Girls' Angle Bulletin

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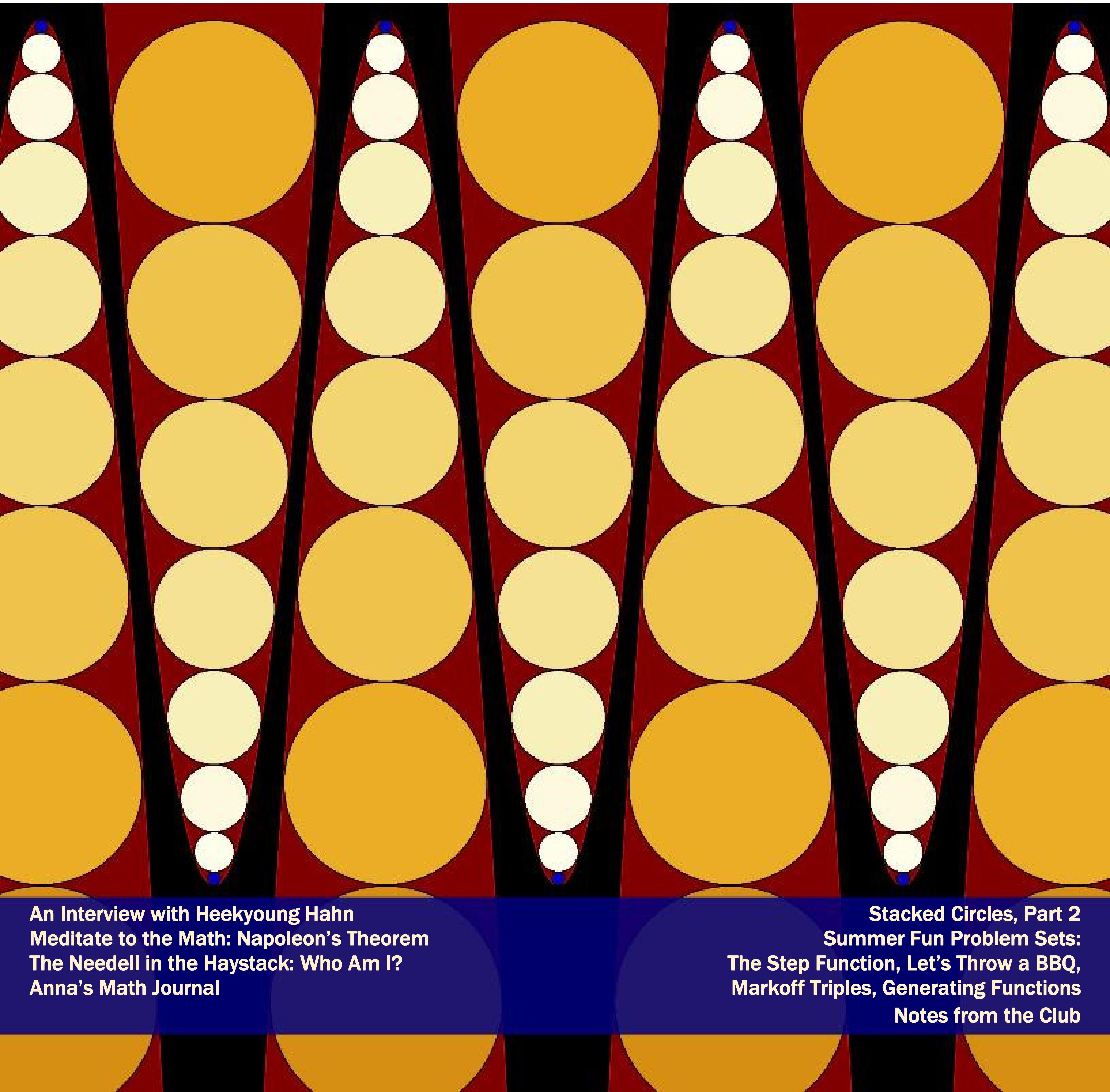
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

The central focus of this Bulletin is doing math. That's the best way to improve and gain appreciation for the subject. What does it mean to do math? How does one go about doing math? How can I get started? Find answers to these questions in here! - Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Parabolic Pods* by C. Kenneth Fan and Amanda Galtman. Arithmetic progressions of circles – see *Stacked Circles, Part 2* on page 14.

An Interview with Heekyoung Hahn

Math is not something to be afraid of. Rather, it is something that you should enjoy playing with.

Heekyoung Hahn is Assistant Research Professor in the Department of Mathematics at Duke University. She received her doctoral degree in mathematics from the University of Illinois at Urbana-Champaign under the supervision of Bruce Berndt.

Ken: What's an early memory of something that excited you to mathematics?

Heekyoung: From my big brother's friend (I had a big family, I am the youngest one) I heard that mathematicians can "count" all the natural numbers (yes, infinitely many numbers). Even more surprising to me, the number of all the natural numbers is equal to that of the integers. I was about a 4th or 5th grader at that time. At that time I knew, for sure, what the natural numbers and the integers were. I thought, if one could "count" them (which was crazy), the number of all the integers should be twice that of the natural numbers + 1 (because you have the positive numbers, negative numbers and zero). I really wanted to know if this is the case and wanted to study math more.

Ken: Could you please describe the journey you traveled to become a mathematician?

Heekyoung: I must say it is unbelievable that I am here. My husband (he is also a mathematician) often says that I should write a book about this.

I was born and raised in South Korea until I moved to the United States to pursue my PhD degree in Math. I grew up in a very small farm in a tiny village (about 70 households in total). There was no electricity available until the time of my 1st grade. I had to walk about an hour to school every day until my 8th grade. Bus service from my home town to the near city was available by then, but it was only twice a day. Even so, often I had to walk to school anyway, because I could not pay the bus fee (10 cents per trip). Back then, it was not mandatory in South Korea to send the kids to the school, so some of my friends did not go to school in order for them to help out their parents, especially female kids. I was lucky enough to have parents who were working hard to send their kids to school.

My parents are the people whom I respect most in my life. They were the hardest working people I have ever known in my entire life (they had to be in order to raise many kids in a very poor household). Back then, the whole country was very poor, especially in the country side. In fact, my dad was one of the young adults who had to fight in the Korean War. My dad had to stop his education when he was a 5th grader when my grandfather passed away. He and my grandmother had to figure out how to make ends meet and provide meals for his younger siblings. My mother never had a formal education, but she was quite a smart person who always desired to learn something. She always used to tell me that, even though I am a woman, I should move beyond high school education and should pursue professional careers.

As you can imagine, back then, almost all women stayed at home (even after their college degree). I am the youngest of all and a Daddy's girl. My father always supported me no matter what I did, although he first thought something was wrong with my brain when he heard about me going to the United States to pursue my PhD. He would never imagine such a thing could possibly happen to his own kid.

I always loved to go to school. There I can learn and sometimes, if I am lucky, I might get to read some story books. During my elementary school years, the only books available to me were almost

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Heekyoung Hahn and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
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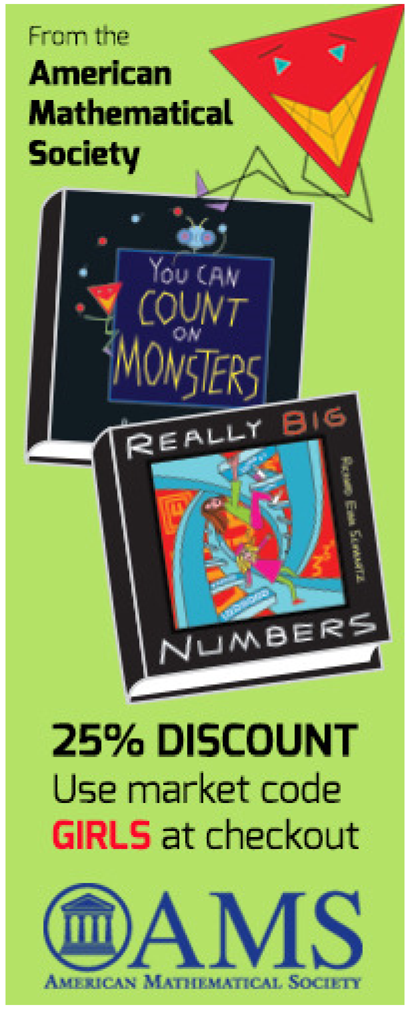
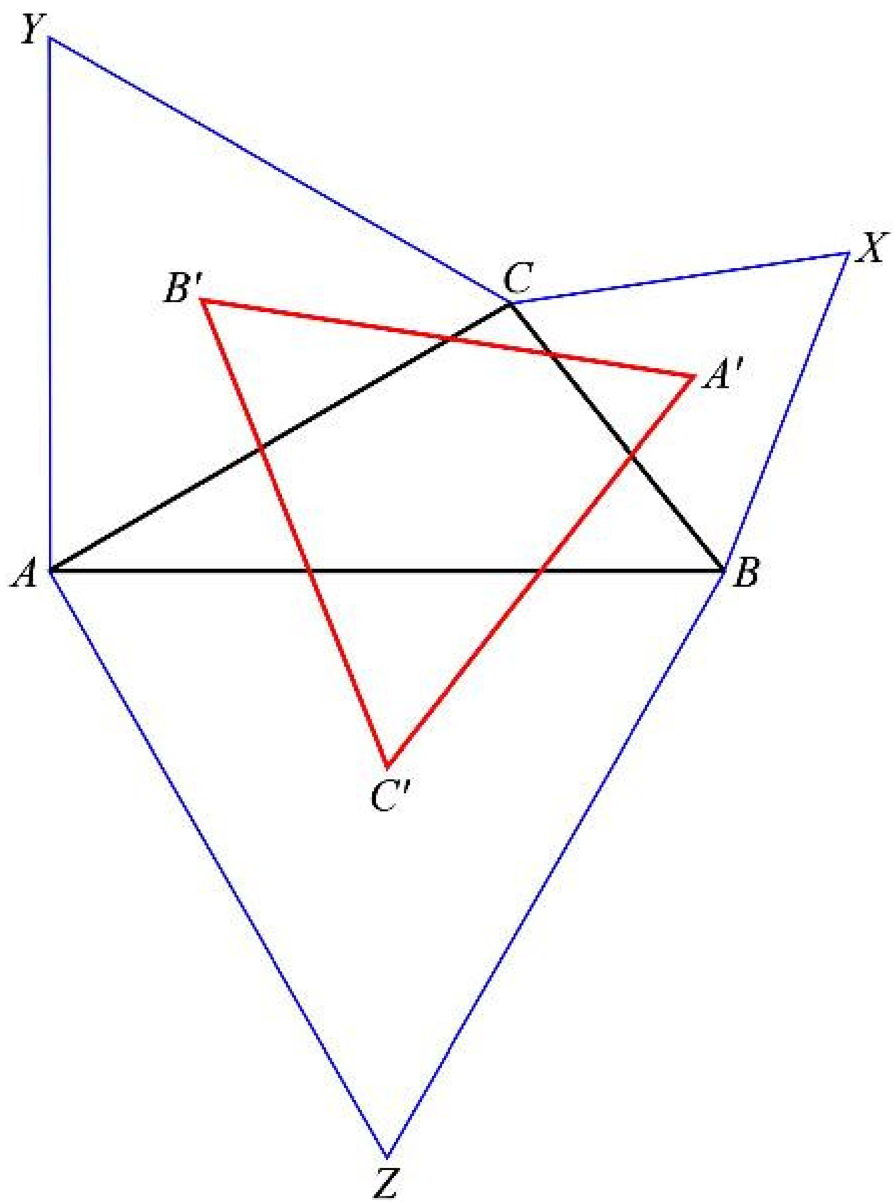


The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code “GIRLS” at checkout.

Meditate[Math]



Napoleon's[1] theorem states the following: Given triangle *ABC*, place equilateral triangles *ABZ*, *BCX*, and *CAY*, externally on its sides. Then the centroids of the equilateral triangles, labelled A', B', and C', are, themselves, vertices of an equilateral triangle.

In this installment of Meditate to the Math, your goal is to prove Napoleon's theorem, doing as much as you can in your head. Look, wonder, think, deduce, and prove!

If you get stuck, here are some things to meditate upon:

1. Recall that the centroid of a triangle is where the three medians of the triangle intersect. (A median of a triangle is any line from a vertex to the midpoint of the opposite side.) In an equilateral triangle, the centroid coincides with the intersection of the angle bisectors and the circumcenter.

2. What is the measure of angle $B'AC$?

3. All equilateral triangles are similar to each other.

4. What is the ratio of the distance between any vertex of an equilateral triangle and its centroid and the triangle's side length?

5. What happens when you rotate $B'C'$ by 30° in the clockwise direction about A?

---

[1] The attribution to Napoleon is unproven. The earliest appearance of this theorem that we are aware of is on an 1820 Gold Medal Examination at the University of Dublin. See *Dublin Problems: Being a Collection of Questions Proposed to the Candidates for the Gold Medal at the General Examinations, from 1816 to 1822 Inclusive*, published by G. and W. B. Whittaker, 1823, page 125.

The Needell in the Haystack

Who Am I? Machine Learning Classification[1]
by Deanna Needell | edited by Jennifer Silva

In previous issues, I wrote about completing missing entries from a low-rank dataset and performing topic modeling to attain feature representations. In many high-dimensional data applications, the goal is to draw inferences from the data, rather than to simply represent the data efficiently or complete missing data. Such analytical techniques can be performed on the large-scale data itself, or from the efficient representations. Techniques are divided into two groups: **supervised** and **unsupervised**.

As the name may suggest, in supervised or (semi-supervised) techniques, a set of labeled **training data** is provided, and models are built based on this known data. Supervised methods are often referred to as **supervised learning** or **machine learning**; the term **learning** is utilized since the machine "learns" a classification rule from the labeled data provided (the cinephile author notes here that although movies like *Terminator* and *I, Robot* are quite entertaining, we are still quite far from that reality!). These methods include the classification problem that we will discuss in detail here, as well as regression, Bayesian or non-Bayesian hypothesis testing, and neural networks; the interested reader can find more information on these by a quick online search. Unsupervised methods, on the other hand, wish to discover natural relationships while assuming no known prior information about the underlying organization of data (an example is topic modeling, discussed in a previous article). Here, we focus on the (supervised) problem of **classification**, for which there are now various types of methods; we will mention an overview of a few classical approaches and then focus in more detail on a new geometric classifier method. In particular, we will introduce and discuss a simple geometric method recently developed by the author and her colleagues in [3] that exhibits extremely efficient computation and storage, as well as interpretability and the ability to analyze mathematically (i.e., prove rigorous theorems about its accuracy).

Consider a $d \times p$ data matrix X, where we visualize each column as corresponding to a data point. Thus there are p data points, each of dimension d. Each data point has a **label**, which we capture in a p-dimensional vector b. There may be two or more labels, and without loss of generality, we may assume the labels are simply 1, 2, 3, . . ., G, where G is the number of possible labels. We call the pair (X, b) **training data**, as it will be used to train a rule or algorithm that can be applied to label future data. That is, given a new unlabeled data point $x \in \mathbb{R}^d$, we want to assign a label from 1, 2, 3, . . ., G that best describes the data point.

As a concrete example, consider a database consisting of multiple images of two different human faces, under various lighting and shadowing conditions. An example is shown in the left-hand image of Figure 3, which is from the YaleB database of face images [1]. Each image is 32×32 pixels, so lives in $d = 32 \times 32 = 1024$ dimensional space, and comes with a label corresponding to which individual it is (for example, "1" could correspond to the individual in the top row and "2" to the individual in the bottom row). Then, given a new image of a face of one of these individuals, we wish to automatically identify which individual it is; such tasks are critical to

[1] This content supported in part by a grant from MathWorks.

modern applications that need **facial recognition**, made popular by your favorite TV crime drama (and from which the title of this article is motivated).

As another example, consider the need for a machine to recognize and read human handwriting. Here, we have images consisting of various handwritten digits, like those shown in the left-hand image of Figure 4, and we seek to automatically identify whether a new image corresponds to a "zero" or a "one" (these images come from a commonly used dataset called MNIST [2])[2]. A human can typically do these identifications fairly easily (although not always, depending on the quality of the image – the author herself consistently fails the "Captcha" tests that various websites use to distinguish humans from machine users!), but this is a hard task for a machine. There are also many applications in which machine recognition of digits is crucial, such as the popular "snapshot" electronic check deposits you can now make from your cell phone. These are some examples of the classification problem we study here.

Figure 1. Left: Example of separating hyperplane (black) for two classes (red and blue), and new unlabeled data point (the "x"). Right: Example of non-linearly separable data.

There are now many approaches to classification; before going into detail about a simple geometric method, we'll briefly discuss more classical approaches. The first such approach is the **support vector machine** (SVM). In its simplest form, for data coming from two distinct classes ($G = 2$), the SVM seeks a linear separator (i.e., a hyperplane or set of hyperplanes) that separates the data in space. The method searches for a separating hyperplane that maximizes the distance between the hyperplane and any training point, as in the left-hand image of Figure 1. Then, given a new unlabeled data point (e.g., the "x" in this same plot), we can use this separating hyperplane to decide what label to assign it (since it falls on the left side of the line in the plot, we would choose to label it blue). If the data is not linearly separable, as in the right-hand image of Figure 1, it is often necessary to find a so-called **kernel**, or a transformation that makes the data linearly separable (as a fun exercise, try to think of a function that maps the points in the right plot into 3-D space where they become linearly separable). Although the SVM approach in general is intuitive and reasonably straightforward, such kernels may sometimes be challenging to find, which is one of the main drawbacks to this method.

On the other algorithmic extreme are so-called **deep learning** methods. These are often hard to understand, analyze, and interpret, but yield astoundingly accurate classification results. State-of-the-art results can be obtained by using **neural nets**, which utilize multiple layers of abstraction to filter learned features across the network, until a single classification label is attained. However, the abstract nature of the layers makes these approaches hard to interpret (for

[2] MNIST stands for Modified National Institute of Standards and Technology; the dataset was modified from the original NIST data to be more amenable to machine learning testing.

example, it is often not clear what each layer represents), as well as difficult to analyze mathematically. Nonetheless, these methods produce some of the most accurate classification to date.

Motivated by the use of neural nets, but wanting to develop a rigorous mathematical framework in which classification methods can be studied, the author and her colleagues recently introduced a simple geometric approach for classification using only binary representations of the data. Concretely, let A be an $m \times d$ random matrix (e.g., suppose the entries are i.i.d. standard normal random variables, and take m much smaller than n). Obtain measurements of the data by applying A to obtain the product AX. Then, keep only the **sign**[3] of each measurement, i.e., $Q = \text{sign}(AX)$. If we imagine each row of A as being the normal vector to a hyperplane in d dimensions, then the sign measurement Q_{ij} simply tells us on which side of the i^{th} hyperplane the j^{th} data point lies. Our goal will then be to use this information along with the training labels b in order to accurately label future data x from its sign information $q = \text{sign}(Ax)$.

Let us build some intuition for the approach. Consider the two-dimensional data X shown in the left plot of Figure 2, consisting of three labeled classes (green, blue, red). Consider the four hyperplanes shown in the same plot, and suppose we had access only to the binary data $Q = \text{sign}(AX)$, where A contains the normal vectors to each hyperplane as its rows. For the new test point x (which by visual inspection should be labeled blue) and its binary data $q = \text{sign}(Ax)$, one could simply cycle through the hyperplanes and decide which class x matches most often. For example, for the hyperplane colored purple in the plot, x has the same sign (i.e., lies on the same side) as the blue and green classes. For the black hyperplane, x only matches the blue class, and so on. Then for this example, x will match the blue class most often, and we could correctly assign it that label. However, next consider the more complicated geometry given in the right plot, where the data consists of only two classes (red and blue), but they are now intermixed. This same strategy will no longer be accurate for the test point x. However, instead of single hyperplanes, consider hyperplane *pairs*, and ask which class label x most often matches (note that in this context, we use "matches" to mean that points lie in the same wedge that the hyperplanes divide the space into). For example, for the pair colored orange and green, x matches both red and blue points, whereas for the pair of hyperplanes colored orange and purple, x only matches the blue class. We could now cycle through all pairs and again ask which class x matches most often. This gives rise to a multi-level approach, using L levels, where level 1 corresponds to using single hyperplanes, level 2 to pairs, level 3 to triples, and so on, so that the l^{th} level uses l-tuples of hyperplanes. Each level then provides a "vote" on which class the new point belongs to, and these votes are aggregated over all L levels.

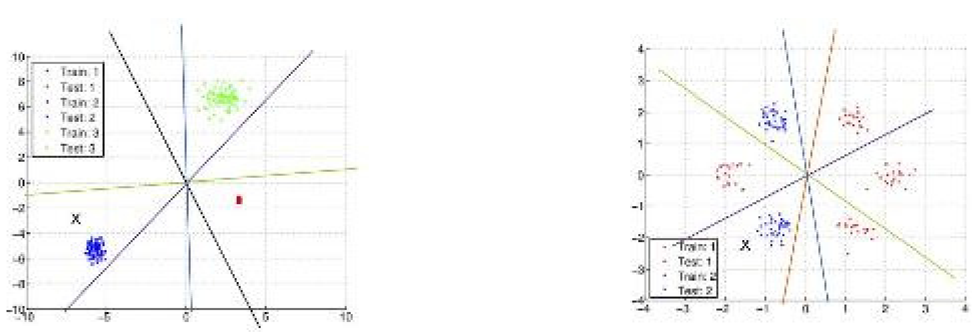


Figure 2. Two motivating examples for the classification method.

[3] For completeness, we define sign(a) to be -1 if $a < 0$ and 1 if $a \geq 0$; for matrices we take the sign of each entry individually. Strictly speaking, zero has no sign, but we attach positive sign to it to obtain a binary representation.

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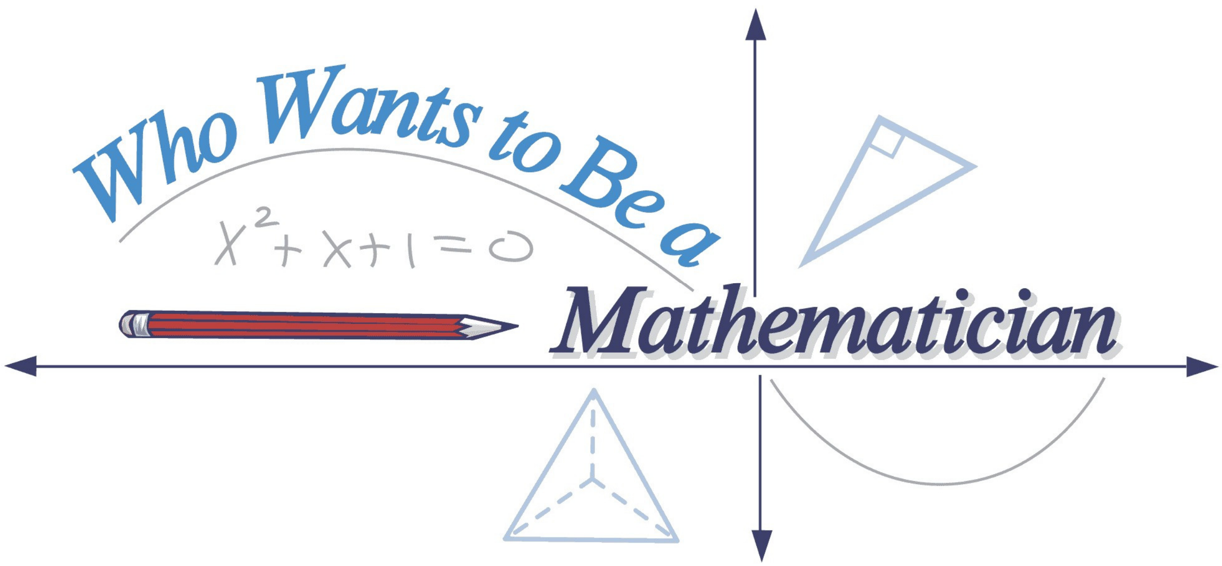


America's Greatest Math Game: Who Wants to Be a Mathematician.

Figure 3 shows classification results from this method on the Yale face data discussed above, where we use p labeled images from each individual as our training data. We then randomly select testing data, assign a label using the method described above, and then compute how often this method assigns the correct label. Figure 4 shows similar results for the MNIST handwritten digit data. Not surprisingly, we see that as the number of measurements m grows (i.e., we capture more "information" about the data), the accuracy of classifying new data correctly also increases. In these examples, the number of training points p used doesn't seem to have much impact; however, if p is too small the accuracy will typically be low, and if it is too large we may start to see "overfitting" along with a decrease in accuracy. Note also that for the face data we used $L = 4$ levels, whereas for the digits we used $L = 1$; it seems that the handwritten digits are more linearly separable in space and less complicated than the face images (which isn't surprising when looking at the images with the human eye). We also note that the approach described here works well on data that is "centered" on the origin, as in Figure 2. For the real data we used in these experiments, we centered the data before running the method. A possible direction for future research related to this method would be to allow for "shifted" hyperplanes that do not need to cross the origin; this would look more like the right-hand image in Figure 1, which is still a challenge for this approach, but would allow for better classification on such data. Nonetheless, the approach gives rise to interesting mathematics and yields promising experimental results.

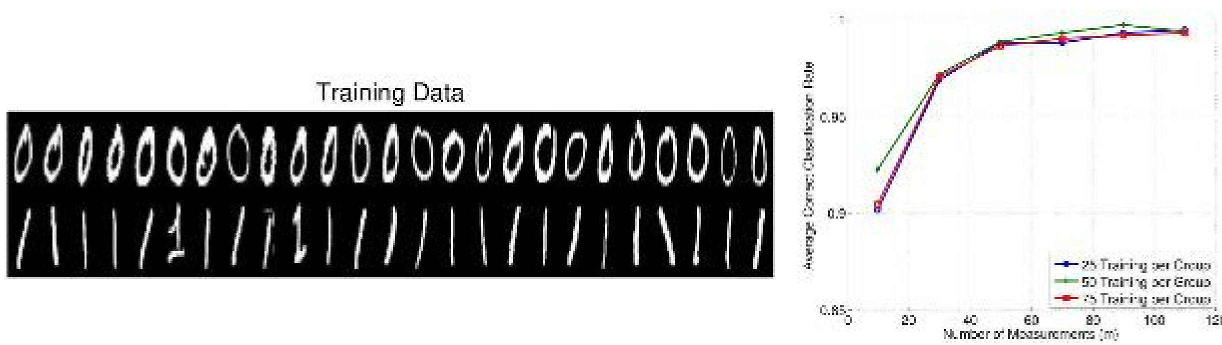


Figure 4. Classification experiment using handwritten "0" and "1" digit images from the MNIST dataset, $L = 1$, $n = 28 \times 28$, 50 test points per group, and 30 trials of randomly generating A. Left: Training data images when $p = 50$. Right: Average correct classification rate versus m and for the indicated number of training points per class.

References

[1] X. He, S. Yan, Y. Hu, P. Niyogi, and H.-J. Zhang, Face recognition using Laplacianfaces. *IEEE Trans. Pattern Anal. Mach. Intelligence*, Vol. 27, Issue 3 (2005), 328-340.

[2] Y. LeCun, The MNIST database of handwritten digits. http://yann.lecun.com/exdb/mnist/.

[3] D. Needell, R. Saab, and T. Woolf, Simple classification using binary data (2017). Submitted for publication.

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Tackling a problem where rows of Pascal's triangle, modulo 3, are read as ternary numbers.

Last time, I figured out that the nth term of the sequence is odd if and only if n has an even number of 2s in its ternary representation and no 1s. So the problem reduces to counting how many of the first 2018 indices have this property.

How many of the first 2018 terms of the sequence obtained by interpreting the rows of Pascal's triangle, reduced modulo 3, as ternary numbers, are odd?

If $S_0, S_1, S_2, \ldots$ is the sequence, then

S_n is odd iff n is even and, when written in base 3, only has digits 0 or 2, and the number of 2's is even.

$\Updownarrow$

n, when written in base 3, only has digits 0 or 2, and the number of 2's is even.

It'll be handy to convert 2018 to base 3, so I'll do that now.

$$2018 = 2 \pmod 3$$

$$\frac{2018-2}{3} = 672 = 0 \pmod 3$$

$$\frac{672}{3} = 224 = 2 \pmod 3$$

$$\frac{224-2}{3} = 74 = 2 \pmod 3$$

$$\frac{74-2}{3} = 24 = 0 \pmod 3$$

$$\frac{24}{3} = 8 = 2 \pmod 3$$

$$\frac{8-2}{3} = 2$$

$$2018 = 2202202_3$$

The base conversion technique I'm using here is to extract the digits from lowest order to highest. The lowest order digit (or ones digit) is the remainder 2018 leaves upon division by 3. I subtract the ones digit to get a multiple of 3, then divide by 3, compute its remainder upon division by 3 to extract the threes digit, and continue the process until the number has been reduced to a digit.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

First, I'll count how many such numbers there are with a fixed number n of ternary digits. To be an n digit ternary number, the first digit must be 2.

How many n-ternary digit numbers are there with an even number of 2-digits and no 1-digits?

$$2\underbrace{\_\ \_\ \_\ \cdots\ \_\ \_}_{n-1}$$

An odd number of the remaining digits must be 2, with the rest 0. Since the 2s can be placed anywhere, I just have to add up $n - 1$ choose k over odd k.

$$\binom{n-1}{1}+\binom{n-1}{3}+\binom{n-1}{5}+\cdots+\binom{n-1}{n-1} \quad \text{if } n \text{ is even}$$

$$\binom{n-1}{1}+\binom{n-1}{3}+\binom{n-1}{5}+\cdots+\binom{n-1}{n-2} \quad \text{if } n \text{ is odd}$$

I've seen these sums before. They add up to a power of 2. To prove it, one can expand $(1 - 1)$ to the mth power and $(1 + 1)$ to the mth power using the binomial theorem, and subtract the former from the latter.

$$0 = (1-1)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k}, \quad m > 0$$

$$2^m = (1+1)^m = \sum_{k=0}^{m} \binom{m}{k}$$

$$2^m - 0 = 2\left(\binom{m}{1}+\binom{m}{3}+\binom{m}{5}+\cdots\right)$$

$$\binom{m}{1}+\binom{m}{3}+\binom{m}{5}+\cdots = 2^{m-1} \quad \text{if } m > 0$$

So, there are 2 to the $n - 2$ n-digit ternary numbers with no 1-digits and an even number of 2-digits, provided $n > 1$. There aren't any such 1-digit numbers, and then there is the number 0.

2^{n-2} such numbers if $n > 1$

0 if $n = 1$

1 if $n = 0$ (i.e. 0)

Before the 1000000_3 term, there are

$1+0+1+2+4+\cdots+2^4$ odd terms

$= 32$.

I can split the indices from 0 to 2017 into groups with specific numbers of digits and specific strings of starting digits. Looks like the answer is 55!

20_ _ _ _ _	⇒	16
2200_ _ _	⇒	4
22020_ _	⇒	2
2202202	⇒	1
		23

$32 + 23 = \underline{\underline{55}}$

ABB 6/28/18

Key:

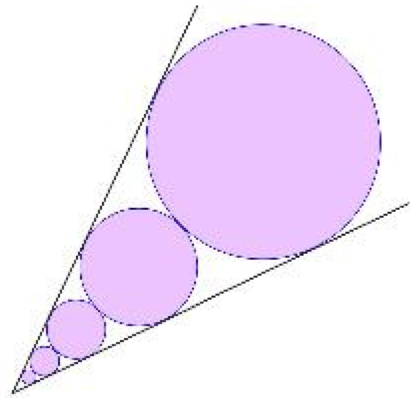
Anna's thoughts

Anna's afterthoughts

Editor's comments

Stacked Circles, Part 2

by Ken Fan | edited by Jennifer Silva

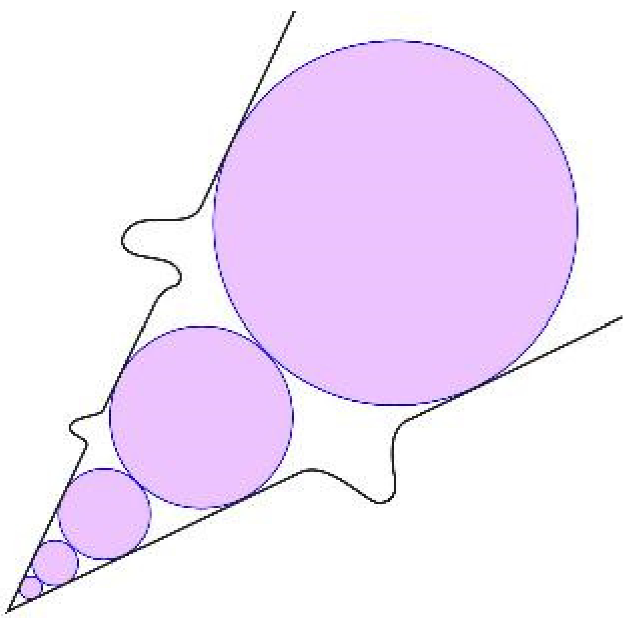


Emily: So what exactly are we trying to do?

Jasmine: Well, we just saw that if we stack circles into an angle, the radii of the circles form a geometric sequence. So now we're asking if there is a shape – analogous to the angle – into which we can stack circles so that the radii will form an arithmetic sequence instead of a geometric sequence.

Emily: And by "stack," you mean to drop circles into the shape so that the circles touch both the sides of the shape and the last circle dropped in, if there is one.

Jasmine: Right.



Emily: That's what I thought we were going to do, but wouldn't there be tons of solutions? After all, with circles stacked into an angle, we could introduce wiggles into the sides of the angle in such a way that we avoid changing anything near points of tangency. The result would be another shape that gives stacked circles with radii in a geometric progression.

Jasmine: Oh, I see what you mean.

Jasmine squiggles up the sides of the angle to make the drawing at left.

Jasmine: You mean like this, for example. This wiggly modification of the angle can be stacked with the same circles as before; since the wiggly parts are so arbitrary, there are infinitely many shapes that contain this stack of circles. Maybe there is a shape that is simplest in some sense and we can look for that.

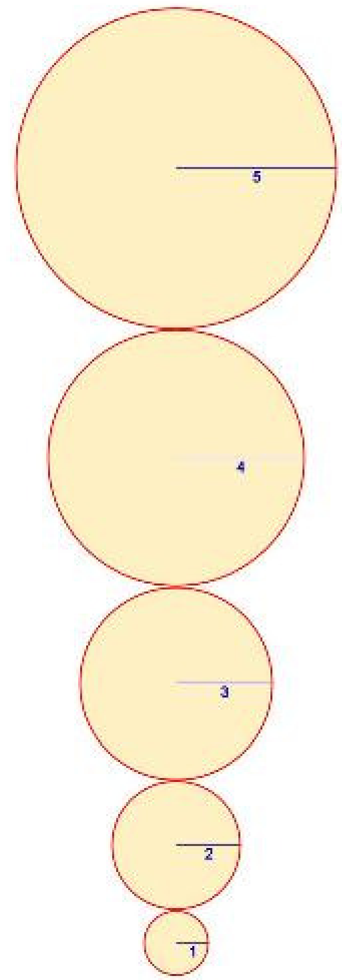
Emily: Wait a sec! One difference between the angle and its wiggly modification is we can stack circles into the angle beginning with a circle of *any* radius we wish, and the resulting stack of circles will have radii in geometric progression. That won't be true of the squiggly shape, because the wiggles will cause discrepancies from a geometric sequence if a circle touches the boundary where we introduced a wiggle. So perhaps we should try to find a shape with the property such that the resulting sequence of stacked circles will have radii that form an arithmetic progression, regardless of the size of the first circle dropped in.

Jasmine: I like that idea. Let's try to find such a shape, or show that it doesn't exist.

Emily: I'm not even sure how to begin. I could propose a shape, but there are so many possibilities that it would be a wild stab in the dark. Can we narrow down the possibilities?

Jasmine: Let's start by drawing a stack of circles whose radii are in arithmetic progression. I'll go ahead and use the straightforward arithmetic progression 1, 2, 3, etc.

Jasmine produces the figure at right.



Emily: Hmm. I still have no idea what curve will neatly contain that stack of circles. It almost seems like these will fit snugly inside an angle, but we know that can't be the case.

Jasmine: Maybe we can get a rough idea by finding a formula for the locations of the centers of these circles. Let's introduce an xy-coordinate system so that the circles are symmetric about the y-axis and the first circle rests upon the x-axis. Since the first circle has radius 1, its center has coordinates (0, 1).

Emily: The y-coordinate of the center of the circle of radius r is r plus the sum of the diameters of all the circles below.

Jasmine: That would be

$$r + 2(1 + 2 + 3 + \ldots + (r - 1)).$$

Since $1 + 2 + 3 + \ldots + (r - 1) = r(r - 1)/2$, the center of the circle of radius r is located at

$$(0, r^2).$$

Emily: That worked out rather nicely!

Jasmine: If the centers are at $(0, r^2)$, then the coordinates of the rightmost points on each circle are (r, r^2).

Emily: Those points sit on the parabola $y = x^2$!

Jasmine: Neat! Unfortunately, though, these circles aren't stacked inside $y = x^2$ since the tangents to circles at their rightmost points are vertical, whereas the parabola doesn't have vertical tangents.

Emily: You're right. The parabola shaves a sliver off of each circle.

Jasmine: Say, what do you think *would* result if we stack circles into a parabola?

Emily: Are you suggesting that the parabola is the curve we're looking for?

Jasmine: Yes, I guess so!

Emily: Let's see… We can always make the vertex of the parabola the origin of coordinates, so we might as well assume that the parabola is $y = ax^2$. Let's drop a circle of radius r into it. When the circle settles in place, where will its center be?

Jasmine: By symmetry, its center will be on the y-axis, so let's say its center settles at $(0, c)$. Then the circle given by the equation $x^2 + (y - c)^2 = r^2$ is tangent to the parabola $y = ax^2$.

Emily: If (u, v) is a point of tangency where the circle touches the parabola, then the slope of the tangent to the parabola at (u, v) must be perpendicular to the slope of the radial line from the center of the circle to (u, v). The slope of the radial line is $(v - c)/u$. Using calculus, the slope of the tangent to the parabola $y = ax^2$ at (u, v) is $2au$. Since the product of the slopes of perpendicular lines is equal to -1, we have

$$-1 = 2au \cdot \frac{v-c}{u} = 2a(v-c).$$

Jasmine: Since the segment from $(0, c)$ to (u, v) is a radial line, we must also have

$$u^2 + (v - c)^2 = r^2.$$

From the first equation, we see that $v - c = -1/(2a)$. Substituting this into the second equation, we find

$$u^2 + 1/(4a^2) = r^2.$$

Therefore, $u = \sqrt{r^2 - \frac{1}{4a^2}} = \frac{\sqrt{4a^2r^2 - 1}}{2a}$.

Emily: That's strange. This formula for u returns a real number only when $4a^2r^2 - 1 \geq 0$, which means that it is only sensible when $r \geq 1/(2a)$. Surely we could drop a circle of smaller radius into the parabola! Why is this formula "rejecting" radii smaller than $1/(2a)$?

Jasmine and Emily imagine a tiny circle dropping into the parabola.

Jasmine: What must be happening is that when the radius is smaller than $1/(2a)$, the circle will drop all the way to the bottom of the parabola, touching it at the origin where $u = 0$. In our derivation, we assumed $u \neq 0$ in order to have a well-defined slope for the radial line connecting the center to (u, v). But when the circle drops all the way to the bottom, the radial line is vertical and its slope is undefined.

Emily: That makes sense. So the quantity $1/(2a)$ is the threshold radius where bigger circles get caught by the sides of the parabola before reaching the bottom and smaller circles make it all the way to the bottom.

Jasmine: Right.

Emily: We have yet to find the coordinates of the center of the circle. If $r \leq 1/(2a)$, then the circle is resting on the bottom of the parabola, so $c = r$. Otherwise, our first formula shows that $c = v + 1/(2a)$; since (u, v) is a point on the parabola $y = ax^2$, we know that

$$v = au^2 = a(r^2 - 1/(4a^2)) = ar^2 - 1/(4a).$$

Therefore, $c = ar^2 - 1/(4a) + 1/(2a) = ar^2 + 1/(4a)$. That means the circle of radius r, when thrown into the parabola $y = ax^2$, will come to rest with its center located at $(0, ar^2 + 1/(4a))$.

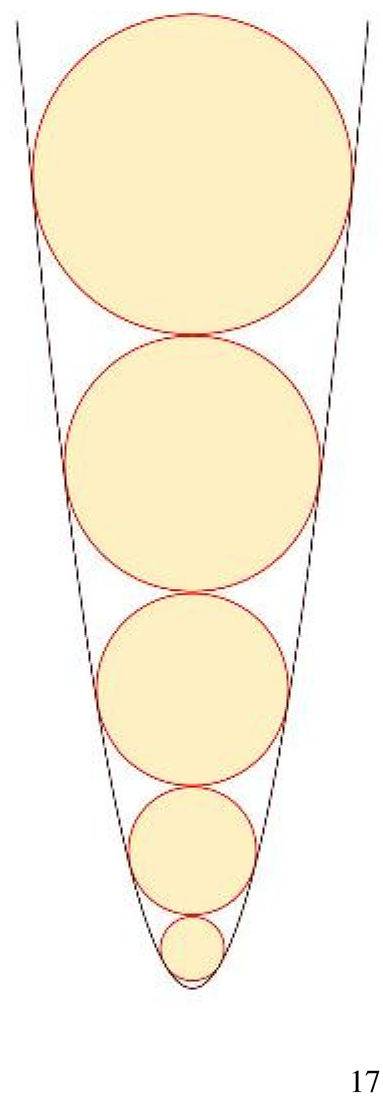
Jasmine: $(0, ar^2 + 1/(4a))$… Hmm…

Emily: Did I make a mistake?

Jasmine: Oh no, I don't think so. It's just that this formula looks a lot like…

Emily: What's up?

Jasmine: Hey! Look what happens when you substitute $a = 1$ into that formula. When $a = 1$, the circle of radius r, provided $r \geq 1/2$, will settle into place with its center at $(0, r^2 + 1/4)$. That's almost like the locations of the centers in our stack of circles whose radii were the positive integers. In that case, we found the centers located at $(0, r^2)$, where r was an integer. The only difference is that the centers are all *shifted by 1/4*!



Emily: Nice observation! You're saying that our stack of circles with radii the positive integers *does* fit snugly into the parabola $y = x^2$.

Jasmine: Exactly! If we shift the parabola down by 1/4, our analysis shows that the circle of radius r will come to rest when its center is at $(0, r^2)$, which is exactly where the circles in our stack of circles (of radii the positive integers) *are* located!

Emily: It's amazing that the same shift adjusts things so that the parabola will suddenly become tangent to every single circle in the stack.

Jasmine: That is stunning!

Emily: But we still need to check if circles stacked in parabolas in general will have radii that are in arithmetic progression.

Jasmine: You're right, let's check…

Emily: If stacking circles into a parabola results in a sequence of circles whose radii are in arithmetic progression, then the difference in radii between successive circles should be a constant independent of the radii of the circles.

Jasmine: I agree. In other words, if we fix some radius r – for now, let's take $r \geq 1/(2a)$ – toss that circle into the parabola, and determine what circle would fall on top of that, we could then compute the difference between their radii; hopefully, this difference will turn out to be independent of r. If it *is* independent, then the resulting stack of circles would have radii in arithmetic progression.

Emily: We just computed that the first circle, which has radius $r \geq 1/(2a)$, will settle into the parabola $y = ax^2$ when its center is located at $(0, ar^2 + 1/(4a))$. But what should the radius of the next circle be?

Jasmine: If we call the radius of the next circle R, then the distance between their centers should equal $r + R$ since R is chosen so that the next circle will just touch the top of the first circle. That is,

$$r + R = aR^2 + 1/(4a) - (ar^2 + 1/(4a)).$$

This simplifies to

$$r + R = a(R^2 - r^2) = a(R + r)(R - r).$$

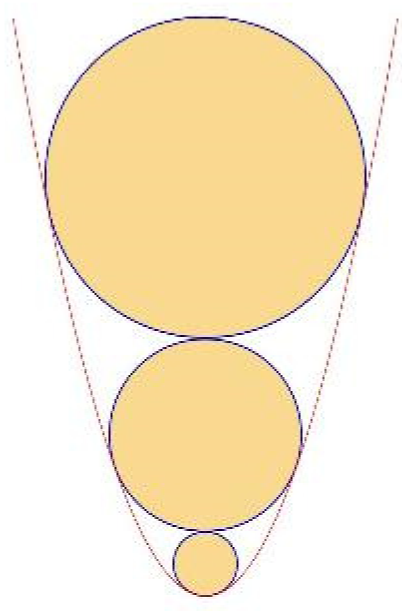
If we divide both sides by $a(R + r)$, we get

$$1/a = R - r.$$

Emily: How about that! The difference in radii *is* independent of r, so we will get a stack of circles illustrating an arithmetic progression.

Jasmine: Yes, and the common difference turns out to be $1/a$.

Emily: As a gets bigger and bigger, the parabola narrows and the radii increase more slowly. But if a is a small, positive number, the parabola will be wide like a bowl, and the radii will increase more quickly.



Jasmine: Yes, but one difference between angles and parabolas is that we could illustrate *any* positive geometric sequence with common ratio greater than 1 as the radii of stacked circles inside some angle, but the first term of one of our arithmetic progressions cannot be smaller than half of the common difference.

Emily: That's true. But that still leaves plenty of arithmetic progressions that *can* be illustrated as the radii of circles stacked into a parabola.

Jasmine: Now that we can illustrate arithmetic and geometric sequences with stacked circles, do you think we can find a shape that will give us harmonic sequences?

To be continued...

Summer Fun!

The best way to learn math is to do math. Here are the 2018 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems on your own.

Some problems are quite a challenge and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can feel frustrating to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

The Step Function

by Whitney Souery

The **unit step function**, which is also known as the **Heaviside step function**, is a useful function with many applications in math and engineering. It is a function that maps real numbers to real numbers and which we will denote by $H(x)$. It is defined as follows:

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1/2, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

1. Draw a graph of $H(x)$.

We will also use the closely related step function, $u(x)$, which is defined as follows:

$$u(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}.$$

Note that $H(x)$ and $u(x)$ differ only at $x = 0$.

2. Draw graphs of the following functions:

A. $H(x + 2)$ B. $3H(-x)$ C. $H(1 - x)$ D. $H(3x - 5)$

E. $-2u(x - 5)$ F. $u(x) - H(x)$ G. $4 - 4H(x)$ H. $H(x) + H(-x)$

3. Show that $2H(x) - 1$ is the sign function, i.e. the function that returns +1 if x is positive, -1 if x is negative, and 0 if x is 0.

In Problems 2 and 3, you explored how our step functions transform under shifts, reflections, and dilations or combinations thereof. Now let's use the step functions to build up other functions.

4. If A is a subset of the real numbers, the **characteristic function** of A is the function that returns 1 if x is in A and 0 otherwise. Show that $u(x) - u(x - 1)$ is the characteristic function of the half-closed half-open unit interval [0, 1) (i.e., the set of real numbers x such that $0 \leq x < 1$).

5. In a similar vein as Problem 4, use $u(x)$ to build the characteristic function of the closed unit interval [0, 1] (i.e., the set of real numbers x such that $0 \leq x \leq 1$).

6. In a similar vein as Problems 4 and 5, use $u(x)$ to build the characteristic function of the open unit interval (0, 1) (i.e., the set of real numbers x such that $0 < x < 1$).

Summer Fun!

7. Show that $\lfloor x \rfloor = u(x) - 1 + \sum_{k=1}^{\infty}(u(x+k) + u(x-k) - 1)$, where $\lfloor x \rfloor$ is the "floor" function, i.e. $\lfloor x \rfloor$ is the largest integer less than or equal to x.

8. Show that $|x| = x(u(x) - u(-x))$.

Here are miscellaneous problems involving $H(x)$ and $u(x)$.

9. Show that $u(x) = 2H(u(x)) - 1$.

10. Show that $H(x) = u(x)(1 - u(-x)/2)$.

11. Show that $u(x) = H(x)(3 - 2H(x))$.

12. Show that $u(x) = \lim_{n\to\infty} H(x)^n$.

13. Fix a real number c. For what subset of the real numbers is $2(u(x - c) - H(x - c))$ the characteristic function?

14. Fix a real number c. For what subset of the real numbers is $4H(x - c)H(c - x)$ the characteristic function?

15. Let a and b be real numbers with $a < b$. For what subset of the real numbers is $u(x - a)u(b - x)$ the characteristic function?

16. Show that $H(x) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(nx)\right)$.

17. Let C be the smallest set of functions that map the real numbers to itself with the following properties:

i. The set C contains $u(x - a)$ for all real numbers a.
ii. If $f(x)$ is in C, then so is $af(x)$ for all real numbers a.
iii. If $f(x)$ and $g(x)$ are in C, then so is $f(x) + g(x)$ and $f(x)g(x)$.

Call a function $f(x)$ that maps real numbers to itself a **mesa vista** function if there exists two finite sequences of real numbers $v_1, v_2, v_3, \ldots, v_n$ and $x_1 < x_2 < x_3 < \ldots < x_n$ such that $f(x) = 0$ if $x < x_1$, $f(x) = v_k$ if $x_k \le x < x_{k+1}$, for $0 < k < n$, and $f(x) = v_n$ if $x \ge x_n$. Let D be the set of mesa vista functions.

Show that $C = D$.

Let’s Throw a BBQ

by Vicky Xu | edited by Amanda Galtman

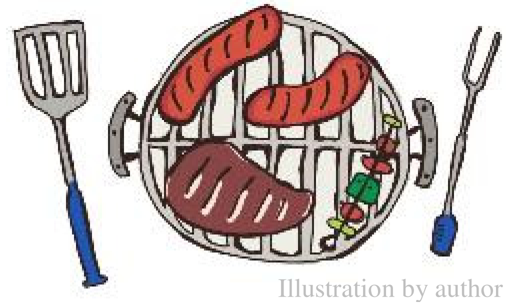


Illustration by author

It’s finally SUMMER—time for some good old backyard barbeque. You’ve invited family and friends, and can’t wait to show off your cooking abilities.

1. You invited 17 people over the age of 50, 15 people who work as teachers, and 20 people who love to play bridge. Exactly 8 people over the age of 50 work as teachers, and exactly 13 people over the age of 50 love to play bridge. Exactly 12 teachers play bridge, and exactly 5 people are over 50, work as teachers, and play bridge. What is the *least* number of people you could have invited to your BBQ?

2. It turns out that you invited 60 people, but you expect only 60% to attend. How many people do you expect to attend your BBQ?

3. Your grandma wants to bring her two dogs and two cats. Unfortunately, she has space in her car for only two of them. Since she loves her pets equally, she will choose the two randomly. For her first choice, she will pick one of her four pets, each with equal probability. For her second choice, she will pick one of the three remaining pets, each with equal probability.

A. What is the probability that she will bring both of her cats?
B. What is the probability that she will bring a cat?
C. What is the probability that she will bring a dog and a cat?
D. What is the probability that she will bring a cat, given that her first choice turned out to be a dog?

4. After solving each problem in this Summer Fun problem set, try to generalize it. For example, one way to generalize Problem 3 is to suppose your grandma has D dogs and C cats. She picks one of them, where all are equally likely to be chosen. Then, she picks a second, with the remaining pets equally likely to be chosen. What are your new answers to parts A, B, C, and D of problem 3 (changing part A to: What is the probability that she will bring two cats?)? If you substitute 2 for D and C in your formulas, do you recover the answers you got for Problem 3?

5. Aunt Sarah, Uncle Tom, and Uncle John each decide independently whether or not they want to attend, but they will drive together if a majority of them vote to come. Aunt Sarah and Uncle John each have probability p of wanting to attend, whereas Uncle John has probability 1/2 of wanting to attend. On the other hand, your friend Emily is biking by herself and will choose to attend with probability p. Will Emily be more likely to come than your aunt and uncles?

6. Uncle John is never late. Aunt Sarah is late with probability 1/10, Uncle Tom is late with probability 3/20, and the probability that both Aunt Sarah and Uncle Tom are late is 1/20. Since Aunt Sarah and Uncle Tom live together, their probabilities of tardiness are not independent. Assuming that your aunt and uncles are attending, what is the probability that their car will arrive late?

Summer Fun!

7. You start grilling as people start to arrive. You have prepared 20 ears of corn when 10 guests have arrived. In each of the following scenarios, how many ways can you distribute the ears?

A. The ears of corn are identical, so you can't distinguish between them.
B. The ears of corn are identical, and each person must get at least one.
C. The ears of corn are not the same.
D. The ears of corn are not the same, and each person must get at least one.

8. More people came to your BBQ than you had expected! A total of 52 people attend. You decide to pick six people to help you cook. You take a standard deck of 52 playing cards and give each guest a different card. From another deck, you pick six cards, with all possible sets of six cards equally likely. What is the probability of the following outcomes?

A. You pick a 2, 3, 4, 5, 6, and 7 of the same suit.
B. You pick six cards of the same suit.
C. You pick six cards in consecutive order (like a straight in poker).
D. You pick at least one card of each suit.
E. You pick three different pairs.

9. Your friend Ruth was about to join the line for food, but the line was so long that she decided to go for a walk and return later. After some time, she came upon a park that happened to be 110 yards from the BBQ. At the park, there was a grandfather clock, which informed her that she had been walking for 25 minutes. Ruth decided she'd better get back to the BBQ or she might not get any food. Unfortunately, she couldn't remember how to get back. She began wandering randomly, hoping to find the BBQ. Every second, she'd end up either 1 yard closer to or 1 yard further from the BBQ, with equal probability. To get food, she'd have to make it within 10 yards of the BBQ within 5 minutes. What is the probability that Ruth makes it back for some good old-fashioned BBQ?

10. At your particular BBQ, the probability that a patty gets burned between time A and time B, with $A < B$, depends only on $B - A$. Also, the probability that a patty gets burned in one time interval is independent of the probability that a patty gets burned in a disjoint time interval. If the probability that a patty gets burned in a 1 hour time interval is 70%, what is the probability that a patty will get burned in a time interval of 10 minutes?

11. You handed a form to each guest, asking them to indicate what drink they want and whether they want beef patties or veggie patties. Assume a person is equally likely to prefer veggie or beef.

A number of people have submitted forms, and two forms indicate a veggie preference. On the other submitted forms, beef was chosen. Then, another guest submits a form, but it gives only a drink preference. Of the people who have given you a form (incomplete or not), you pick one uniformly at random and ask that person to give you a patty preference; the person indicates veggie.

What is the probability that the person who forgot to give you her patty preference wants a veggie burger?

Markoff Triples

by Matthew de Courcy-Ireland

The Markoff equation is

$$x^2 + y^2 + z^2 = 3xyz,$$

to be solved in positive integers x, y, and z. There is no general method for finding all the integer solutions of a cubic equation in three variables like this. However, this equation has a very special structure that can help us. Although it is a cubic equation because of the term xyz, it is only a quadratic equation when we consider one variable at a time.

1. Show that the two solutions of a quadratic equation $x^2 + bx + c = 0$ add up to $-b$, the negative of the coefficient of x.

(Problem 1 is a special case of Vieta's formulas, which relate the coefficients of a polynomial to its roots.)

2. Using Problem 1, show that if (x, y, z) is a solution to the Markoff equation, then so is

$$(3yz - x, y, z).$$

3. Find "moves" for the second and third coordinates similar to the "move" in the first coordinate described in Problem 2.

The "moves" in Problems 2 and 3 are called **Markoff moves**.

4. Observe that (1, 1, 1) is a solution of the Markoff equation. By repeatedly applying the moves above, we can make many more solutions. Draw a picture connecting each solution to the other ones you can obtain using Markoff moves.

5. Markoff proved that this procedure gives *all* solutions to the equation. Can you show this by using the moves above "in reverse" to get from any solution back to our basepoint (1, 1, 1)?

6. Do you recognize any of the numbers in the tree? Recall that the Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13, etc., are obtained by adding the previous two numbers at each stage.

References

Despite its very special properties, or maybe because of them, the Markoff equation occurs in many different contexts. For more information, we recommend the book *Markov's Theorem and 100 Years of the Uniqueness Conjecture*, by Martin Aigner.

Summer Fun!

Generating Functions

by Laura Pierson | edited by Amanda Galtman

Number sequences abound. Think of the counting numbers, 1, 2, 3, ..., or the perfect squares, 1, 4, 9, ..., or the Fibonacci numbers, 1, 1, 2, 3, 5, 8, ..., for example. We can package any sequence of numbers a_0, a_1, a_2, . . . into a **generating function**:

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n \ .$$

In this problem set, we'll explore how such packaging lets us use tools from algebra to solve counting problems. A generating function is an example of a **formal power series**. The word "formal" indicates that we are not thinking of $F(x)$ as a function of x. We don't care for what values of x the series converges or even whether the series converges. What we're interested in is organizing the numbers in the sequence as coefficients in a power series.

We define the sum of two formal power series by the formula

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n + b_n) x^n \ ,$$

and the **product** of two formal power series by the formula

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_m b_{n-m}\right) x^n \ .$$

Let $A(x)$, $B(x)$, and $C(x)$ be generating functions. If $A(x) + B(x) = C(x)$, we can equivalently write $A(x) = C(x) - B(x)$. Similarly, if $A(x)B(x) = C(x)$, we can also write $A(x) = C(x)/B(x)$.

1. Show that the definition of addition and multiplication of formal power series is compatible with interpreting formal power series as polynomials of infinite degree in the variable x. That is, when you add and multiply polynomials together, how do the coefficients in the sum and the product relate to the coefficients in the addends and factors, respectively?

By the way, notice that if all but finitely many of the coefficients are zero, then the power series becomes a polynomial.

2. Show that addition and multiplication of formal power series are commutative and associative. Show that multiplication distributes over addition.

Now we'd like to apply the algebra of generating functions to help us solve counting problems!

3. You have infinitely many apples, all identical. Thus, for each n, there is effectively only one way to choose n apples. So if a_n is the number of ways we can choose n apples, then $a_n = 1$ for all n, and the associated generating function is $1 + x + x^2 + x^3 + \ldots$. (It's convenient to think of there being one way to pick zero apples.) Show that this generating function is equal to $1/(1 - x)$. If apples come only in bags of five, show that the generating function whose coefficients are the number of ways to choose n apples becomes $1/(1 - x^5)$. How does the generating function change if apples come only in bags of N?

4. Now, you have infinitely many apples and infinitely many oranges. All fruits of the same kind are identical. Let a_n be the number of ways to choose n fruit. (For instance, $a_2 = 3$.) Find a formula for a_n. Let $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ be the associated generating function. Show that $F(x) = (1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots) = 1/(1 - x)^2$. Notice that the factors can be thought of as the generating functions for the number of ways to pick n fruit if there were only apples or if there were only oranges. Explain why it makes sense that the generating function for picking n apples and/or oranges is such a product.

5. Now suppose you have infinitely many apples, oranges, and bananas. Can you see that the generating function associated with the number of ways to choose n fruit is $1/(1 - x)^3$? Using this formula for the generating function, can you find a formula for its coefficients?

6. Suppose there are unlimited numbers of each of T different types of fruit. Find an expression for the generating function associated with the number of ways n fruit can be chosen.

7. Now imagine that there are still infinitely many apples and oranges, but only five bananas (still identical!). Also, apples are sold only in pairs. Express the generating function associated with the number of ways of choosing n fruit under these circumstances as a product of generating functions, where each factor in the product is a generating function with a natural combinatorial interpretation pertaining to each fruit separately. Expand the product to find the first few terms.

8. Shopping for fruit is getting more and more complicated! Now there are infinitely many apples and bananas, but apples are sold only in pairs and bananas are sold only in bunches of five. Also, there are only four oranges and one pear. Use generating functions to find a formula for the number of ways you can choose n fruit.

9. Let A and B be collections of objects. Let $A(x)$ and $B(x)$ be the generating functions associated with the number of ways you can choose n objects from collections A and B, respectively. Show that the generating function associated with the number of ways you can choose n objects from the combination of collections A and B is equal to $A(x)B(x)$.

10. Alice rolls two standard six-sided dice and Beth rolls two funny dice: one has the numbers 1, 2, 2, 3, 3, and 4 on its six faces, and the other has the numbers 1, 3, 4, 5, 6, and 8. Write the generating function associated with the number of ways to roll a sum of n for each player's pair of dice. They play a game where they both roll their two dice and whoever has the higher sum wins. Show that they are both equally likely to get each sum. (You should be able to do this by factoring the two generating functions.) Beth's dice are known as **Sicherman** dice.

11. Write the first few coefficients of the generating function $1/(1 - x - x^2)$. What sequence of numbers do you think this generating function is associated with? Can you prove it? Can you use the expression $1/(1 - x - x^2)$ to find a formula for the nth term of the sequence?

12. Express the generating function for the perfect squares as a rational function (i.e., a ratio of polynomials).

13. A **partition** of a positive integer n is a way to write n as a sum of positive integers (called the **parts**), where order does not matter. For instance, the number 4 has five partitions:

$$4,\ 3 + 1,\ 2 + 2,\ 2 + 1 + 1, \text{ and } 1 + 1 + 1 + 1.$$

Let $\pi(n)$ be the number of partitions of n and let $P(x)$ be the associated generating function.

Write the first few terms of $P(x)$ and verify the following table.

$\boldsymbol{n}$	1	2	3	4	5	6	7	8	9	10
$\boldsymbol{\pi(n)}$	1	2	3	5	7	11	15	22	30	42

14. Show that $P(x)$ is equal to

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)\cdots.$$

15. Let $O(x)$ be the generating function for the number of partitions of n whose parts are all odd. Write the first few terms of $O(x)$.

16. Let $N(x)$ be the generating function for the number of partitions of n with no repeating part. That is, the coefficient of x^n in $N(x)$ is the number of ways to write n as a sum of distinct positive integers, where order does not matter. Write the first few terms of $N(x)$.

17. Use the generating functions $O(x)$ and $N(x)$ from Problems 15 and 16 to show that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

18. Try to do Problem 17 without using generating functions. To do so, try to find a one-to-one correspondence by turning a partition of n into odd parts into a partition of n into distinct parts, and vice versa. (Hint: Think about what you can do to get rid of even parts or repeated parts in a partition.)

Notes from the Club

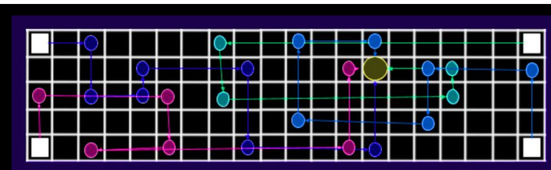
These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 22 - Meet 11 May 3, 2018	Mentors:	Rachel Burns, Anna Ellison, Jacqueline Garrahan, Amber Guo, Elise McCormack, Kate Pearce, Samantha Russman, Christine Soh, Shohini Stout, Jane Wang

To underscore the universality of mathematics, some members read a math lesson and answered associated math problems all written in Romanian. Despite the fact that none knew a word of Romanian, they were still able to succeed.

Session 22 - Meet 11 May 10, 2018	Mentors:	Rachel Burns, Alexandra Fehnel, Jacqueline Garrahan, Amber Guo, Elise McCormack, Kate Pearce, Shohini Stout, Jane Wang, Josephine Yu

For the second time at Girls' Angle, our traditional end-of-session Math Collaboration was created by mentors, namely, Jacqueline Garrahan and Elise McCormack. The two dreamed up a Star Wars themed spectacular, complete with scrolling text mission delivery! The large table in our meeting room was transformed into a star chart. The girls had to navigate four starships to a centrally located planet where they could join forces and defeat the evil empire.



The star chart designed by Jacqueline and Elise.

Here's a sampling of the problems from the space opera. Can you solve them?

Let $f(x) = 1 + x^2/4$. Let $h(x) = f(f(f(f(f(x)))))$. What is $h'(2)$, where $h'(x)$ is the derivative of $h(x)$ with respect to x?

What is the greatest integer n that satisfies $3^n < 2^{n+3}$?

If you trim off a thin portion of a Möbius strip that has n half-twists, cutting carefully around the edge, the resulting trimmed portion will have 4 full twists. What is n?

A fair 6-sided die has three faces marked 1, two faces marked 2, and one face marked 3. If you take two of these dice and roll them, what is the probability of getting a sum of 4?

A triangle has side lengths 26.5, 50, and 70.5 units. What is the length of the altitude whose foot is on its longest side?

Reduce Pascal's triangle modulo 2 and interpret each row as a binary number. You get a sequence that begins 1, 3, 5, 15, …. What is the smallest prime factor of the 21st of these numbers?

Calendar

Session 22: (all dates in 2018)

February	1	Start of the twenty-second session!
	8	
	15	
	22	No meet
March	1	Melody Chan, Brown University
	8	
	15	
	22	
	29	No meet
April	5	
	12	Anna Frebel, MIT
	19	No meet
	26	
May	3	
	10	

Session 23: (all dates in 2018)

September	13	Start of the twenty-third session!
	20	
	27	
October	4	
	11	
	18	
	25	
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors strive to get members to do math through inspiration and not assignment. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where are Girls' Angle meets held? Girls' Angle meets take place near Kendall Square in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ______________________________________