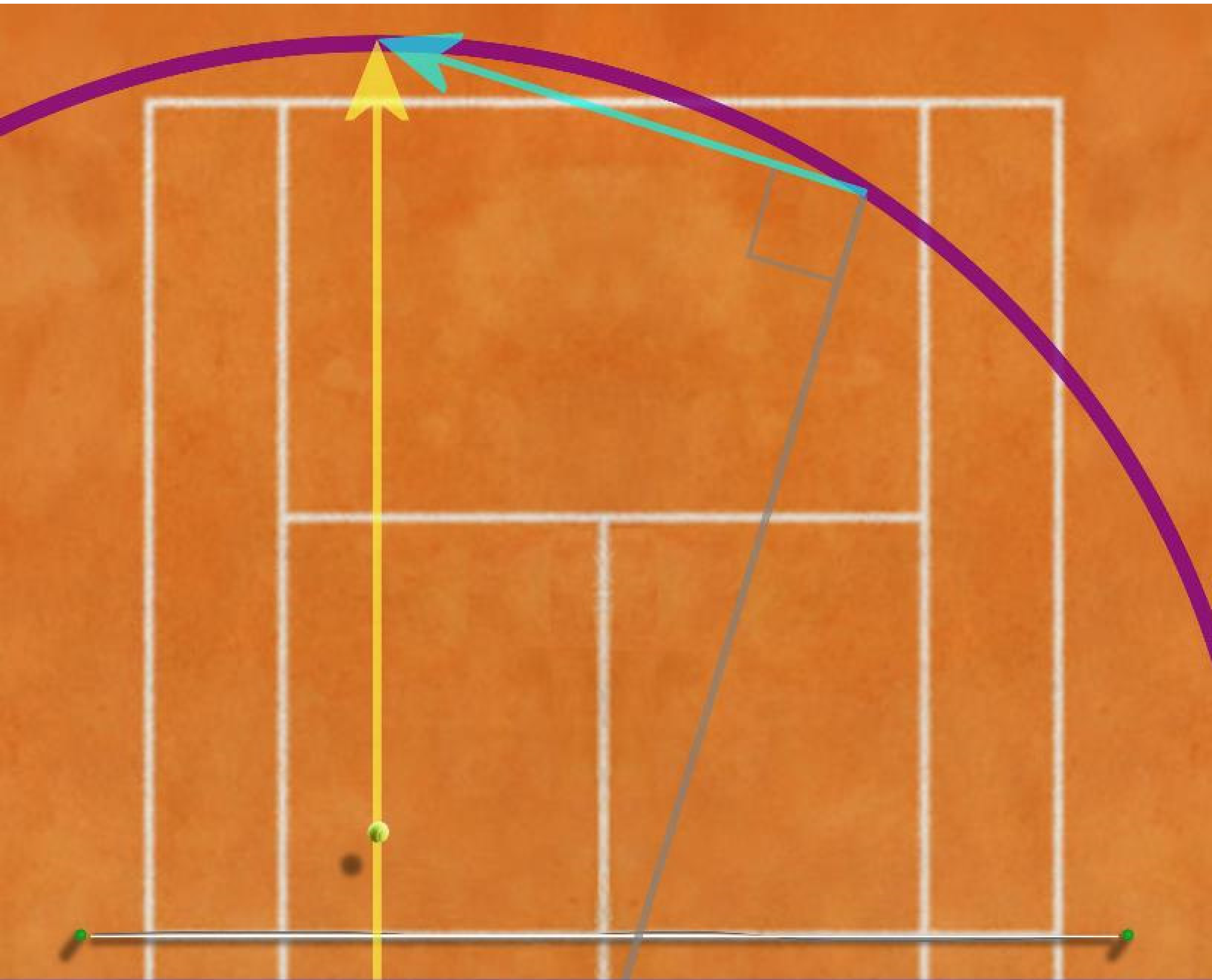
Girls' Angle Bulletin

April/May 2018 • Volume 11 • Number 4

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

This Bulletin also serves as a venue for students to show math work they've done beyond school expectations, like **Allie**'s proof in Member's Thoughts (p. 22). We need more ways for students to show mathematical achievement, and this magazine is one. - Ken Fan, President and Founder

Girls' Angle Donors

Girls' Angle thanks the following for their generous contribution:

Individuals

Uma Achutha
Dana Albert
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Bill Bogstad
Ravi Boppana
Lauren Cipicchio
Merit Cudkowicz
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Kim Deltano
Eleanor Duckworth
Concetta Duval
Glenn and Sara Ellison
John Engstrom
Lena Gan
Courtney Gibbons
Vanessa Gould
Rishi Gupta
Andrea Hawksley
Delia Cheung Hom and Eugene Shih
Mark and Lisel Macenka
Brian and Darlene Matthews
Toshia McCabe
Mary O'Keefe
Stephen Knight and Elizabeth Quattrocki Knight
Junyi Li
Alison and Catherine Miller
Beth O'Sullivan
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Christian Rudder
Craig and Sally Savelle
Eugene Shih
Eugene Sorets
Sasha Targ
Diana Taylor
Waldman and Romanelli Family
Marion Walter
Andrew Watson and Ritu Thamman
Brandy Wiegers
Brian Wilson and Annette Sassi
Lissa Winstanley
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathematical Sciences Research Institute
The Mathenaeum Foundation
Orlando Math Circle

Corporate Donors

Adobe
Akamai Technologies
Big George Ventures
John Hancock
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
Mathenaeum
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle

For Bulletin Sponsors, please visit girlsangle.org.

Girls' Angle Bulletin

The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *No Time To Lose!* by C. Kenneth Fan. See *Math In Your World* on page 18.

An Interview with Betsy Stovall

Betsy Stovall is an Assistant Professor of Mathematics at the University of Wisconsin-Madison. She earned her doctoral degree in mathematics in 2009 from the University of California, Berkeley under the supervision of Michael Christ.

Ken: In a brief interview with the University of Wisconsin-Madison, you explained that you originally wanted to major in political science, but found your math homework too compelling to put aside, so ended up majoring in math. What is it about math that pulls you so?

Betsy: I love and I hate that period when I know enough about a problem to carry it around in my head, but not enough to solve it, when it's almost physically painful to make progress. I like the feeling of chipping away at some problem and seeing the way that the different pieces fit together (or fall apart, depending on your viewpoint). I find that I like math more when I work harder at it, though I have to remind myself of this on the days, weeks, months when it's really hard. And I love writing mathematics.

Ken: How aware were you of math and mathematics as a profession prior to college? When did you start to like math?

Betsy: To be honest, the process by which I ended up where I am is still a bit mysterious to me. It never occurred to me until well into college that there might be people still developing new mathematics, much less that this is what ordinary college math professors spend much of their time doing. Not in my wildest dreams would I have a career that involved proving mathematical theorems.

When I was a kid, I always liked my math classes and enjoyed logic puzzles, but my passion was for reading, pretty much anything I could get my hands on. When it came time to choose a college, I thought I wasn't good enough at math to pass calculus at the engineering school in the state, so I opted to attend a liberal arts university and focus on political science or possibly philosophy. It's strange to say, but a defining moment for me was getting a C on a calculus test my first semester in college. I realized that I hadn't been studying hard enough, so I spent hours and hours in the remaining weeks of the semester solving problems, and I found that I loved it. The more basic calculus problems I did, the more I wanted to do, to the exclusion of everything else. After that, I was never ready to stop doing math, and I somehow now find myself as a math professor.

Ken: That's so neat! Could you please describe in detail some specific math problem from your student years which you remember finding particularly compelling?

Betsy: When I took my undergraduate "How to Write Proofs" class, we were asked to determine for which positive integers n the relationship $(n + 1)^n \leq n^{n+1}$ holds, using only induction and elementary facts about multiplication. I was very fortunate to have a math professor whose philosophy was to assign difficult problems and then make us resubmit and resubmit and resubmit our "solutions" until we finally had a correct, readable proof. I remember struggling with this particular problem for weeks and submitting a number of incorrect solutions, but once I really understood mathematical induction things just sort of clicked. Now I always assign this problem to my honors analysis students.

Ken: What questions in mathematics intrigue you today?

Betsy: I work in a field called harmonic analysis, whose basic philosophy is to study mathematical objects and operations by breaking them up into small pieces and

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

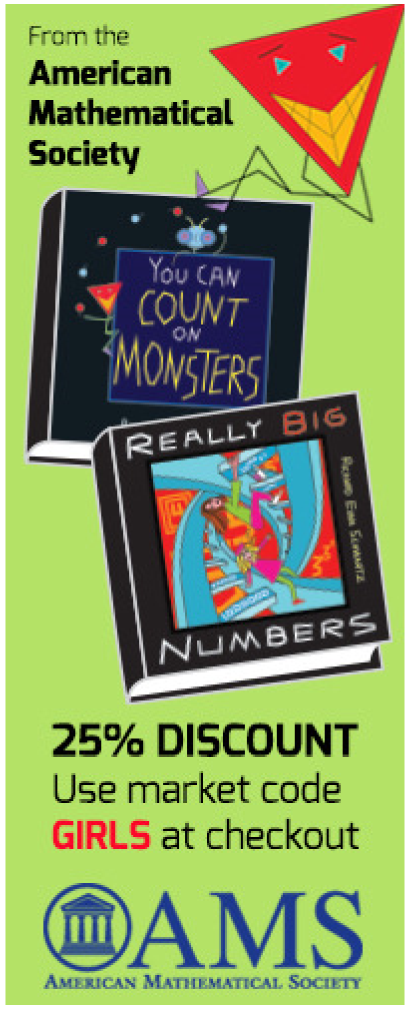
For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Betsy Stovall and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

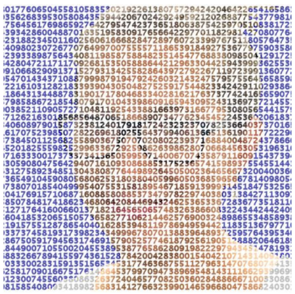
So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.



The Needell in the Haystack

How to Fill in the Blanks[1]

by Deanna Needell | edited by Jennifer Silva

In the previous issue, I wrote an article about making inferences on very large-scale data, such as inferring class labels on unlabeled data points. In this article, we discuss a related topic whose goal is to infer missing entries in a large data matrix. Although it can also be viewed as an inference problem, this setting is quite different from other inference problems like classification, where the data itself is known but only one attribute of the data needs to be inferred.

Although most data is now extremely large-scale, it is also highly *incomplete*. We view the underlying data as a matrix; we refer to the full matrix as the **data matrix**, whose rows and columns will correspond to particular variable types, depending on the application. The examples of incomplete data are abundant. In survey data, where the rows of the data matrix correspond to users taking the survey and the columns correspond to the survey questions, participants do not or cannot answer every question in the survey; this leads to missing entries in the data matrix. In imaging applications where the data matrix may be the image itself (so each entry corresponds to a *pixel* in the image), artifacts or obstructions may appear that result in an incomplete matrix. In geophysical, environmental, and medical sensor data, missing data results from sensor malfunction, user removal, or power/memory restrictions. The goal in all of these settings will be to accurately fill in these missing entries, thereby *completing* the matrix.

Such so-called **matrix completion** is clearly impossible without further assumptions; given only a partially-observed matrix without any assumptions, there are infinitely many ways to "fill in the blanks." However, there are examples that demonstrate situations where a unique solution does exist. For example, the popular game of *Sudoku* requires the player to complete a 9-by-9 matrix of integers from 1 to 9, inclusive, from only a fraction of filled-in entries. The game is set up so that every row, column, and 3-by-3 block of the matrix should contain every digit 1 through 9. With this additional information, the game has a unique and discoverable solution, as long as enough of the entries are revealed – and are in an appropriate pattern. These latter conditions beg the following questions: How many entries does the player need to see before she can determine a unique solution? And in what pattern must those revealed entries be? It turns out these questions will also be relevant to our problem of mathematical matrix completion.

To better motivate the setting we will discuss further, consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & ? & ? & ? & ? \\ ? & 4.8 & ? & ? & ? \\ ? & ? & ? & ? & -1 \\ ? & ? & ? & \pi & ? \end{bmatrix},$$

[1] This content supported in part by a grant from MathWorks.

supposing you also know that this matrix is rank 1; thus, all rows are linearly dependent, meaning that each row is a multiple of one of the others. Can the missing entries (marked by "?") be uniquely determined? In this case, we can easily see that the answer is yes (I've left this as an exercise for the reader). Although this is a highly simplified example since an entire row is visible, it nonetheless motivates the more realistic problems discussed here.

State-of-the-art methods in matrix completion allow one to accurately complete a data matrix by observing only a few of its entries, *under the assumption that the data is intrinsically low-rank*. Low-rankness is a reasonable assumption in most applications. For example, in survey data, the matrix is (exactly or approximately) low-rank when a small number of underlying variables or trends determine how users answer the questions. In the now famous Netflix problem, where mathematicians aim to predict Netflix user ratings from the matrix whose rows correspond to users and columns correspond to movies (and entries to the corresponding ratings), the low-rankness stems from the fact that only a few parameters are needed to describe the tastes of most users (e.g., whether someone likes romance movies or sci-fi, etc.). In imaging, the images tend to be approximately low-rank through appropriate representation transforms, and sensor applications lead to low-rank data due to correlations across space or time.

Mathematically, then, the problem is formulated as recovering a matrix M from linear measurements of the form $y_i = \langle A_i, M \rangle \equiv \text{trace}(A_i^* M)$ (where A^* denotes the adjoint or conjugate transpose of a matrix A) for $i = 1 \ldots m$ and where A_i are matrices of the same dimension as M. For example, when A_i are matrices of all zeros and a single 1, this corresponds to the classical matrix completion problem when only a subset of entries are observed. We then seek a low-rank matrix M consistent with these measurements. Formally, we wish to solve for the minimizer of

(1) $$\hat{M} \equiv \underset{B}{\text{argmin}}\, \text{rank}(B) \quad \text{such that } y_i = \langle A_i, B \rangle \text{ for all } i.$$

Let's analyze this program more carefully. Recall that a singular value of a matrix A is the square root of an eigenvalue of the matrix A^*A, which is called the **gram** matrix. Singular values are used in so-called **singular value decompositions**, and they provide a way of understanding how much a matrix can shrink or stretch a vector. For example, if $\sigma_{\max}(A)$ denotes the largest singular value of the matrix A, then it is always true that $\|Ax\|_2 \le \sigma_{\max}(A)\|x\|_2$; that is, the largest singular value quantifies how much the matrix can stretch a vector. Similarly, if $\sigma_{\min}(A)$ denotes the smallest singular value of a full-rank square matrix A, then $\|Ax\|_2 \ge \sigma_{\min}(A)\|x\|_2$; so the smallest singular value quantifies how much a vector can be shrunk by applying A.

Now consider a rank-r matrix and note that any such matrix has only r non-zero singular values. In fact, the rank of a matrix can be defined as the number of non-zero singular values. One may write this using ℓ_0-notation as

$$\text{rank}(A) = \|\sigma(A)\|_0 \equiv \left|\{j : \sigma_j(A) \ne 0\}\right|,$$

where $\sigma_j(A)$ denotes the jth singular value of A, and $\|\cdot\|_0$ counts the number of non-zeros in its argument. Note that although we use norm notation, this function is not actually a norm (a fun exercise to check). Unfortunately, this also makes solving (1) computationally challenging. For that reason, we consider a *relaxation* of the program. Instead of counting the number of non-zero singular values to compute the rank of a matrix, let's instead consider *summing* them (in absolute value), as a proxy to the rank. This leads to what we call the *nuclear norm* of a matrix:

$$\|A\|_* \equiv \operatorname{trace}(\sqrt{A^* A}) = \sum_j |\sigma_j(A)|.$$

Thus, the nuclear norm of a matrix is the relaxation of the rank – where the rank counts the number of non-zero singular values and the nuclear norm sums them. Since rank-minimization is not computationally tractable, we may now instead solve its semi-definite relaxation:

(2) $$\hat{M} \equiv \underset{B}{\operatorname{argmin}} \|B\|_* \quad \text{such that } y_i = \langle A_i, B \rangle \text{ for all } i.$$

Fortunately, this relaxation yields a *convex program*, which can be solved computationally efficiently. Due to recent work in matrix completion and nuclear norm minimization (see, e.g., [3], [2]), it is now well known that when m is on the order of nr, nuclear norm minimization accurately recovers any rank-r $n \times n$ matrix from m linear measurements under mild assumptions on the type of measurements. For example, the measurements may be single-entry observations of the matrix, selected uniformly at random. This means that if the matrix is very low-rank, i.e., $r \ll n$, only a small fraction of the observations need to be observed for one to be able to correctly fill in the blanks! We now turn to a real-world example, considering practical aspects of the typical assumptions.

Lyme Data

Lyme disease is the most common vector-borne disease in the United States. The Centers for Disease Control (CDC) estimates that there are 300,000 people in the U.S. diagnosed with Lyme disease each year. A significant proportion of patients with Lyme disease develop chronic, debilitating symptoms that often mimic other illnesses such as multiple sclerosis and amyotrophic lateral sclerosis (also known as Lou Gehrig's disease). Founded over 25 years ago, LymeDisease.org (LDo) is a national non-profit dedicated to advocacy, research, and education. In November 2015, LDo announced the launch of MyLymeData, a patient-powered research project. MyLymeData has over 5,000 patients enrolled, includes several phases of initial and follow-up survey responses, and asks patients questions about diagnosis, treatment, symptoms, and quality of life. Like many large-scale surveys, this data is noisy, incomplete, and has a tree-like structure that makes it challenging to mathematically analyze it.

Lyme data, like most real-world data, has missing entries that are not random, much less uniformly selected at random. For example, in the Lyme data, entries may be missing because the user chose not to answer, the survey structure deemed the question irrelevant, or there were corruptions/errors in acquisition. An example of a portion of the data is shown in Figure 1, which displays the sampling pattern, the original data, and the completed data.

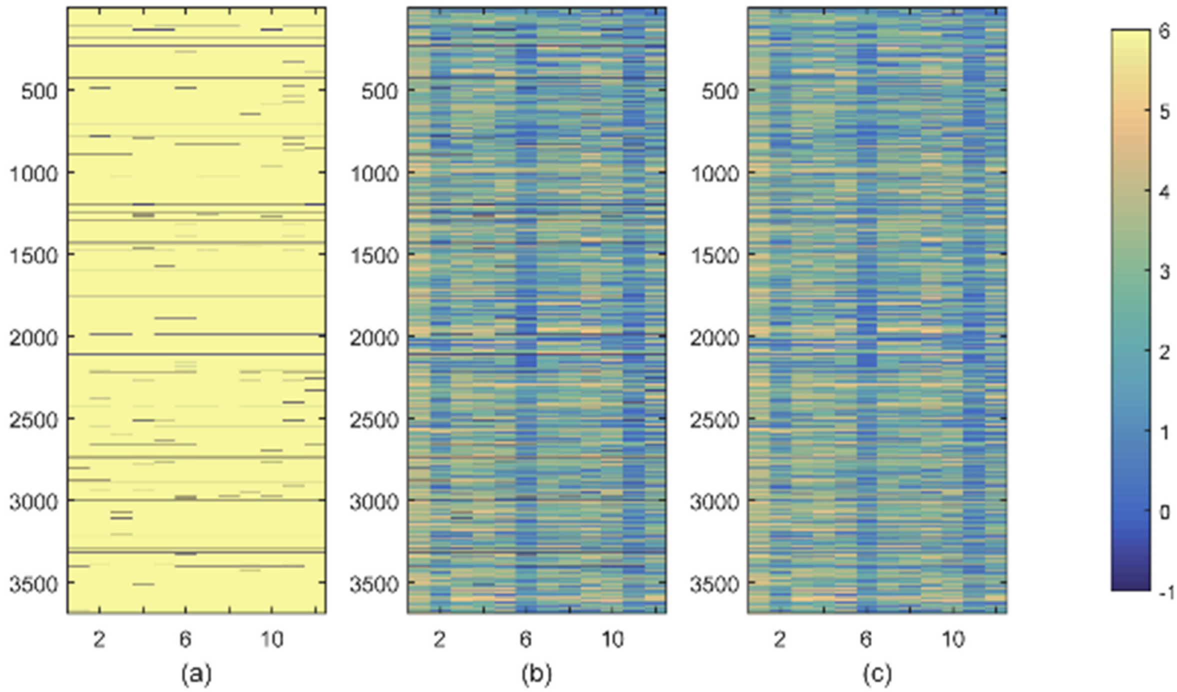


Figure 1. Matrix completion on the full (incomplete) Lyme symptom data. The actual sampling pattern from the survey for these questions is shown in (a); yellow denotes an observed entry and blue denotes an unobserved one. The full matrix is shown in (b) and the reconstruction is shown in (c).

The astute observer will quickly notice that the sampling pattern displayed in Figure 1(a) looks far from randomly uniform. Indeed, in the majority of real applications, the patterns arise from some sort of data *structure*, rather than randomness. In fact, not only are these sampling patterns non-random, but the pattern itself yields a lot of untapped information. In both of these examples, most of the unobserved entries are likely due to irrelevance or participant disinterest. This information is useful! Motivated by this setting, we consider a nuclear-norm matrix completion program with an added regularizer that promotes small values for unobserved entries.

Let $M \in \mathbb{R}^{n_1 \times n_2}$ be the unknown matrix we would like to recover and Ω be the set of indices of the observed entries. Let $\wp_\Omega : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$, where

$$[\wp_\Omega]_{ij} = \begin{cases} M_{ij} & (i,j) \in \Omega \\ 0 & (i,j) \notin \Omega \end{cases}.$$

Recall the nuclear-norm minimization,

(3) $$\hat{M} \equiv \operatorname*{argmin}_A \|A\|_* \quad \text{such that } \wp_\Omega(A) = \wp_\Omega(M).$$

Motivated by applications in which the unobserved entries tend to have small values, we instead solve

(4) $$\tilde{M} \equiv \operatorname*{argmin}_A \|A\|_* + \alpha \| \wp_{\Omega^c}(A) \|_1 \quad \text{such that } \wp_\Omega(A) = \wp_\Omega(M),$$

where $\alpha > 0$ and the entrywise L_1 norm $\|M\|_1$ is defined as $\sum_{ij} |M_{ij}|$. This L_1 term, denoted a **regularizer**, aims to penalize matrices that have many large entries in the unobserved locations. The minimizer, therefore, will tend to have small entries in those locations.

We again consider the Lyme data discussed above. We complete this subsampled matrix with both (3) and (4) using L_1 regularization (for a reasonably selected choice of α) and report $\|\tilde{M} - M\|_F / \|\hat{M} - M\|_F$, averaged over ten trials in Figure 2. As expected, when most of the unobserved entries are small and the bulk of the observed entries are large, we see the most improvement in using the regularizer. Preliminary theoretical results can be found in [4], which are motivated by work in robust principal component analysis [1], but future work is needed to clearly quantify the theoretical gains as a function of the sampling rates.

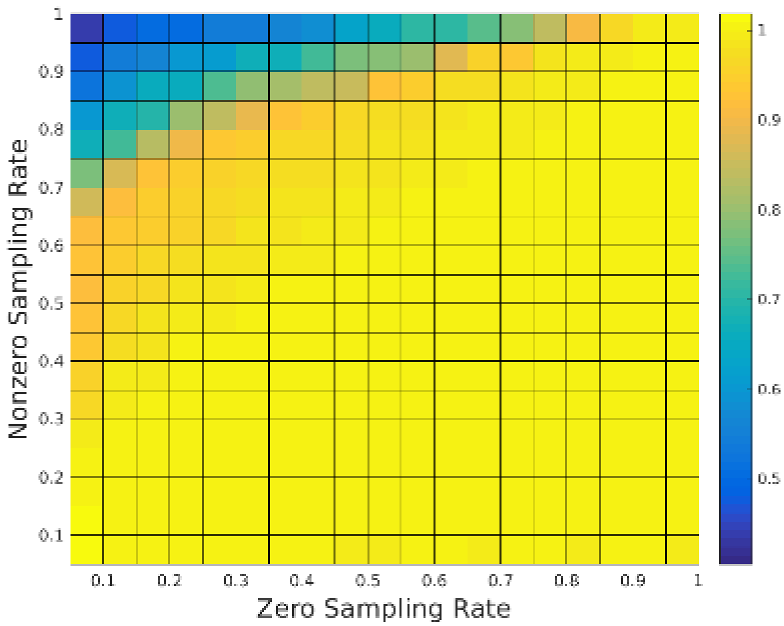


Figure 2. For $\tilde{M}$ and $\hat{M}$ given by (4) and (3), respectively, with L_1 regularization on the recovered values for the unobserved entries, we plot $\|\tilde{M} - M\|_F / \|\hat{M} - M\|_F$. We consider 50 patient surveys with 65 responses each, chosen randomly from 2126 patient surveys.

References

[1] E. J. Candès, X. Li, Y. Ma, and J. Wright, Robust principal component analysis? *Journal of the ACM*, Vol. 58, Issue 3 (May 2011), 1-37.

[2] E. J. Candès and Y. Plan, Matrix completion with noise. *Proceedings of the IEEE*, Vol. 98, Issue 6 (2010), 925-936.

[3] E. J Candès and B. Recht, Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, Vol. 9, Issue 6 (2009), 717-772.

[4] D. Molitor and D. Needell, Matrix completion for structured observations. *Proc. Information Theory and Approximation* (to appear 2018).

Matrices

by Addie Summer | edited by Jennifer Silva

In order to fully understand Prof. Needell's installments of "The Needell in the Haystack," you need to have a familiarity with matrices. However, even if you aren't familiar with matrices, her articles are worth reading because one can still learn a lot without knowing the complete details.

The reality is that we lack complete details in almost everything we do. For example, when you first learned the formula for the area of a circle, namely πr^2, where r is the radius, you probably didn't know how the formula is actually derived. You may not even have had an idea about how the numerical value of π is computed. Yet, despite that lack of information, you could still use the formula and make interesting inferences about circles and related shapes.

In this article, I'd like to say a few words about matrices for those of you who haven't worked with them before. My hope is that you'll then be more comfortable reading Prof. Needell's articles.

To start off, what is a matrix? A matrix is just a rectangular array of numbers.

Rectangular arrays of numbers have many uses. For example, the record of race completion times of a group of athletes in multiple races, the prices of various items when sold in different stores, and the distances between cities all furnish examples of information that can be nicely organized as a matrix. If you are reading this on a computer monitor, then you are looking at a visual representation of a matrix. In fact, you're seeing three matrices, which describe the intensity of red, green, and blue in each pixel on your monitor.

Matrix Notation

Here's a matrix with 3 rows and 5 columns:

$$\begin{bmatrix} 15 & 4 & 7 & 2 & 99 \\ -10 & -9 & 8 & 0 & 6 \\ 11 & 15 & 13 & 14 & 15 \end{bmatrix}.$$

People often place the entire array inside square brackets as above, but some prefer to use large parentheses.

It's convenient to use variables to name matrices. For instance, we might call the above matrix M. We can then refer to M without having to write down all the numbers inside of it each time. It's also useful to be able to refer to specific numbers in the array. For example, we might want to refer to the number in row 2 and column 4 of M, which happens to be 0. Mathematicians have agreed that we can simply write $M_{2,4}$ to refer to the number in row 2 and column 4 of M. By convention, the first number in the subscript tells us which row the number is in and the second number tells us which column, with the top row being row 1 and the leftmost column being column 1. It's also a convention to omit the comma in the subscript when there's no ambiguity about what the two numbers are.

We can describe a matrix with r rows and c columns as an r by c (or an $r \times c$) matrix. Again, the first number specifies the number of rows, and the second number tells the number of columns. This decision to make the first number correspond to the row is arbitrary and has to be memorized. It's simply the price we pay for compact notation.

Matrix Types and Operations

Through the years, matrices have been applied to hundreds, perhaps even thousands, of different applications. With all of these different uses, vocabulary has been created to refer to specific kinds of matrices.

For example, a **column matrix** is a matrix with a single column. A **square matrix** is a matrix that has the same number of rows as it has columns. A **diagonal matrix** is a square matrix A for which $A_{ij} = 0$ if $i \neq j$. A **zero-one matrix** is a matrix whose entries can be 0 or 1 only. A **permutation matrix** is a zero-one square matrix where each row and each column contain precisely one 1. There are many other types of matrices; if you're curious to see dozens more, check out the list of matrices on Wikipedia.

Just as there are many types of matrices, there are also many operations that can be performed on matrices. The various operations arose as matrices were used for different purposes. For example, suppose you oversee a number of convenience stores. You have a matrix X that represents the net profit made on various items sold at the convenience stores under your charge in the year 2016. The rows represent different items sold and the columns represent the store locations. Suppose you have another matrix Y representing the same information for the year 2017. You might be interested to know the net profit on the various items at the different store locations over the two-year span of 2016 and 2017. That would correspond to a third matrix whose entries are the sum of the corresponding entries in X and Y. This operation is used so often that it has a name, **matrix addition**, which is denoted using the plus sign. Thus, $X + Y$ is the matrix whose i, j entry is equal to $X_{ij} + Y_{ij}$.

An important application of matrices that gave birth to many types of matrices and operations on matrices is solving systems of linear equations. Here's a system of 3 linear equations in 3 unknowns:

$$\begin{gathered} 5x + 4y + z = 10 \\ 2x - 3y + 2z = 8 \\ x + y + z = 3 \end{gathered}$$

It has proven extremely useful to organize the coefficients in such equations as a matrix:

$$\begin{bmatrix} 5 & 4 & 1 \\ 2 & -3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

If you play around with linear equations and use matrices to represent coefficients in this manner, you will end up recreating the subject of **linear algebra and matrices**. Many of the matrix operations that Prof. Needell uses have their origins in this application.

If you are new to matrices and you come upon a matrix-dense section of one of Prof. Needell's articles, don't let that deter you. It's perfectly fine if you don't understand all the details on a first reading; you can worry about those details later. Just aim to get an overall sense of what she is doing. Or, you can look up the various terms as you encounter them to gain basic knowledge of the matrix types or operations she is using. The significance may not be immediately clear, but you can always read more about the operations or play with the concepts to develop a working understanding of the material. And if you're a Girls' Angle member, you can email us or ask a mentor about matrices at the club!

Stacked Circles, Part 1

by Ken Fan | edited by Jennifer Silva

Jasmine: How's your graphic design class going?

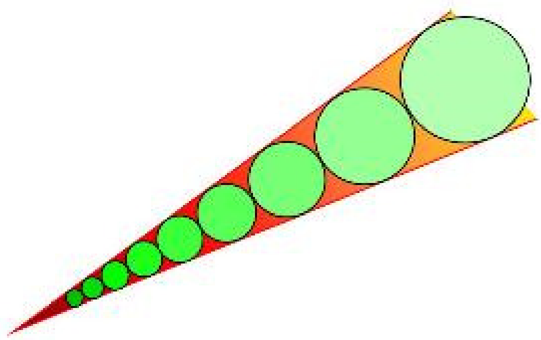
Emily: It's fun, but I'm not too happy with how my last assignment came out.

Jasmine: What was it?

Emily: We had to create a design based on circles. I tried to make a radiating pattern of circles.

Jasmine: Circles and radial lines – intriguing!

Emily: I picked a point and drew radial lines all emanating from it to make a bunch of angles. Then I drew a series of circles within each angle, so that each circle was tangent to its circular neighbors as well as to the sides of the angle. Here's an example of one circle-filled angle.



Emily sketches the drawing shown at left.

Jasmine: Like peas in an angular pod. That's neat!

Emily: Thanks, but the result wasn't as exciting as I'd hoped. Actually, I found the mathematics I had to figure out to create the design more interesting than the design itself. For example, the radii of the circles form a geometric sequence.

Jasmine: Really?

Emily: Yeah. Initially, I found that out using a lot of algebra and trigonometry. But I kept wondering about it. In fact, I was going to ask you, but then I thought of a nifty way to see it.

Jasmine: Please tell!

Emily: Imagine gradually scaling the design by stretching everything outward equally in all directions from the apex of the angle. The angle maps onto itself because the radial lines map onto themselves.

Jasmine: I think I see where you're going. As you scale, the circles will grow as they slide outward, farther and farther away from the apex of the angle. In fact, the radius of the circle determines how far away from the apex it must be in order to be tangent to both sides of the angle.

Emily: Right, so when the figure has been dilated so that the smallest circle becomes as large as the second smallest circle, the dilation must map the smallest circle right smack on top of the second smallest circle. The second smallest circle, by the *same dilation*, must map onto the third smallest circle. Put another way, consecutive circles must map to consecutive circles.

Jasmine: Since all of this happens with the same dilation factor, it means that each successive radius will be the previous radius times this one universal dilation factor. That's precisely what it means to be a geometric sequence!

Emily: Exactly! A geometric sequence is a sequence of numbers where each successive term is obtained from the previous term by multiplying by the same factor, called the **common ratio**.

Jasmine: Did you figure out how the common ratio depends on the measure of the angle?

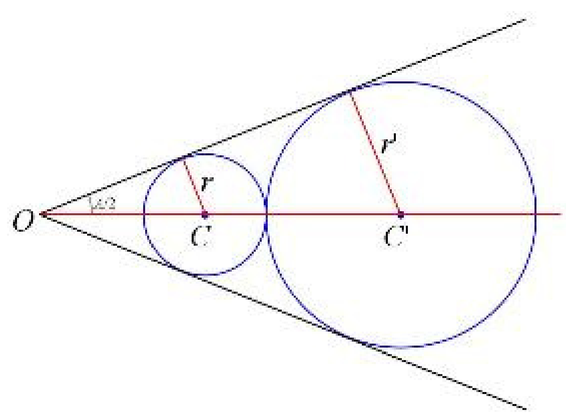
Emily: Yes. I had to compute that common ratio in order to make the design. If *A* is the measure of the angle, then the common ratio turns out to be

$$\frac{1+\sin(A/2)}{1-\sin(A/2)}.$$

Jasmine's eyes dart back and forth.

Emily: You want to derive that formula for yourself, don't you?

Jasmine: You know me too well! Let's see. It suffices to look at two touching circles and figure out the ratio of their radii. Since the circles are tangent to the sides of the angle, the radii that meet the points of tangency meet the sides at right angles. Therefore,



$$r/OC = r'/OC' = \sin(A/2).$$

Emily: Right.

Jasmine: Those equations only use the fact that the circles are tangent to the sides of the triangle. I should also need to use the fact that the two circles are tangent to each other.

Emily: Correct.

Jasmine: The fact that the two circles are tangent to each other means that the distance between their centers is equal to the sum of their radii: $CC' = r + r'$. Combining these facts, we have

$$r + r' = CC' = OC' - OC = r'/\sin(A/2) - r/\sin(A/2).$$

Multiplying throughout by $\sin(A/2)$, we have $\sin(A/2)(r + r') = r' - r$. Rearranging so that r' is isolated, we get

$$r' = \frac{1+\sin(A/2)}{1-\sin(A/2)}r,$$

just as you said!

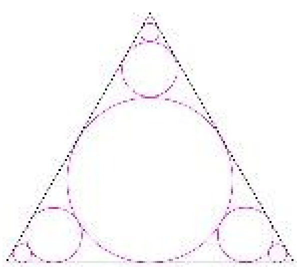
Emily: That's it! Isn't it neat that as A tends to 0, the common ratio tends to 1, and as A tends to 180°, the common ratio tends to infinity? That means that *every* geometric sequence can be illustrated by the radii of a sequence of such circles.

Jasmine: I'm curious, did you invert this formula to find out how the angle measure depends on the common ratio?

Emily: I did. If you solve for A in terms of the common ratio r'/r, which I called s, you get

$$A = 2\ \arcsin\left(\frac{s-1}{s+1}\right).$$

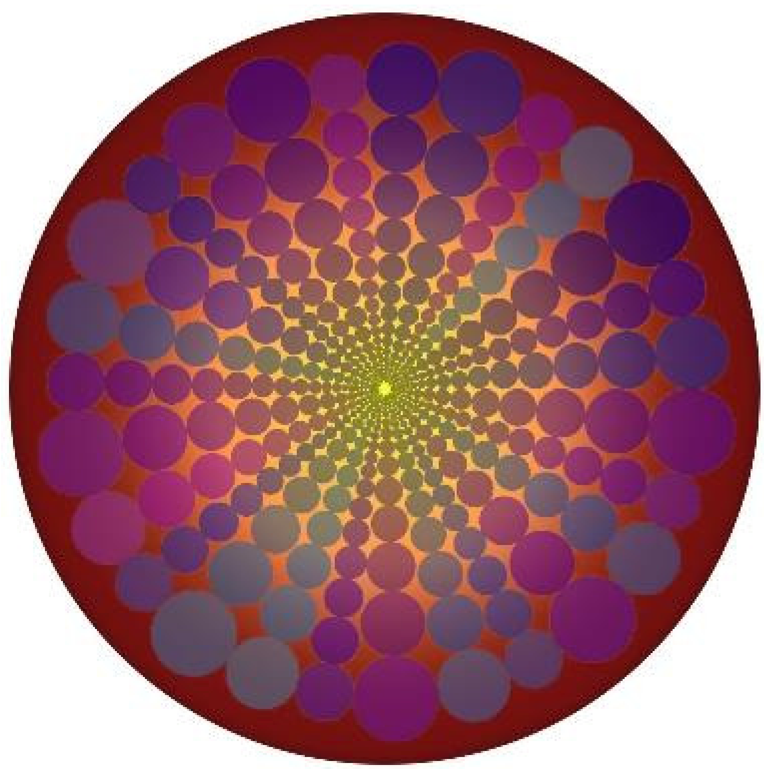
Jasmine: Beautiful! The expression inside the arcsine must map the interval [1, ∞] onto the interval [0, 1], since that corresponds to being able to geometrically realize every common ratio.



Emily: And it does. By the way, the common ratio happens to be precisely 3 when the angle is 60°. This fact made me play around with circles inside an equilateral triangle, but I abandoned that and returned to the radial design concept.

Jasmine: May I see your design?

Emily: Sure, here it is. I don't really like it.



Jasmine: Wow, *I* do!

Emily: The circles next to each other in the outer ring are also tangent to each other. In theory, the circles could be continued inward toward the center indefinitely, but I stopped drawing them when they started to get so small they began to look like dots.

Jasmine: I can't stop staring at it.

Emily: Thanks. Say, Jasmine? A question just occurred to me. Using this stacked circle concept, is there an elegant way to illustrate arithmetic sequences instead of geometric sequences?

Jasmine: Hmm, I wonder. Let's figure that out!

To be continued...

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Tackling a problem where rows of Pascal's triangle, modulo 3, are read as ternary numbers.

Last time, I boiled down the problem to determining when $2m$ choose m is equal to 1, modulo 3. I also saw that $2m$ choose m is divisible by 3 if and only if the base 3 expansion of m has a 2 digit. So I'll assume that the only digits in the base 3 expansion of m are 0 and 1.

When is $\binom{2m}{m} \equiv 1 \pmod 3$?

The base 3 expansion of m must consist of only the digits 0 and 1.

In this case, I don't have to concern myself with factors of 3. So let's define a modified factorial as $n!$ divided by all factors of 3.

Define $n!_3 = \dfrac{n!}{3^{\nu_3(n!)}}$.

I'll compute the first few values of this modified factorial to get a feel for how it goes.

$$1!_3 = 1 \equiv 1 \pmod 3$$
$$2!_3 = 2\cdot 1 \equiv 2 \pmod 3$$
$$3!_3 = 2\cdot 1 \equiv 2 \pmod 3$$
$$4!_3 = 4\cdot 2\cdot 1 = 8 \equiv 2 \pmod 3$$
$$5!_3 = 5\cdot 4\cdot 2\cdot 1 = 40 \equiv 1 \pmod 3$$
$$6!_3 = 2\cdot 5\cdot 4\cdot 2\cdot 1 = 80 \equiv 2 \pmod 3$$
$$7!_3 = 7\cdot 6!_3 \equiv 2 \pmod 3$$

I only care what the result is modulo 3. So really, I'm multiplying a string of ones and twos, and the ones and twos comes in a regular pattern.

$$n!_3 \equiv 1\cdot 2\cdot {}^{1}\cdot 1\cdot 2\cdot {}^{2}\cdot 1\cdot 2\cdot {}^{1}\cdot 1\cdot 2\cdot {}^{1}\cdot 1\cdot 2\cdot {}^{2}\cdot 1\cdot 2\cdot {}^{2} \cdots$$

These are NOT exponents... the level represents the number of factors of 3 in the corresponding factor in $n!$. The highest 1 and 2, for example, correspond to 9 and 18.

I can organize the factors in $n!$ by the number of 3's that divde the factor...it's kind of fractal-like!

$$\{1,2,3,\dots,n\} = \{1\le k\le n \mid 3\nmid k\}$$
$$\sqcup \{1\le k\le n \mid 3\mid k \text{ but } 3^2\nmid k\}$$
$$\sqcup \{1\le k\le n \mid 3^2\mid k \text{ but } 3^3\nmid k\}$$
$$\vdots$$

I'll define sets that split the factors according to the largest power of 3 that divides it.

Define $S_{n,m} = \{1\le k\le n \mid 3^m \mid k \text{ but } 3^{m+1}\nmid k\}$.

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Organizing the factors according to the power of 3 that divides the corresponding factor in $n!$ allows me to find the remainder, modulo 3, more easily.

$$\prod_{k\in S_{n,0}} k \equiv 2^{\lfloor \frac{n}{3} \rfloor} \cdot (\text{remainder of } n \div 3) \pmod 3$$

$$\prod_{k\in S_{n,1}} k \equiv 2^{\lfloor \frac{n}{9} \rfloor} \cdot \begin{pmatrix} 1 \text{ if remainder of } n\div 9 = 0,1,2,3,4, \text{ or } 5 \\ 2 \text{ if remainder of } n \div 9 = 6,7, \text{ or } 8\end{pmatrix} \pmod 3$$

I goofed here with the notation. The product should be over k divided by the largest power of 3 that divides k. These products are meant to be free of factors of 3.

These computations strongly suggest working in base 3. I'll use an underline to indicate that this should be interpreted as a base 3 number adn not a product.

Suppose n written in base 3 is

$$\underline{b_d b_{d-1} b_{d-2} \cdots b_3 b_2 b_1 b_0}$$

So each $b_x \in \{0,1,2\}$.

$$\prod_{k\in S_{n,m}} k \equiv 2^{\underline{b_d b_{d-1} b_{d-2}\cdots b_{m+1}}} \cdot \begin{pmatrix} 2 \text{ if } b_m = 2 \\ 1 \text{ if } b_m = 0,1 \end{pmatrix} \pmod 3$$

$$\equiv (-1)^{\underline{b_d b_{d-1} b_{d-2}\cdots b_{m+1}}} \cdot \begin{pmatrix} 2 \text{ if } b_m = 2 \\ 1 \text{ if } b_m = 0,1 \end{pmatrix} \pmod 3$$

Suppose n is $\underline{b_d b_{d-1} b_{d-2} \cdots b_2 b_1 b_0}$ in base 3 where $b_x \in \{0,1\}$.

Then

$$\prod_{k\in S_{n,m}} k \equiv (-1)^{\underline{b_d b_{d-1} b_{d-2}\cdots b_{m+1}}} \pmod 3$$

$$\equiv (-1)^{b_d + b_{d-1} + \cdots + b_{m+1}} \pmod 3$$

$$n!_3 \equiv \prod_{m=0}^{d} (-1)^{b_d+b_{d-1}+\cdots+b_{m+1}} \pmod 3$$

$$\equiv (-1)^{(d+1)b_d + d b_{d-1} + (d-1) b_{d-2} + \cdots + 2b_1 + b_0} \pmod 3$$

$$(2n)!_3 \equiv \prod_{m=0}^{d} \prod_{k\in S_{n,m}} k = \prod_{m=0}^{d} (-1)^{\underline{(2b_d)(2b_{d-1})\cdots(2b_{m+1})}} \begin{pmatrix} 2 \text{ if } b_m = 1 \\ 1 \text{ if } b_m = 0 \end{pmatrix} \pmod 3$$

$$= 2^{b_d+b_{d-1}+\cdots+b_1+b_0} \pmod 3$$

$$\binom{2n}{n} \equiv \frac{(2n)!_3}{n!_3\, n!_3} \equiv (-1)^{b_d+b_{d-1}+\cdots+b_2+b_1+b_0} \pmod 3$$

$$\equiv \begin{cases} 1 & \text{if } n \text{ has an even number of nonzero ternary digits} \\ 2 & \text{otherwise.} \end{cases}$$

ABB 4/23/18

Terrific! So $2m$ choose m is 1, modulo 3, when the base 3 expansion of $2m$ has an even number of two digits and no one digits. So to answer the original question, I must count how many such numbers there are in the desired range.

Key:
- Anna's thoughts
- Anna's afterthoughts
- *Editor's comments*

Tennis Directions

by Lightning Factorial
edited by Jennifer Silva

I love tennis and often find myself wondering the following: What is the ideal direction to run in order to intercept an incoming shot?

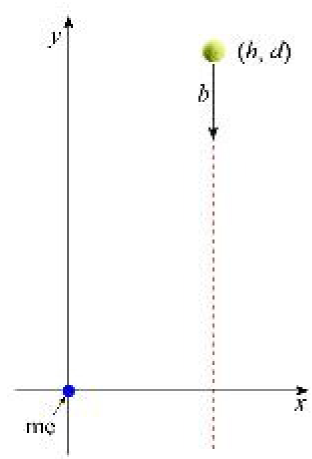
I've heard that one should run in the direction perpendicular to the path of the ball. This advice sounds reasonable because it corresponds to sprinting the shortest distance. However, when I watch professionals chase down balls, they don't seem to heed it.

I decided to use mathematics to answer the question.

Framing the Problem

In order to apply mathematics, I must create a mathematical model of the situation. I'll simplify the tennis ball motion by assuming that the ball travels at a constant velocity. This means that for now, I'm not going to worry about air resistance slowing the ball, nor about spin, nor the bounce. Even though real life is more intricate, this simplified model will still give me an approximation to the truth and a better understanding of how to play.

In tennis, the ball could be coming in from a variety of different angles. However, I'm only interested in figuring out how to get to the ball as quickly as possible. So for this analysis, I'm not concerned with the court geometry. In my mind's eye, I let the lines that define the playing area and the net fade away, leaving only the ball and me. In my picture, I even replace myself with a geometric point and make the simplifying assumption that this point can instantaneously move at a speed s. For ease of computation, I place my initial position at the origin of an xy-coordinate plane and rotate the plane as necessary; thus, I may also assume that the ball starts at some point (h, d) and travels along a line perpendicular to the y-axis at a speed of b (b for "ball") in the direction of decreasing y-coordinates. It therefore makes sense to assume that it begins in the upper half plane, i.e., $d > 0$. I'll also assume that there is a symmetry between forehand and backhand, allowing me to assume $h > 0$. [See the figure at right.] By the way, I've deliberately avoided mention of units of measurement. In the end, we will be able to use whatever units of measurement we wish, provided that we use them consistently within the same formula.



In order to answer which direction I should run to intercept the ball, I have to be precise about what I am trying to achieve. Do I want to reach the ball as quickly as possible? Or do I want to give myself the most time to set up for a shot? As an amateur tennis player, I certainly can use all the time I can get to set up for my shot and make a good swing. With this latter goal in mind, I'll figure out which direction I should run in order to maximize my set-up time.

Content Removed from Electronic Version

America's Greatest Math Game: Who Wants to Be a Mathematician.

Content Removed from Electronic Version

Girls!

Learn Mathematics!

Make new Friends!

Meet Professional Women who use math in their work!

Improve how you Think and Dream!

Girls' Angle

A math club for ALL girls, grades 5-12.

girlsangle@gmail.com
girlsangle.org

Girls' Angle

Member's Thoughts

The Greatest Number From *N* Ones

by **Allie**

Last semester, **Allie** wondered what is the largest number one can express using N ones, addition, multiplication, and parentheses. After some experimentation, she formulated a conjecture. Here, she proves her conjecture. Her proof is an excellent example of a relatively advanced inductive proof. Much of this work was done under the guidance of Girls' Angle mentor Josephine Yu. - *Editor*

We find the greatest number you can make with N ones, addition, multiplication, and parentheses. We call this maximal value $L(N)$. We assume N is a positive integer.

Theorem. Let N be a positive integer and let $L(N)$ be the largest number expressible using N ones, addition, multiplication, and parentheses. Then $L(1) = 1$ and

$$L(N) = \begin{cases} 3^{N/3}, \text{ if } N = 0 \pmod 3, \\ 2^2 3^{\lfloor N/3 \rfloor - 1}, \text{ if } N = 1 \pmod 3, \\ 2 \cdot 3^{\lfloor N/3 \rfloor}, \text{ if } N = 2 \pmod 3, \end{cases}$$

if $N > 1$.

We prove this by induction, splitting the proof of the inductive step into cases depending on the remainder N leaves upon division by 3. We will consider how the last binary operation splits the expression in two with one part involving a ones and the other involving b ones, where $N = a + b$. By the induction hypothesis, we will know what $L(a)$ and $L(b)$ are. We use $L(a)$ and $L(b)$ to determine $L(N)$.

We shall make frequent use of the following lemma:

Lemma. Let m and n be positive integers. If m and n are both greater than 1, then $mn \geq m + n$, with equality if and only if $m = n = 2$. Otherwise, $mn < m + n$.

Proof. First, suppose $m > 1$ and $n > 1$. Then $1/m \leq 1/2$ with equality if and only if $m = 2$. Similarly, $1/n \leq 1/2$ with equality if and only if $n = 2$. Therefore, $1/n + 1/m \leq 1$, with equality if and only if $m = n = 2$. Multiplying both sides of this inequality by mn, we find $m + n \leq mn$, with equality if and only if $m = n = 2$.

Now suppose $n = 1$. Then $mn = m$ and $m + n = m + 1$. Since $m < m + 1$, we have $mn < m + n$. A similar argument shows that $mn < m + n$ if $m = 1$. □

Now, we establish the base cases. We will consider the cases where $N = 1$ and $N = 2$ as base cases.

Base case 1. Suppose $N = 1$. The only expression that can be formed with a single 1 is 1. Therefore $L(1) = 1$.

Base case 2. Suppose $N = 2$. Essentially the only expressions we can form with 2 ones, addition, multiplication, and parentheses, are $1 + 1$ and 1×1. Of these, the larger is 2, hence $L(2) = 2$. This is consistent with our expression in the statement of the theorem when $N = 2$, since $2 \cdot 3^{\lfloor 2/3 \rfloor} = 2$.

Next, we prove the inductive step by splitting it into cases depending on whether $N = 0$, 1, or 2 modulo 3.

Inductive case 1. Suppose $N = 0 \pmod 3$ and $N > 1$.

Our expression must have the form $(A) + (B)$ or $(A)(B)$, where A is an expression that uses only addition, multiplication, parentheses, and a ones, and B is an expression that uses only addition, multiplication, parentheses, and b ones, where a and b are both positive integers satisfying $a + b = N$. Note that both a and b must be smaller than N.

Without loss of generality, the possible cases for a and b are:

Case 1a. We have $a = b = 0 \pmod 3$.

Case 1b. We have $a = 1 \pmod 3$ and $b = 2 \pmod 3$ and $a > 1$.

Case 1c. We have $a = 1$ and $b = 2 \pmod 3$.

We do not need to consider the cases where $a = 2 \pmod 3$ and $b = 1 \pmod 3$ because both addition and multiplication are commutative.

We will first consider whether $(A) + (B)$ or $(A)(B)$ produces the largest possible value. In either case, to maximize the value of the expression, we need to pick A and B to evaluate to their largest possible values, which are, by the induction hypothesis, $L(a)$ and $L(b)$, respectively.

Case 1a and 1b. Since $a > 1$ and $b > 1$, we know that $L(a) > 1$ and $L(b) > 1$. Hence

$$L(a)L(b) \geq L(a) + L(b).$$

Case 1c. In this case, $a = 1$, so $L(a) = 1$. Then

$$L(a)L(b) = L(b) < 1 + L(b) = L(a) + L(b).$$

Using the formulas for $L(a)$ and $L(b)$ by the induction hypothesis, we find the largest candidates in each case to be:

Case 1a. The largest value of $L(a)L(b) = 3^{a/3}3^{b/3} = 3^{N/3}$.

Case 1b. The largest value of $L(a)L(b) = 2^2 3^{\lfloor a/3 \rfloor - 1} \cdot 2 \cdot 3^{\lfloor b/3 \rfloor} = 2^3 3^{(a-1)/3 - 1 + (b-2)/3} = 2^3 3^{(N-6)/3}$.

Case 1c. The largest value of $L(a) + L(b) = 1 + 2 \cdot 3^{\lfloor b/3 \rfloor} = 1 + 2 \cdot 3^{(N-3)/3}$.

Comparing these values, we see that case 1a yields the largest possible value. (In detail, we have $3^{N/3} = 9\cdot 3^{(N-6)/3} > 2^3 3^{(N-6)/3}$ and $3^{N/3} = 3\cdot 3^{(N-3)/3} = 3^{(N-3)/3} + 2\cdot 3^{(N-3)/3} \geq 1 + 2\cdot 3^{(N-3)/3}$. For the remaining cases, we omit these details.) We conclude that if $N = 0 \pmod 3$, then $L(N) = 3^{N/3}$.

Inductive case 2. Suppose $N = 1 \pmod 3$ and $N > 1$.

As in case 1, our expression must have the form $(A) + (B)$ or $(A)(B)$, where A is an expression that uses only addition, multiplication, parentheses, and a ones, and B is an expression that uses only addition, multiplication, parentheses, and b ones, where a and b are both positive integers satisfying $a + b = N$. Note that both a and b must be smaller than N.

Without loss of generality, the possible cases for a and b are:

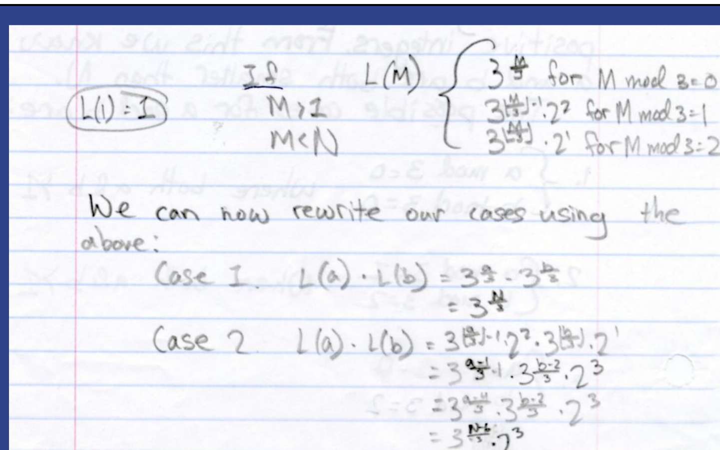
Case 2a. We have $a = 0 \pmod 3$ and $b = 1 \pmod 3$ and $b > 1$.

Case 2b. We have $a = b = 2 \pmod 3$.

Case 2c. We have $a = 0 \pmod 3$ and $b = 1$.

We do not need to consider the cases where $a = 1 \pmod 3$ and $b = 0 \pmod 3$ because both addition and multiplication are commutative.

We will first consider whether $(A) + (B)$ or $(A)(B)$ produces the largest possible value. In either case, to maximize the value of the expression, we need to pick A and B to evaluate to their largest possible values, which are, by the induction hypothesis, $L(a)$ and $L(b)$, respectively.



An excerpt from one of **Allie**'s drafts.

Case 2a and 2b. Since $a > 1$ and $b > 1$, we know that $L(a) > 1$ and $L(b) > 1$. Hence

$$L(a)L(b) \geq L(a) + L(b).$$

Case 2c. In this case, $b = 1$, so $L(b) = 1$. Then

$$L(a)L(b) = L(a) < L(a) + 1 = L(a) + L(b).$$

Using the formulas for $L(a)$ and $L(b)$ by the induction hypothesis, we find the largest candidates in each case to be:

Case 2a. The largest value of $L(a)L(b) = 3^{a/3}\cdot 2^2 3^{\lfloor b/3\rfloor -1} = 2^2 3^{a/3+(b-1)/3-1} = 2^2 3^{(N-4)/3}$.

Case 2b. The largest value of $L(a)L(b) = 2\cdot 3^{\lfloor a/3\rfloor}\cdot 2\cdot 3^{\lfloor b/3\rfloor} = 2^2 3^{(a-2)/3+(b-2)/3} = 2^2 3^{(N-4)/3}$.

Case 2c. The largest value of $L(a) + L(b) = 3^{a/3} + 1 = 3^{(N-1)/3} + 1 = 1 + 3\cdot 3^{(N-4)/3}$.

Case 2a and 2b yield the same value and both are greater than or equal to the value in case 2c. We conclude that if $N = 1 \pmod 3$, then $L(N) = 2^2 3^{(N-4)/3} = 2^2 3^{\lfloor N/3 \rfloor - 1}$.

Inductive case 3. Suppose $N = 2 \pmod 3$ and $N > 2$.

As in cases 1 and 2, our expression must have the form $(A) + (B)$ or $(A)(B)$, where A is an expression that uses only addition, multiplication, parentheses, and a ones, and B is an expression that uses only addition, multiplication, parentheses, and b ones, where a and b are both positive integers satisfying $a + b = N$. Note that both a and b must be smaller than N.

Without loss of generality, the possible cases for a and b are:

Case 3a. We have $a = 0 \pmod 3$ and $b = 2 \pmod 3$.

Case 3b. We have $a = b = 1 \pmod 3$ and both $a > 1$ and $b > 1$.

Case 3c. We have $a = 1$ and $b = 1 \pmod 3$ with $b > 1$.

We do not need to consider the cases where $a = 2 \pmod 3$ and $b = 0 \pmod 3$ or where $b = 1$ because both addition and multiplication are commutative. Note that the case $N = 2$ was covered as a base case.

We will first consider whether $(A) + (B)$ or $(A)(B)$ produces the largest possible value. In either case, to maximize the value of the expression, we need to pick A and B to evaluate to their largest possible values, which are, by the induction hypothesis, $L(a)$ and $L(b)$, respectively.

Case 3a and 3b. Since $a > 1$ and $b > 1$, we know that $L(a) > 1$ and $L(b) > 1$. Hence

$$L(a)L(b) \geq L(a) + L(b).$$

Case 3c. In this case, $a = 1$, so $L(a) = 1$. Then

$$L(a)L(b) = L(b) < 1 + L(b) = L(a) + L(b).$$

Using the formulas for $L(a)$ and $L(b)$ by the induction hypothesis, we find the largest candidates in each case to be:

Case 3a. The largest value of $L(a)L(b) = 3^{a/3} \cdot 2 \cdot 3^{\lfloor b/3 \rfloor} = 2 \cdot 3^{a/3 + (b-2)/3} = 2 \cdot 3^{(N-2)/3}$.

Case 3b. The largest value of $L(a)L(b) = 2^2 3^{\lfloor a/3 \rfloor - 1} \cdot 2^2 3^{\lfloor b/3 \rfloor - 1} = 2^4 3^{(a-4)/3 + (b-4)/3} = 2^4 3^{(N-8)/3}$.

Case 3c. The largest value of $L(a) + L(b) = 1 + 2^2 3^{\lfloor b/3 \rfloor - 1} = 1 + 2^2 3^{(b-4)/3} = 1 + 2^2 3^{(N-5)/3}$.

Of these 3 cases, case 3a is the biggest. We conclude that if $N = 2 \pmod 3$, then

$$L(N) = 2 \cdot 3^{(N-2)/3} = 2 \cdot 3^{\lfloor N/3 \rfloor}.$$

This concludes the proof of the induction hypothesis and the theorem follows. □

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 22 - Meet 4
March 1, 2018

Mentors: Rachel Burns, Sarah Coleman, Anna Ellison, Alexandra Fehnel, Molly Humphreys, Elise McCormack, Suzanne O'Meara, Samantha Russman, Christine Soh, Shohini Stout, Elizabeth Tso, Jane Wang, Josephine Yu

Visitor: Melody Chan, Brown University

"*Think Globally, Act Locally*" That's the motto of Support Network visitor Melody Chan, an assistant professor of mathematics at Brown University, who treated us to a wonderful presentation on distributed computing.

In distributed computing, one builds networks that consist of many "simple" sensors, which are computing devices that have limited capability, but together, can accomplish nontrivial tasks. There can be millions of sensors in a single such network.

To illustrate, Melody challenged us to create a series of networks that accomplished tasks of increasing difficulty. The first task was to design a network with a single sensor with a memory capacity of 1 bit. The sensor has a binary input that it could sample, and based on the input, change its memory. A stream of bits is sent to the sensor. The task for this network is to answer the question, "Was there a 1 in the input stream?"

A protocol for a distributed system is a set of rules dictating the initial state (i.e. what its memory will contain at the beginning) and how the sensors will change state depending on the input and current state. For example, the following protocol solves the problem in the previous paragraph:

Start in 0 state

Input	**State**	**New State**
0	0	0
0	1	1
1	0	1
1	1	1

We interpret the state "1" as giving a "yes" answer and "0" as giving a "no" answer.

For the next challenge, we had a sensor with a binary input, but it could be in any one of three states: 0, 1, or 2. What protocol will answer the question: Were there two 1's in the input stream?

After solving that, Melody asked us to design a protocol for a 2-state sensor that answered the question: Was the number of 1's in the input stream even?

Melody concluded by performing a card trick. She explained that to perform it, she played the role of a sensor with a specific protocol and challenged us to figure out that protocol.

Session 22 - Meet 5
March 8, 2018

Mentors: Karia Dibert, Anna Ellison, Amber Guo, Suzanne O'Meara, Kate Pearce, Christine Soh, Shohini Stout, Elizabeth Tso, Jane Wang, Josephine Yu

A group of members engaged in a long discussion about the meaning of the decimal number $0.\overline{9}$. Is it or is it not equal to 1? What are sensible ways of defining what $0.\overline{9}$ is? Can a system of numbers be created that can be represented by sequences of decimal digits in which $0.\overline{9}$ is not equal to 1? What aspects of such a discussion are actually about numbers and what aspects are about representations of numbers? These are good questions to think about!

Session 22 - Meet 6 March 15, 2018	Mentors:	Ivana Alardin, Rachel Burns, Sarah Coleman, Danielle Fang, Jacqueline Garrahan, Amber Guo, Elise McCormack, Kate Pearce, Samantha Russman, Christine Soh, Shohini Stout, Josephine Yu

Some members worked on solving problems pertaining to paper folding, which is a rich source of mathematical material. Other members worked on solving problems involving jumping frogs or checker-like moves, another rich source of problems. For example, there is the following problem of John Horton Conway: Checkers are placed on the lattice points of a coordinate plane on or below the horizontal axis. These checkers can only move if they can jump over an adjacent checker into an unoccupied lattice point, at which point the checker that has been leapt over is removed. However, unlike in the game checkers, these checkers can only move horizontally and vertically. The question is: What is the minimum number of checkers needed if the goal is to get a checker with a vertical coordinate of n, where n is a given positive integer, and how should they be initially positioned?

For $n = 1$, two checkers are needed. For example, you could place them at the locations (0, 0) and (0, -1). With one move, the checker at (0, -1) leaps over the checker at (0, 0) to (0, 1).

For $n = 2$, it can be done with four checkers placed at (2, 0), (1, 0), (0, 0), and (0, -1). The checker at (0, -1) leaps the checker at (0, 0) landing in (0, 1) and freeing up (0, 0). Then the checker at (2, 0) leaps the checker at (1, 0) landing in (0, 0) and freeing (1, 0). Finally, the checker now in (0, 0) leaps the checker at (0, 1) landing in (0, 2), achieving the goal.

What is the answer for any positive integer n?

This problem is also described in Ross Honsberger's *Mathematical Gems II*.

Session 22 - Meet 7 March 22, 2018	Mentors:	Rachel Burns, Sarah Coleman, Danielle Fang, Alexandra Fehnel, Elise McCormack, Kate Pearce, Jane Wang, Josephine Yu

Some members found beautiful formulas for the lengths of the altitudes of a triangle in terms of the side lengths of the triangle. A few meets prior, one of these members, **Barry Allen**, had solved this problem for a specific triangle. So the task was essentially to redo that computation using variables in order to obtain a general formula. We then discussed the advantages of having the general formula, such as being able to see how the altitude lengths depend on the lengths of each side, seeing the symmetries of the problem reflected in the formulas, and being able to apply dimensional analysis.

"*...so it can't be actual magic... it must be math!*" – a member at Meet 7

Session 22 - Meet 8 April 5, 2018	Mentors:	Danielle Fang, Alexandra Fehnel, Amber Guo, Kate Pearce, Samantha Russman, Christine Soh, Jane Wang, Josephine Yu, Annie Yun

We opened this meet by playing a game due to Alain Ledoux. Everybody was asked to write down an integer between 0 and 100, inclusive, which represented their best guess as to what two-thirds of the average of all turned in numbers would be. We collected the numbers, then at the end of the meet, revealed the answer, declaring those who got the answer, rounded down to the nearest integer, as the "winners." What number would you have submitted?

In this experiment, zero satisfies the condition that if everybody submitted it, everybody would win. For any other valid number, one must realize that if everyone else did the same, then everybody would lose, because the average would be this one number, and two-thirds of that would be less.

If you're interested in participating in such an experiment, there is a large scale ongoing one being conducted by Leonhardt and Quealy at the New York Times. To find it, search on the internet for "new york times leonhardt quealy two-thirds".

Session 22 - Meet 9 April 12, 2018	Mentors:	Rachel Burns, Jacqueline Garrahan, Amber Guo, Elise McCormack, Kate Pearce, Samantha Russman, Christine Soh, Jane Wang, Josephine Yu
	Visitor:	Anna Frebel, Department of Astronomy, MIT

Today, we were fortunate to enjoy not only Prof. Frebel's performance, but also a visit from Beth Malmskog, professor of mathematics from Colorado College and the author of *Quilt-doku!* which appeared in Volume 10, Numbers 2 and 3 of the Girls' Angle Bulletin.

Prof. Frebel gave us a special performance of her play "Pursuit of Discovery: Lise Meitner & Nuclear Fission." Anna drew us all into the difficult circumstances of Meitner's life surrounding her discovery that the uranium nucleus spontaneously breaks apart. Meitner had just fled Berlin for Sweden to escape the holocaust where she found herself bereft of her colleagues, laboratory, money, personal items, and home, carrying with her only what she could pack into a suitcase in a couple of hours. She was in the midst of experiments designed to create new, heavy elements, like uranium (which has atomic number 92). But surprisingly, the experiments seemed to only produce barium (atomic number 56). Anna conveyed the mystery and thought process that led Meitner from the experimental data to the then breakthrough idea that heavier elements are so unstable that they actually spontaneously break apart into lighter elements.

After her performance, Prof. Frebel explained how the discovery of nuclear fission impacts her research into the chemical composition of stars. She gave an overview of element formation in stars. Briefly, fusion creates elements up to iron (atomic number 26). When neutron stars collide, much heavier elements are produced, and the heaviest of these decay into lighter elements producing all the elements in the periodic table up to Uranium. For more details, read her book, *Searching for the Oldest Stars*.

Prof. Frebel also pointed out that despite being nominated for a Nobel prize 48 times over a roughly 40 year span, Meitner was never given the distinction, although the element meitnerium (atomic number 109) is named after her.

Session 22 - Meet 10 April 26, 2018	Mentors:	Rachel Burns, Sarah Coleman, Anna Ellison, Alexandra Fehnel, Amber Guo, Suzanne O'Meara, Kate Pearce, Jane Wang, Josephine Yu

Allie finished her proof of her conjecture on the largest number that can be expressed using *N* ones, addition, multiplication, and parentheses. For her proof, see page 22.

Calendar

Session 22: (all dates in 2018)

February	1	Start of the twenty-second session!
	8	
	15	
	22	No meet
March	1	Melody Chan, Brown University
	8	
	15	
	22	
	29	No meet
April	5	
	12	Anna Frebel, MIT
	19	No meet
	26	
May	3	
	10	

Session 23: (all dates in 2018)

September	13	Start of the twenty-third session!
	20	
	27	
October	4	
	11	
	18	
	25	
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle holds its meets within 5 minutes of the Kendall Square T stop in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________