Illinois at Chicago under the supervision of Schmuel Kantorovitz.

Ken: I read a biographical sketch of you in Notable Women in Mathematics: A Biographical Dictionary. I was really struck by your zigzag path into academia. According to that biography, students at your high school had to choose between an academic track, which included math and physics, and a business track, which included secretarial skills such as shorthand and typing. You initially chose the secretarial track. I'm curious to know, how did you come to that decision?

Rhonda: My mother was a secretary, and many of my friends intended to be secretaries. No one in my family had gone to college at that point, and I did not think about it as an option until later in my senior year of high school. As you can tell, I was very much influenced by my peers!

Ken: Soon after choosing the business track, you switched to the academic track and thrived, and yet, despite acing all your classes and being selected valedictorian, you were not recognized as an outstanding student nor encouraged to go to college. Yet, eventually, you chose to try college. What I would like to know is, how did you come to that decision and what should a person in a similar situation ask of herself to make the right decision for herself between pursuing further study or entering the work force? You went against the prevailing advice. How does one navigate through all the conflicting advice and opinion to identify the right choice for oneself?

…as an undergraduate, I had been passive in my understanding. In graduate school, I realized there were many concepts that I did not really understand, so I started all over again. I held myself accountable for understanding, and being able to explain to others, and myself. For me, that made all the difference.

Rhonda: After abandoning the secretarial track (because I was very bad at it and worried that my grades would suffer), I decided to consider college. Some of my friends were applying to the University of Illinois at Urbana-Champaign. I got an application and filled it out. It was the only school I applied to and if I did not get in, I'm not sure what I would have done. Everything was very hit or miss, despite the fact that I took my studies seriously. High school was a very social time for me, I was a cheerleader and loved it. Based on my own experiences, I guess my motto is, "it's never too late."

Ken: In college, after a year and a half, you returned home and got a job, only to return to college half a year later as a math major. What led you to leave and then return? And what made you decide to return as a math major?

Rhonda: I was homesick and did not like the culture of a large residential school. I grew up in a rather sheltered environment and was somewhat intimidated by the more sophisticated, suburban students at college. I came home and got a job in an office knowing I would eventually return to school. I applied to transfer to Northwestern University and the University of Illinois at

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Rhonda Hughes and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Ken Fan
President and Founder
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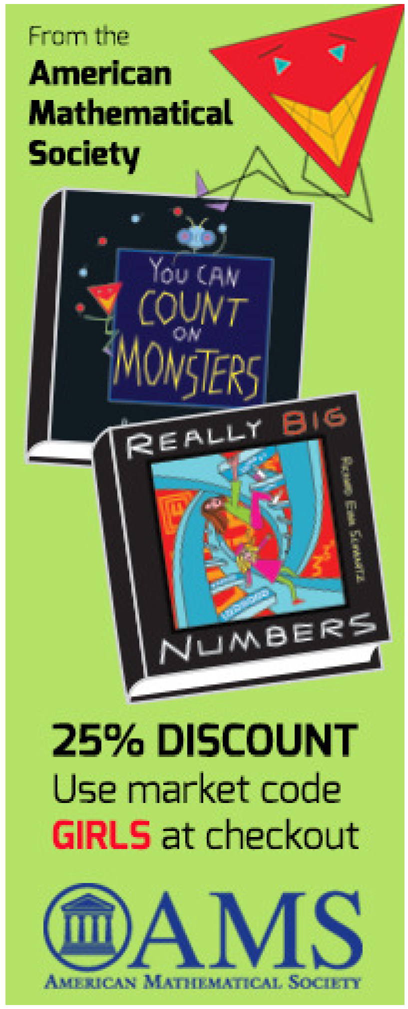

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Partitions from Mars, Part 2[1]

by Pamela E. Harris, Alexander Pankhurst, Cielo Perez, and Aesha Siddiqui[2]
edited by Jennifer Silva

In memory of Bertram Kostant (May 24, 1928 – February 2, 2017)

In Part 1, we introduced Kostant's partition function [2] as the counting function that gives the number of different ways we could spend martian currency, modeled by vectors, on a particular list of martian arcade prizes. The currency on Mars was given by the vectors

$$\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \text{and}\ \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

As we saw, these currencies could not be exchanged for one another. The items we could purchase at the martian arcade were:

Item	**Martian Price**
Eclipse sunglasses	α_1
Deimos rocks	α_2
Astronaut bobblehead	α_3
Solar system marble set	$\alpha_1 + \alpha_2$
Pocket telescope	$\alpha_2 + \alpha_3$
Antigravity socks	$\alpha_1 + \alpha_2 + \alpha_3$

The question remains: In how many ways can we exchange $m\alpha_1 + n\alpha_2 + k\alpha_3$ for the martian arcade prizes when m, n, and k are nonnegative integers? Recall that we let $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$ denote the answer. Last time, we obtained a partial result that was restricted to the case where $m, k \geq n$, which we repeat here:

Proposition 1. If m, n, and k are nonnegative integers with $m, k \geq n$, then

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = (n + 1)(n + 2)(n + 3)/6. \tag{1}$$

The proof of Proposition 1 used the observation that the answer was equivalent to a so-called balls-in-urns problem. Unfortunately, the balls-in-urns counting technique does not generalize to the case where there are no conditions between the nonnegative integers m, n, and k, so a new proof technique is needed.

However, we do have the following general result:

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[1] This content is made possible by a grant from MathWorks.
[2] All authors are affiliated with the Department of Mathematics and Statistics at Williams College.

Proposition 2. If m, n, and k are nonnegative integers, then

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = \sum_{f=0}^{\min(m,n,k)} \sum_{d=0}^{\min(m-f,n-f)} \sum_{e=0}^{\min(n-f-d,k-f)} 1.$$

Proof. Let a, b, c, d, e, and f be the number of eclipse sunglasses, Deimos rocks, astronaut bobbleheads, solar system marble sets, pocket telescopes, and antigravity socks we are purchasing, respectively. Note that f is the only one of these six variables that involves all three currencies, so let's focus first on f. Observe that f can never be greater than $\min(n, m, k)$. Hence, $0 \le f \le \min(n, m, k)$, and any choice of f in this range is possible. Once f is selected, we can next consider possible values for d.

After we purchase f antigravity socks, we will have $m - f$ α_1's, $n - f$ α_2's, and $k - f$ α_3's left. Therefore, the largest number of solar system marble sets that we can purchase is equal to the smaller of $m - f$ and $n - f$; that is, $0 \le d \le \min(m - f, n - f)$, and any choice of d in this range is possible.

After purchasing f antigravity socks and d solar system marble sets, the amount of currency we will have left over will be $(m - f - d)\alpha_1 + (n - f - d)\alpha_2 + (k - f)\alpha_3$. Therefore, the largest number of pocket telescopes we could purchase with what is left of our currency would be the smaller of $n - f - d$ and $k - f$; that is, $0 \le e \le \min(n - f - d, k - f)$, and any choice of e in this range is possible.

Once we've settled on how many antigravity socks, solar system marble sets, and pocket telescopes to purchase, there is only one way to spend the rest of our martian currency on eclipse sunglasses, Deimos rocks, and astronaut bobbleheads, namely $a = n - d - f$, $b = m - d - e - f$, and $c = k - e - f$.

In the sum $\sum_{f=0}^{\min(m,n,k)} \sum_{d=0}^{\min(m-f,n-f)} \sum_{e=0}^{\min(n-f-d,k-f)} 1$, each term corresponds to a valid choice of f, d, and e. Since for each valid choice of f, d, and e there's only one way to spend the rest of the martian currency, we can set each term in the sum to 1 (as it is) to get $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$. Thus,

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = \sum_{f=0}^{\min(m,n,k)} \sum_{d=0}^{\min(m-f,n-f)} \sum_{e=0}^{\min(n-f-d,k-f)} 1. \qquad \square$$

The formula for $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$ in Proposition 2 looks nice and is valid in general; unfortunately, though, it isn't very helpful when we actually need to use it. For example, computing $\wp(4\alpha_1 + \alpha_2 + 3\alpha_3)$ using equation (1) would be much more straightforward than using the summation formula. Can we simplify the summation formula to obtain a formula more like that of equation (1) in general?

While there isn't a simple formula that works in general, we can split the possible values of m, n, and k into a few cases for which there *is* a formula analogous to equation (1) that is easier to apply than the summation formula.

For example, consider the case where $k \ge n \ge m \ge 0$. Under these conditions, note that $\min(m, n, k) = m$, $\min(m - f, n - f) = m - f$ (since $n \ge m$), and $\min(n - f - d, k - f) = n - f - d$ since $k \ge n$ and d is nonnegative. Then

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = \sum_{f=0}^{\min(m,n,k)} \sum_{d=0}^{\min(m-f,n-f)} \sum_{e=0}^{\min(n-f-d,k-f)} 1 = \sum_{f=0}^{m} \sum_{d=0}^{m-f} \sum_{e=0}^{n-f-d} 1 = \sum_{f=0}^{m} \sum_{d=0}^{m-f} (n - f - d + 1).$$

Notice that

$$\sum_{d=0}^{m-f}(n-f-d+1)=\sum_{d=0}^{m-f}(n-f+1)-\sum_{d=0}^{m-f}d=(n-f+1)(m-f+1)-(m-f+1)(m-f)/2 .$$

Thus,

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = \sum_{f=0}^{m}((n-f+1)(m-f+1)-(m-f+1)(m-f)/2) .$$

After some algebraic manipulation, we see that

$$(n - f + 1)(m - f + 1) - (m - f + 1)(m - f)/2 = (m + 1)(n + 1 - m/2) - (n + 3/2)f + f^2/2.$$

Thus, $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = \sum_{f=0}^{m}(m+1)(n+1-m/2)-(n+3/2)\sum_{f=0}^{m}f+\frac{1}{2}\sum_{f=0}^{m}f^2$. Let's consider each of the three sums in this expression separately.

The first sum has terms that are all constant with respect to f, hence

$$\sum_{f=0}^{m}(m+1)(n+1-m/2) = (m + 1)^2(n + 1 - m/2).$$

The second sum is the sum of the first m positive integers, hence

$$(n+3/2)\sum_{f=0}^{m}f = (n + 3/2)m(m + 1)/2.$$

The third sum is the sum of the first m positive perfect squares. It has a well-known formula, namely $1^2 + 2^2 + 3^2 + \ldots + m^2 = m(m + 1)(2m + 1)/6$, hence

$$\frac{1}{2}\sum_{f=0}^{m}f^2 = m(m + 1)(2m + 1)/12.$$

Substituting these back into our expression for $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$, we find

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = (m + 1)^2(n + 1 - m/2) - (n + 3/2)m(m + 1)/2 + m(m + 1)(2m + 1)/12;$$

after some algebra, this simplifies to

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = (m + 1)(m + 2)(3n - 2m + 3)/6. \tag{2}$$

So in the case where $k \geq n \geq m \geq 0$, we've found a formula analogous to the one in Proposition 1.

As an example, let's use it to compute how many ways we can spend $3\alpha_1 + 4\alpha_2 + 5\alpha_3$ units of martian currency. We find $\wp(3\alpha_1 + 4\alpha_2 + 5\alpha_3) = (3 + 1)(3 + 2)(3\cdot 4 - 2\cdot 3 + 3)/6 = 30$.

In a similar manner, one can derive formulas for other conditions between the integers m, n, and k that cover all possible cases. Such formulas were given by De Loera and Sturmfels in reference [1], and we provide them below adapted to the notation we've been using.

Conditions on m, n, and k	$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$
$m, k \geq n$	$(n + 1)(n + 2)(n + 3)/6$
$k \geq n \geq m$	$(m + 1)(m + 2)(3n - 2m + 3)/6$
$m \geq n \geq k$	$(k + 1)(k + 2)(3n - 2k + 3)/6$
$n \geq k \geq m$ and $n \geq m + k$	$(m + 1)(m + 2)(3k - m + 3)$
$n \geq m \geq k$ and $n \geq m + k$	$(k + 1)(k + 2)(3m - k + 3)$
$n \geq m \geq k$ and $m + k \geq n$	$(6 - k^3 - 5m^3 + 3k^2(n - 1) + 2n - 3n^2 + n^3 + m^2(9n - 3))/6 + (m(5 + 6n - 3n^2) + k(4 + 3m + 6n - 3n^2))/6$
$n \geq k \geq m$ and $m + k \geq n$	$(6 - m^3 - 5k^3 + 3m^2(n - 1) + 2n - 3n^2 + n^3 + k^2(9n - 3))/6 + (k(5 + 6n - 3n^2) + m(4 + 3m + 6n - 3n^2))/6$

Notice that the first case corresponds to Proposition 1 and the second case corresponds to equation (2). We invite you to supply the details for deriving the remaining cases.

Further directions

Mathematicians continue to uncover new behavior and patterns in the study of partition functions, following in the footsteps of Kostant. Yet there is still much we do not know. One avenue of research in vector partition functions is to provide useful formulas, called closed formulas, akin to those we presented above. The problem we considered is associated with something known as the Lie algebra of type A_3. De Loera and Sturmfels [1] provide the corresponding polynomials for the Lie algebra of type A_4 and A_5, but the polynomials for the general case type A_n, for $n > 5$, are not known explicitly. In type A_n, there are n different, non-interchangeable currencies α_1 through α_n and there are $n(n + 1)/2$ items we can purchase with values $\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \ldots + \alpha_j$, for all possible pairs $1 \leq k \leq j \leq n$. In fact, we still do not even know how many sets of inequalities we must consider to give the specific polynomials! Notice that we needed seven sets of inequalities in type A_3 to be able to produce polynomial formulas. In type A_4, A_5, and A_6, we need to consider 4, 480, and 44,288 sets of inequalities, respectively. But knowing how many sets of inequalities are needed for type A_n for $n > 6$ is an unsolved problem.

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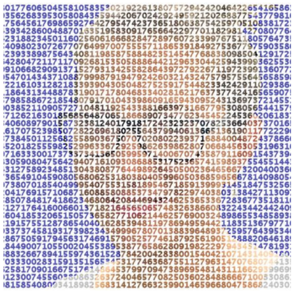
[1] J. A. De Loera and B. Sturmfels, Algebraic Unimodular Counting. *Mathematical Programming, Series B*, 96 (2003), 183 –203.

[2] B. Kostant, A formula for the multiplicity of a weight, *Proceedings of the National Academy of Sciences of the United States of America,* 44 (1958), 588–589.

[3] P. E. Harris, A. Pankhurst, C. Perez, and A. Iddiqui, Partitions from Mars, Part 1, *Girls' Angle Bulletin*, Vol. 11, No. 2 (2017), 7–11.



The Needell in the Haystack

Topic Modeling[1]
by Deanna Needell | edited by Jennifer Silva

Recent advances in technology have led to a monumental increase in large-scale data across many platforms. One may think that more data means more *information*, but the large-scale nature of modern data actually ends up choking classical analytical methods, making information extraction more challenging than ever before. There is a serious shortage of reliable, accurate, and efficient techniques for analyzing this data. Such mathematical techniques are imperative in order to synthesize the abundance of information in this data and truly harness its power to promote significant advances.

Large-scale data takes many forms, ranging from high resolution images to sensor data, to text and graphical data. For concreteness, imagine that your data consists of millions of scanned books, each containing thousands of pages of text. Suppose your goal is to locate books about basketball, in particular the history of basketball, from this huge dataset. Your first thought might be to run a search on your computer for the word “basketball.” You try this and – tada! Your computer instantaneously finds the word “basketball.” All good, right? Not quite; the book is a fictional murder-mystery novel where the antagonist happens to play basketball in one of the scenes. This clearly isn’t what you were looking for; you want books that are entirely or mostly about basketball! The reality is that you would be hard-pressed to find a book on basketball through such a search, much less one specifically focused on its history. You could search for a *very* long time and would likely give up before finding even one book that actually pertains to your topic. Fortunately, mathematical tools like *topic modeling* can save the day.

Topic modeling is a suite of mathematical techniques that find hidden thematic structures in data. Consider the example above about books and basketball. Suppose you represent this dataset in a large matrix where each column of the matrix represents a book and each row represents a word. Imagine that the cells of the matrix contain the number of times that word appears in that book; for example, if the 243^{rd} column corresponds to the book *Charlie and the Chocolate Factory* and the $9{,}727^{\text{th}}$ row corresponds to the word “blueberry,” the entry in the (9727, 243) location would contain the number 12, meaning that the word “blueberry” appears twelve times in the novel[2]. We’ll call this frequency matrix X. Let’s say it is of size $w \times b$, where w stands for the total number of possible words and b is the total number of books in our dataset.

The goal of topic modeling is to find a representation of the data as a set of objects from a *latent topic distribution*, where each topic (or theme) is a learned distribution over words. Methods include non-negative matrix factorization (NMF) with re-weighting, randomization approaches, point process models, and many others. We will examine the mathematics of the simplest form of NMF, which gives some really nice results in practice.

Just as integers can be factored, so can matrices. Different matrix factorizations can be used to highlight different representations. The goal of the NMF method is to factor the $w \times b$ data matrix X into a product $X = AS$. Here, the matrix A is of size $w \times t$; it has rows representing the words and columns representing the *topics* or *themes* within the data. The matrix S is of size $t \times b$, and it has rows representing the topics and columns representing the books. Concretely, the entry in the $(i, j)^{\text{th}}$ location of A, written A_{ij}, represents the level of involvement that the i^{th}

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[1] This content supported in part by a grant from MathWorks.
[2] The author has shamelessly made up this statistic.

word has in the j^{th} topic. Similarly, S_{jk} represents how well the k^{th} book represents the j^{th} topic. For example, suppose we had a large data matrix X that was factored as $X = AS$, where A and S contained the following entries:

A:

	Topic 1	Topic 2	Topic 3
blueberry	5	0.2	0.15
chocolate	8	0.01	0.07
fertilizer	0	9.5	0
wand	0.03	0	32
miracle	0	0	16

S:

	Charlie and the Chocolate Factory	*Gardening for Dummies*	*Harry Potter*	*The Constitution*
Topic 1	14	1.1	0.8	0
Topic 2	0	18	0	0
Topic 3	4.2	0.03	57	0

Let's now take a closer look at the topics this (piece of a) factorization reveals. Notice we are showing a snippet of five words, four books, and three revealed topics. Let's first look at the snippet of matrix A. The three columns represent the three revealed topics, and the values in the cells should give us some information about what each topic denotes. For example, the first column has large values corresponding to "blueberry" and "chocolate," so perhaps this topic captures a theme of candy or food. The second column has a large value for "fertilizer," so this topic may have something to do with gardening. The third column has large values in "wand" and "miracle," so it likely represents something to do with magic. Now let's examine the snippet of the S matrix. The first row represents the first topic, which we just suggested signifies a candy or food theme, the second row is a gardening theme, and the third row is about magic. If we now look at the first column of S, it has its largest value of 14, corresponding to the candy theme. This tells us that the book *Charlie and the Chocolate Factory* contains a theme about candy! Similarly, the relatively large value of 4.2 in that same column suggests that it also contains a – somewhat less-pronounced – theme about magic. Analysis of the other columns in S suggests that the book *Gardening for Dummies* contains a theme about gardening and a small amount of candy/food, whereas *Harry Potter* has a significant theme about magic.

The key observation here is that these "themes" were revealed to us by this factorization and were *not* input to the method; they were revealed automatically by the NMF method itself. Therefore, we can run the NMF method on any large dataset of books (or any other type of data), and it will *automatically* detect themes and identify which books correspond to which themes!

So how does NMF work? Note that the values in the cells of the matrices A and S should be nonnegative, since negative values would have no meaning in our context. In addition, we want the product AS to accurately represent our original data matrix X. Mathematically, we ask that $\| X - AS \|_F$ be small, where $\| M \|_F$ denotes the Frobenius norm of M (equivalent to the Euclidean norm of the matrix M unraveled into a vector; i.e., if $M = (M_{ij})$, then $\|M\|_F = \sqrt{\sum_{i,j} M_{ij}^2}$).

Thus, the NMF method solves as follows:

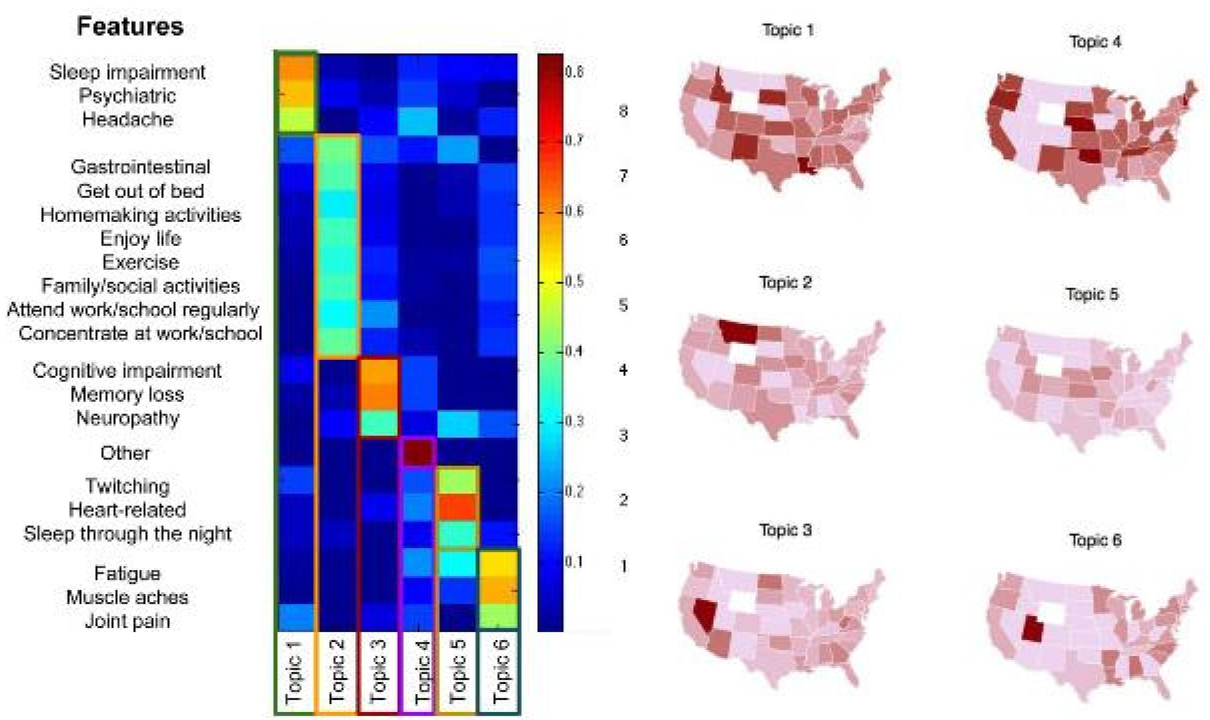
$$(A, S) := \text{argmin}_{A,\,S} \| X - AS \|_F \text{ s.t. } A \geq 0, S \geq 0, A \in \mathbf{R}^{w \times t}, S \in \mathbf{R}^{t \times b},$$

where the use of $\geq$ here denotes the entry-wise inequality and $\mathbf{R}$ is the set of real numbers. Note that the number of topics, t, must be an input to the method. One need take some care in selecting this parameter; if t is too large, the method will "overfit" (e.g., in the extreme case of

$t \approx w$, each topic will simply be its own word), whereas if t is too small, topics will be unnaturally clumped together. This parameter can be selected by hand or by using techniques such as cross-validation; computing the optimal value of t remains an interesting direction of study.

Note that the NMF optimization problem above can be solved by various algorithms, including alternating iterative and gradient descent algorithms. However, even for a fixed value of t, the factorization need not be unique. For example, if AS is a minimizing factorization, then so is $(AB)(B^{-1}S)$ for any invertible nonnegative monomial matrix B. In this case, the transformation B represents a scaling and/or permutation matrix, and corresponds to a re-ordering or re-scaling of the topics. Of course, this would not affect the interpretability of the revealed topics.

So what does NMF look like on real data? Recently, we acquired large-scale survey data from lymedisease.org, which launched a patient survey asking patients diagnosed with Lyme disease about symptoms, treatment, diagnosis and general well-being. We took a portion of this data, about symptomology, and ran NMF. The results are shown in the figure below.



The color bar on the left displays six topics revealed by NMF, with associated words listed on the left-hand side. We notice that the first topic seems related to sleep and headache, the second to life activities, the third neurologic, and so on. The colors in the color bar represent the strength of that variable (word) within each topic. Next, for each topic we plot its prevalence by geographic state across the country. We find some interesting surprises, for example, that Topics 2, 3, and 6 are much more significant in Montana, Nevada and Utah, respectively. Topic 4 seems more prevalent on the coasts, and Topic 1 appears fairly uniformly. This type of investigation can lead to greater understanding of the illness, its proper diagnosis and treatment, and geographic characteristics. And all from a matrix factorization!

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna tackles a problem where rows of Pascal's triangle, modulo 3, are read as ternary numbers.

Interpret the rows of Pascal's triangle, reduced mod 3, as base 3 numbers. How many of the first 2018 are odd?

```
                1
              1   1
            1   2   1
          1   0   0   1
        1   1   0   1   1
      1   2   1   1   2   1
    1   0   0   2   0   0   1
  1   1   0   2   2   0   1   1
1   2   1   2   1   2   1   2   1
```

$1_3 = 1$
$11_3 = 4$
$121_3 = 16$
$1001_3 = 28$
$11011_3 = 112$
$121121_3 = 448$

I'll go ahead and work out the first few numbers in the sequence.

Let $T_n = \sum_{k=0}^{n} \overline{\binom{n}{k}} 3^k$, where $\overline{\binom{n}{k}}$ is the remainder $\binom{n}{k}$ leaves upon division by 3.

In a base 3 number, the values of the different places are all powers of 3, which are all odd numbers. For a sum of numbers to be odd, there have to be an odd number of odd numbers. That means there have to be an odd number of 1's in the reduced row of Pascal's triangle.

T_n is odd $\Leftrightarrow$ an odd number of entries in the n^{th} reduced row of Pascal's triangle are odd.

By symmetry, T_n is odd $\Leftrightarrow$ n is even, say $n = 2m$, and

$$\binom{2m}{m} \equiv 1 \pmod 3.$$

But Pascal's triangle is symmetric, so any off-center 1's will come in pairs. That means I only have to see if the middle term is 1.

$$\binom{2m}{m} = \frac{(2m)!}{m!m!}$$

Let $p_3(x)$ be the exponent of 3 in the prime factorization of x.

When is $2m$ choose m congruent to 1 mod 3? I guess I have to try to figure out if all factors of 3 in $(2m)!$ are cancelled by the factors of 3 in $m!$ squared.

$$p_3((2m)!) = \left\lfloor \frac{2m}{3} \right\rfloor + \left\lfloor \frac{2m}{3^2} \right\rfloor + \cdots$$

$$p_3(m!m!) = 2p_3(m!) = 2\left(\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{m}{3^2} \right\rfloor + \cdots\right)$$

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Compare $\lfloor \frac{2m}{3} \rfloor$ with $2\lfloor \frac{m}{3} \rfloor$.

$m = 3k + j$, $j \in \{0, 1, 2\}$.

$2\lfloor \frac{m}{3} \rfloor = 2k$

$\lfloor \frac{2m}{3} \rfloor = \lfloor \frac{6k+2j}{3} \rfloor = 2k + \lfloor \frac{2j}{3} \rfloor$

So $\lfloor \frac{2m}{3} \rfloor = 2\lfloor \frac{m}{3} \rfloor \iff m = 0, 1 \pmod 3$

$\lfloor \frac{2m}{3} \rfloor = 2\lfloor \frac{m}{3} \rfloor + 1 \iff m = 2 \pmod 3$

Compare $\lfloor \frac{2m}{3^p} \rfloor$ with $2\lfloor \frac{m}{3^p} \rfloor$

Say $m = 3^p k + j$, $j \in \{0, 1, 2, \ldots, 3^p - 1\}$

$2\lfloor \frac{m}{3^p} \rfloor = 2k$

$\lfloor \frac{2m}{3^p} \rfloor = \lfloor \frac{2 \cdot 3^p k + 2j}{3^p} \rfloor = 2k + \lfloor \frac{2j}{3^p} \rfloor$

So $\lfloor \frac{2m}{3^p} \rfloor = 2\lfloor \frac{m}{3^p} \rfloor \iff m = 0, 1, 2, \ldots, \frac{3^p - 1}{2} \pmod{3^p}$

$\lfloor \frac{2m}{3^p} \rfloor = 2\lfloor \frac{m}{3^p} \rfloor + 1 \iff m = \frac{3^p + 1}{2}, \ldots, 3^p - 1 \pmod{3^p}$

In order for $p_3\left(\binom{2m}{m}\right) = 0$, we must have

$m \equiv 0, 1 \pmod 3$

Hm. I think I know where this is going . . .

$\equiv 0, 1, 2, 3, 4 \pmod 9$

$\equiv 0, 1, 2, \ldots, 13 \pmod{27}$

etc.

Write m in base 3.

Units digit must be 0 or 1.

Threes digit must be 0 or 1.

⋮

In base 3, m consists only of the digits 0 and 1.

Of the first 3^N pos. integers, 2^N have this property, so only $\left(\frac{2}{3}\right)^N$ of the first 3^N positive integers m have $\binom{2m}{m} \not\equiv 0 \pmod 3$.

That's RARE!

But of those m whose base 3 expansion consist of only digits 0 and 1, which satisfy $\binom{2m}{m} \equiv 1 \pmod 3$?

Great! So now I know when $2m$ choose m is not divisible by 3. So next, I have to figure out which are congruent to 1, mod 3.

ABB 2/23/18

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Errorbusters!

by Ken Fan / edited by Jennifer Silva

Numerical computation errors are like mosquitos: they're seemingly unavoidable and really annoying. How can we minimize them?

A number of neat ideas have been developed to help detect numerical computation errors. For example, if you're adding a column of numbers, you can redo the computation but only pay attention to the units digits. Similar to a checksum, if the result of your units digit calculation doesn't match your tally's units digit, you made a computational error, which you can go back and fix. You can also check for the reasonableness of your answer. For example, if you're computing the length of some segment and your answer turns out to be negative, you'd better go back and double-check your work!

In this installment of "Errorbusters!," we're going to illustrate the best way to avoid numerical computation errors, which is to *avoid making numerical computations* in the first place! How? *Use variables for as long as possible*.

If you have to compute a specific answer, some numerical computation will be unavoidable; however, the more computation you can avoid, the less likely you'll succumb to a computational error. I'll first illustrate with a simple example, then give a more complex one where the power of the technique is even more evident.

Consider the following typical geometry problem:

> *What is the ratio of the volume of a sphere with a 5-inch radius to that of its circumscribed cylinder? Use 3.14 for π.*

Let's begin by showing a solution that we aim to *avoid*. The strategy will be to compute the volume of the sphere, then the volume of the cylinder, then take the ratio:

The volume of a sphere is given by the formula $4\pi r^3/3$, where r is the radius. Therefore, the volume of the given sphere is

$$\frac{4}{3}\pi 5^3 = \frac{4}{3}(3.14)5^3 = \frac{4}{3}(3.14)(125) = \frac{500}{3}(3.14) = \frac{1570}{3} = 523\frac{1}{3} \text{ cubic inches.}$$

Note that the volume we just computed is really an approximation because π is not, in fact, equal to 3.14. But 3.14 is a very common approximation to π used in schools.

The formula for the volume of a cylinder is $\pi r^2 h$, where r is the base radius and h is the height. The circumscribed cylinder of the given sphere has base radius 5 inches and height 10 inches. Therefore, the volume of the cylinder in the problem is

$$\pi 5^2(10) = (3.14)(25)(10) = 250(3.14) = 785 \text{ cubic inches.}$$

Again, note that this volume is an approximation.

Finally, we compute the ratio (523 1/3)/785 = (1570/3)/785 = 2/3.

That's correct, so what is there to criticize? To answer, let's see how this problem can be solved in a way that minimizes numerical computation. Rather than plugging numbers into formulas and evaluating, let's use variables for as long as possible.

The volume of a sphere is given by $4\pi r^3/3$, where r is the radius. The circumscribed cylinder about a sphere of radius r has base radius r and height $2r$. The volume of a cylinder is given by $\pi r^2 h$, where r is the base radius and h is the height. Therefore, the volume of our cylinder is $\pi r^2(2r)$. The desired ratio is:

$$\frac{4\pi r^3/3}{\pi r^2(2r)}.$$

There is a factor of π in both the numerator and denominator, which means it cancels! Not only does this signify that the multiplications by 3.14 in the previous solution were completely unnecessary, but we also don't need to use an approximation for π at all; our solution is going to be exact.

Furthermore, there are three factors of r in both numerator and denominator, so those also cancel. After canceling, we are left with (4/3)/2. At this point, to simplify further, we do have to perform a numerical computation. But note that all we have to do is divide 4 by 2, and we're far less likely to miscompute 4/2 than we are to err when we compute 250×3.14.

Beyond minimizing computational error, there are even greater benefits to using variables until the last moment. By using variables, we learned that the ratio of the volume a sphere to its circumscribed cylinder is *independent* of the sphere's radius. This ratio will be the same regardless of the size of our sphere. Observing this enables us to ponder *why*. *Why* does this ratio not depend on the size of the sphere? If you think about that, you will increase your understanding of the important geometric concept of similarity. By contrast, it would be easy to miss this opportunity for increased understanding if our instinct is to grab numbers and start munching on them right away.

Let's tackle a more elaborate problem to underscore the power of the technique, problem 15 from the 2017 American Invitational Mathematics Exam I. Here's the problem (slightly modified to eliminate the AIME contest requirement for the answer to be an integer):

> *What is the area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths* $2\sqrt{3}$, 5, *and* $\sqrt{37}$?

We want to avoid numerical computations. To help us resist that temptation, let's solve this problem for a general right triangle instead of the specific one indicated in the problem. So let a, b, and c be the side lengths of a right triangle with hypotenuse c. By introducing variables, not only will we obtain a more general solution, we will also be able to use the resulting formula as a kind of high-level checksum to give us confidence that we didn't err. More on this later.

To avoid computation, we're going to conceptualize as much as we can. We're asked to find the area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle. For the sake of convenience, let's introduce an xy-coordinate system so that the vertices of the given right triangle are located at (0, 0), (a, 0), and (0, b).

Suppose we have an equilateral triangle with one vertex on each side of the right triangle and that two of its vertices are located at $(m, 0)$ and $(0, n)$. The values m and n are related by the condition that they are vertices of an equilateral triangle with a third vertex on the hypotenuse. What is this relationship?

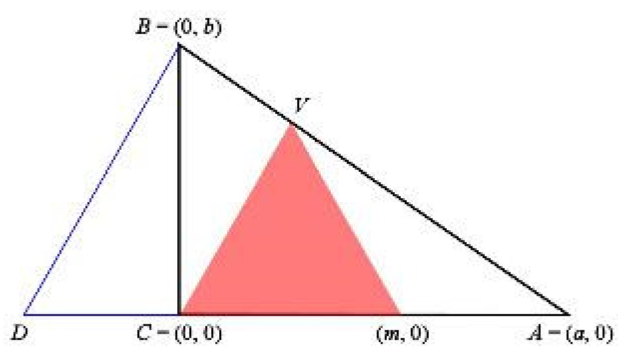
The third vertex of the equilateral triangle can be found by rotating the vector that points from $(m, 0)$ to $(0, n)$ by 60° (clockwise) and adding the rotated vector to $(m, 0)$. At this stage, we could carry out this computation, then require that the third vertex satisfy the equation of the line that defines the hypotenuse in order to obtain the relationship between m and n. But we're trying to avoid computation, so let's think on this.

Rotation is a linear transformation in $(m, 0)$ and $(0, n)$, and so is adding $(m, 0)$. So the coordinates of the third vertex of the equilateral triangle will be linear functions in m and n. Therefore, when these expressions are substituted into the linear equation that defines the hypotenuse, the result will be a linear equation of the form $Am + Bn = 1$. (We can assume a constant term of 1; if the constant term were 0, it would mean that two of the vertices of the equilateral triangle could both be at the origin, which isn't possible. And once you have a nonzero constant term, you can normalize it.) Thus, solutions (m, n) to the line $Ax + By = 1$ correspond to two vertices, $(m, 0)$ and $(0, n)$, of an equilateral triangle with its third vertex on the hypotenuse.

The area of an equilateral triangle is proportional to the square of its side length, which is $m^2 + n^2$ in this case. Since (m, n) are confined to the line $Ax + By = 1$, the minimum value of $m^2 + n^2$ is simply the square of the distance of the origin from the line $Ax + By = 1$. Applying the distance formula of a point from a line, we find that the minimum value of $m^2 + n^2$ is

$$\frac{1}{A^2+B^2}.$$

So we must find A and B. We could do this by performing the computation described a few paragraphs ago, rotating the vector that points from $(m, 0)$ to $(0, n)$ and adding $(m, 0)$ to it, then substituting into the equation of the line that defines the hypotenuse. But we'll find A and B more directly to avoid computation. The coefficients A and B can be obtained by finding the two equilateral triangles that have a vertex on the origin, i.e., when $m = 0$ or $n = 0$.



The figure shows the case where $n = 0$, along with an auxiliary line that passes through $B = (0, b)$ and intercepts the horizontal axis at D where it makes a 60° angle. When $n = 0$, the value of m is the side length of the equilateral triangle, and we can find that side length by noting that triangles AVC and ABD are similar. Thus, $CV : BD = AC : AD$.

Triangle BCD is a 30-60-90 right triangle with long leg of length b, hence $CD = \frac{b}{\sqrt{3}}$ and $BD = \frac{2b}{\sqrt{3}}$. Thus, we have $m : \frac{2b}{\sqrt{3}} = a : (a + \frac{b}{\sqrt{3}})$. Solving for m, we find

$$m = \frac{2ab}{\sqrt{3}a+b}.$$

The coefficient A is the reciprocal of this, so $A = \dfrac{\sqrt{3}a+b}{2ab}$.

Next, we need to find the coefficient B. We could determine B by finding the side length of the appropriate equilateral triangle that shares a side with the vertical side of the right triangle, but we don't have to! By symmetry, all we need to do is take our formula for A and interchange the roles of a and b:

$$B = \frac{a+\sqrt{3}b}{2ab}.$$

The general answer to the problem is therefore

$$\frac{\sqrt{3}}{4}\frac{1}{A^2+B^2} = \frac{\sqrt{3}}{4}\frac{1}{\left(\dfrac{\sqrt{3}a+b}{2ab}\right)^2+\left(\dfrac{a+\sqrt{3}b}{2ab}\right)^2} = \frac{\sqrt{3}(ab)^2}{(\sqrt{3}a+b)^2+(a+\sqrt{3}b)^2} = \frac{\sqrt{3}}{4}\frac{(ab)^2}{a^2+\sqrt{3}ab+b^2}.$$

Only towards the end did we have to do a few numerical computations, but these numerical computations are relatively simple and not particularly error-prone. Perhaps you could do them in your head?

Having this general formula affords us at least two additional benefits. One is that we can use dimensional analysis to see that this formula returns units of length squared, which is a good sign since we are computing an area. The other is that we can see the answer is symmetric in a and b, which jives with the fact that a right triangle with leg lengths a and b is congruent to a right triangle with leg lengths b and a. These are the "high-level" checksums alluded to earlier. If either didn't check out, we'd know that we messed up the derivation.

By contrast, if you solved this problem by using the given values of the side lengths right off the bat, you'd lose track of how the final answer depends on those values. You'd end up with some number and not be sure if your units were correct or, if you switched a and b, that you would get the same answer. And if you made a mistake, it would be much harder to figure out where you went awry because you'd have to check for errors not just in the derivation, but also in the numerical computations.

We derived the general answer without doing any laborious, error-prone numerical computation at all. So now when we actually plug in the given values, we will know that any computational error likely occurred at this particular stage. Taking advantage of the fact that $a^2 + b^2$ is the square of the hypotenuse, we finally compute

$$\frac{\sqrt{3}}{4}\frac{(ab)^2}{a^2+\sqrt{3}ab+b^2} = \frac{\sqrt{3}}{4}\frac{(5\cdot 2\sqrt{3})^2}{37+\sqrt{3}\cdot 5\cdot 2\sqrt{3}} = \frac{\sqrt{3}}{4}\frac{(5\cdot 2\sqrt{3})^2}{37+30} = \frac{75\sqrt{3}}{67}.$$

Notice that we refrained from multiplying out the numerator until we evaluated the denominator. This approach allows you to potentially avoid unnecessary numerical computation, as it did here when we canceled the 2^2 in the numerator with the 4 in the denominator, though unfortunately 67 is prime so won't likely yield to cancelation. Still, it was worth the wait as it spared us from unnecessarily multiplying 5 by 2.

Avoid numerical computational errors by eschewing numerical computation!

Implications

by Ken Fan | edited by Jennifer Silva

The world is not a jumble of isolated facts.

That is quite possibly the most profound realization ever had.

We live in a world where the night sky is randomly sliced by streaks ablaze, where rain spigots shut off for unpredictable lengths of time, where your kitty refuses to eat the same frisky tuna fare he devoured just yesterday. What's wrong, little kitty?

In such a world, it may well seem that nothing is certain, and that we have to learn the quirks of life one by one. It can be astonishing to realize that sometimes, a set of facts will combine to imply, by sheer force of logic, another fact.

Mathematics revels in this realization. Math is a web of implications.

The story of mathematics is the story of humble facts, woven neatly together into ever more astonishing facts, which become the theorems of far-reaching theories. Thus, we can understand a lot by knowing just a little. If one were to present a fact in isolation – a fact that actually belongs at the end of a long chain of happy implications – the effect could be quite startling and often amusing. Part of the joy of mathematics is discovering these startling results, building the series of implications from humble facts to surprising conclusion.

For example, consider the two rather large numbers $11^{9,998} - 5^{9,998}$ and $11^{10,002} - 5^{10,002}$. Please don't bother calculating what each number is as a decimal. If you must know, the first number is a decimal with 10,411 digits and the second one has even more.

Here's my question: What is the largest whole number that divides evenly into both of these massive numbers? That is, what is their greatest common factor?

Without even reaching for scratch paper, I can tell you the following fact: The answer is 96.

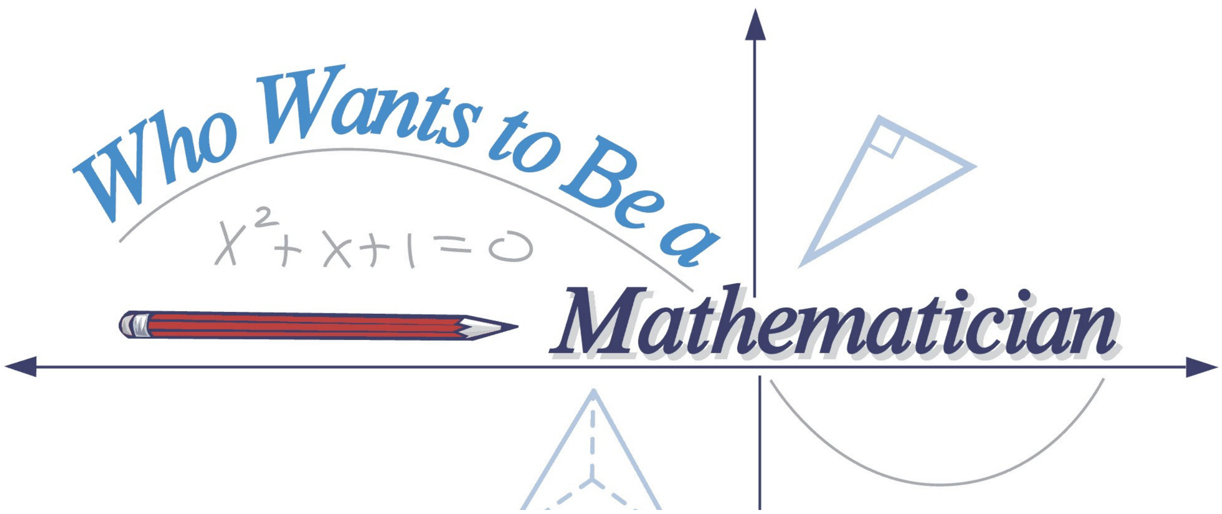
Now be honest: are you a bit startled?

That is the power reaped by mathematics for embracing implications. Let's have a look at the humble facts that begin the journey of implications that lead to 96.

Humble Facts

Consider two numbers. Let's call them X and Y. Suppose that both are divisible by the whole number d. One way to think of this is that if you had X chocolates and needed to package them into boxes that each held d pieces, you could neatly fill up some number of boxes using all the chocolates and have none left over. Algebraically, we can write $X = md$ for some integer m. Similarly, we can write $Y = nd$ for some integer n.

If you had $X + Y$ chocolates, could you pack them away into boxes of d chocolates each without having any stray pieces left over?



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Learn by Doing

Fibonacci Numbers
by Girls' Angle Staff

In this Learn by Doing, we'll explore Fibonacci numbers.

The Fibonacci numbers are a sequence of numbers that begin with two 1's, then each successive term is obtained by adding the previous two. The first few Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

1. Calculate the next few Fibonacci numbers after 34.

Let F_n be the n^{th} Fibonacci number, with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. The relation $F_{n+1} = F_n + F_{n-1}$ for $n > 1$ is known as the **Fibonacci recurrence relation**.

2. Consider the geometric sequence $1, \gamma, \gamma^2, \gamma^3, \gamma^4, \ldots$. Find the two values of γ for which this geometric sequence satisfies the Fibonacci recurrence relation, that is, find γ so that

$$\gamma^{n+1} = \gamma^n + \gamma^{n-1},$$

for all $n > 1$.

The values of γ that you found in Problem 2 are the roots of the quadratic equation $x^2 - x - 1$. Let γ_+ be the larger of the two roots and let γ_- be the smaller. The root γ_+ is known as the **golden mean**.

3. Suppose G_n is a sequence that satisfies the Fibonacci recurrence relation. Show that the sequence whose terms are aG_n for a fixed constant a also satisfies the Fibonacci recurrence relation.

4. Suppose G_n and H_n are sequences that both satisfy the Fibonacci recurrence relation. Show that the sequence $G_n + H_n$ also satisfies the Fibonacci recurrence relation.

Let A and B be constants. From Problems 2-4, it follows that $A\gamma_+^n + B\gamma_-^n$ satisfies the Fibonacci recurrence relation.

5. Find A and B so that $A + B = 0$ and $A\gamma_+ + B\gamma_- = 1$. For this A and B, show that $A\gamma_+^n + B\gamma_-^n$ is the Fibonacci sequence.

6. How many decimal digits does the 100^{th} Fibonacci number have?

It's convenient to extend the Fibonacci sequence to include $F_0 = 0$.

7. Show that $\gamma_+^n = F_n\gamma_+ + F_{n-1}$ for all positive integers n.

8. Express $x/(1 - x - x^2)$ as a power series. How does this relate to the Fibonacci numbers?

Let's prove some nifty identities involving Fibonacci numbers.

9. Show that $1 + F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2}$ for all positive integers n.

10. Show that $(-1)^n = F_{n+1}F_{n-1} - F_n^2$ for all positive integers n.

11. Make a conjecture about the sum of the squares of two consecutive Fibonacci numbers. Prove your conjecture.

12. Make a conjecture about the value of $F_{n+1}^2 - F_{n-1}^2$. Prove your conjecture.

13. Make a conjecture about the sum of the first n odd-indexed Fibonacci numbers. Prove your conjecture.

14. Make a conjecture about the sum of the first n even-indexed Fibonacci numbers. Prove your conjecture.

15. Make a conjecture about the sum of Fibonacci numbers weighted by the binomial coefficients. Specifically, what is ${}_nC_0 \cdot F_0 + {}_nC_1 \cdot F_1 + {}_nC_2 \cdot F_2 + {}_nC_3 \cdot F_3 + \ldots + {}_nC_n \cdot F_n$, where ${}_nC_k$ is the binomial coefficient "n choose k". Prove your conjecture.

16. Prove that $F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_nF_{n+1}$.

17. Prove that $1 \cdot F_1 + 2 \cdot F_2 + 3 \cdot F_3 + \ldots + n \cdot F_n = 2 + F_n + (n-2)F_{n+2}$.

18. Make a conjecture about the numbers $5F_n^3 + 3(-1)^nF_n$. Prove your conjecture.

Now let's examine divisibility properties of the Fibonacci numbers.

19. Make a conjecture about which Fibonacci numbers are even. Prove your conjecture.

20. Make a conjecture about which Fibonacci numbers are multiples of 3. Prove your conjecture.

21. Make a conjecture about which Fibonacci numbers are multiples of 5. Prove your conjecture.

22. Prove that F_k divides evenly into F_{kn} for all positive integers k and n.

23. Make a table whose columns and rows are headed by Fibonacci numbers. In the entry with column headed by F_n and row headed by F_m, place the greatest common factor of F_n and F_m. Look for as many patterns as you can in this table. How many of the patterns can you prove?

24. For instance, can you prove that consecutive Fibonacci numbers are relatively prime?

25. (Spoiler alert!) Can you prove that the greatest common factor of F_m and F_n is F_d, where d is the greatest common factor of m and n? This is known as Lucas' theorem.

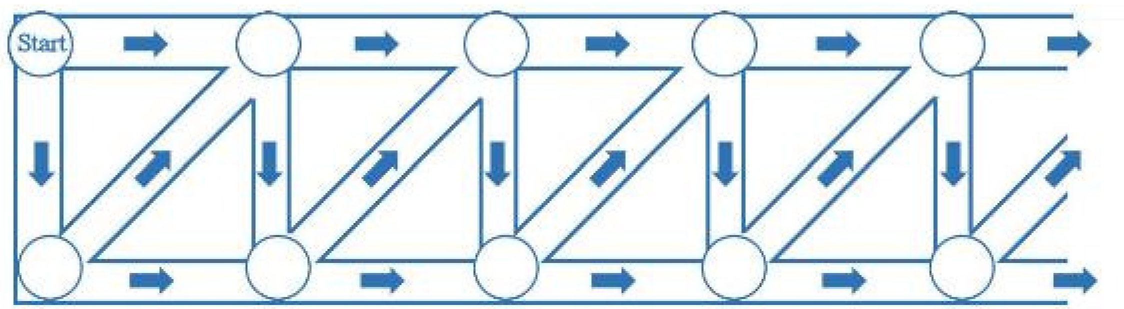
Now let's explore combinatorial properties of the Fibonacci numbers. For Problems 24-28, show that the answer to each is the Fibonacci numbers.

26. How many ways are there to tile a 2 by n rectangle with 2 by 1 dominoes?

27. How many ways are there to tile a 1 by n rectangle with 1 by 1 squares and 1 by 2 dominoes?

28. You have an unlimited supply of 1 by k dominoes for each odd k. Using these tiles, how many ways are there to tile a 1 by n rectangle?

29. In the map of one-way streets below, how many different paths are there from the intersection marked "Start" to each of the other intersections?



30. How many subsets of the set $\{1, 2, 3, \ldots, n\}$ are there which contain no two consecutive integers?

31. (This result is known as Zeckendorf's theorem, though it was noticed before him by Lekkerkerker.) Prove that every positive integer can be written uniquely as a sum of nonconsecutive Fibonacci numbers. For example, $17 = 13 + 3 + 1$.

The picture below is of an echinacea flower and originally appeared in Volume 1, Number 4 of this Bulletin.

Photo by C. Kenneth Fan

Count the number of clockwise spirals and the number of counterclockwise spirals and you'll get two consecutive Fibonacci numbers.

32. Can you find other occurrences of the Fibonacci numbers in nature?

Notation

by Ken Fan

Good notation facilitates communication. To learn notation, use it. Practice makes perfect!

Function Notation

The concept of a function is one of the most important concepts in mathematics. To create a function, you need two sets, which we'll call S and T. One set, S, is called the domain and the other set, T, is called the codomain. A function associates an element of the codomain T to each element in the domain S. For example, S could be the set of items for sale in a store and T could be the set of whole numbers. Associating to each item in S its price in cents is an example of a function with domain S and codomain T.

Functions are typically given names, and in mathematics, any symbol that can be used as a variable can also be used as the name for a function.

If f is a function with domain S and codomain T, we write $f : S \to T$, and say "f is a function from S to T". If s is an element of S, then the element in T that the function f associates with s is denoted $f(s)$. If you didn't realize that f is a function and thought it was a variable, then you would misinterpret "$f(s)$" as "f times s". Be careful of this pitfall!

Here's a typical example from school: Let $\mathbf{R}$ be the set of real numbers and let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 2x + 1$. In other words, f associates a real number to each real number. Specifically, f associates the real number $2x + 1$ to the real number x. Make sure you agree with each of the following statements involving this function f:

$$f(5) = 11.$$
$$f(a + 2) = 2(a + 2) + 1 = 2a + 5.$$
$$f(x^2) = 2x^2 + 1, \text{ but } f(x)^2 = (2x + 1)^2 = 4x^2 + 4x + 1.$$
$$f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3.$$
$$3f(10) = 3(2(10) + 1) = 3(21) = 63.$$

While every element of the domain has some element in the codomain associated with it, it *does not have to be the case* that every element in the codomain be associated by the function to an element in the domain. The subset of elements in the codomain that are associated to an element in the domain by a function f is called the **image** (or **range**) of f.

Also, elements in the codomain *can* be associated to multiple elements in the domain.

A word of caution

In math, when you create a function, you *must* explicitly specify the function's domain and codomain. However, in high school, there is often a standing assumption that all functions have codomain the real numbers and domain some subset of the real numbers. In fact, it's common in high school for a function to be defined by providing a **function rule**, such as $f(x) = 1/x$. One is meant to understand that the domain of this f is the largest subset of the real numbers for which the function rule makes sense, which, in this case, is all nonzero real numbers.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 22 - Meet 1 February 1, 2018	Mentors:	Ivana Alardin, Karia Dibert, Anna Ellison, Danielle Fang, Alexandra Fehnel, Molly Humphreys, Jennifer Matthews, Kate Pearce, Samantha Russman, Josephine Yu, Annie Yun

Some members are exploring modular arithmetic using the Fibonacci sequence as a rich source of material. For problems related to the Fibonacci numbers, see this issue's *Learn by Doing* on page 24.

Session 22 - Meet 2 February 8, 2018	Mentors:	Sarah Coleman, Anna Ellison, Danielle Fang, Alexandra Fehnel, Molly Humphreys, Jennifer Matthews, Kate Pearce, Samantha Russman, Jane Wang, Josephine Yu, Annie Yun

One of our members found the height of a triangle with side lengths 25, 52, and 63, regarding the side of length 63 as the base. In the spirit of this issue's *Errorbusters!* (see page 16), she is now working on finding a formula for the height in terms of generic side lengths a, b, and c.

Some of our members constructed beautiful histograms showing the number of ways to roll different numbers using 2 and 3 dice. For n dice, where will the peak be? What is the general shape of the associated histogram?

Session 22 - Meet 3 February 15, 2018	Mentors:	Ivana Alardin, Rachel Burns, Anna Ellison, Danielle Fang, Alexandra Fehnel, Jacqueline Garrahan, Molly Humphreys, Jennifer Matthews, Elise McCormack, Suzanne O'Meara, Kate Pearce, Samantha Russman, Christine Soh, Sarah Tammen, Jane Wang, Annie Yun

One of our members enjoys cryptarithmetic problems, so we made up a couple. In each equation below, replace the X's with the digits 1 through 9 using each digit exactly once to construct a valid equation.

$$\frac{\mathrm{XX}}{\mathrm{XX}} + \frac{\mathrm{XX}}{\mathrm{XX}} = \mathrm{X}.$$

$$\frac{\mathrm{X}}{\mathrm{X}} + \frac{\mathrm{X}}{\mathrm{X}} + \frac{\mathrm{X}}{\mathrm{X}} + \frac{\mathrm{X}}{\mathrm{X}} = \mathrm{X}.$$

Calendar

Session 21: (all dates in 2017)

September	7	Start of the twenty-first session!
	14	
	21	No meet
	28	
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Session 22: (all dates in 2018)

February	1	Start of the twenty-second session!
	8	
	14	
	22	No meet
March	1	Melody Chan, Brown University
	8	
	15	
	22	
	29	No meet
April	5	
	12	
	19	No meet
	26	
May	3	
	10	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle holds its meets within 5 minutes of the Kendall Square T stop in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________