Girls' Angle Bulletin

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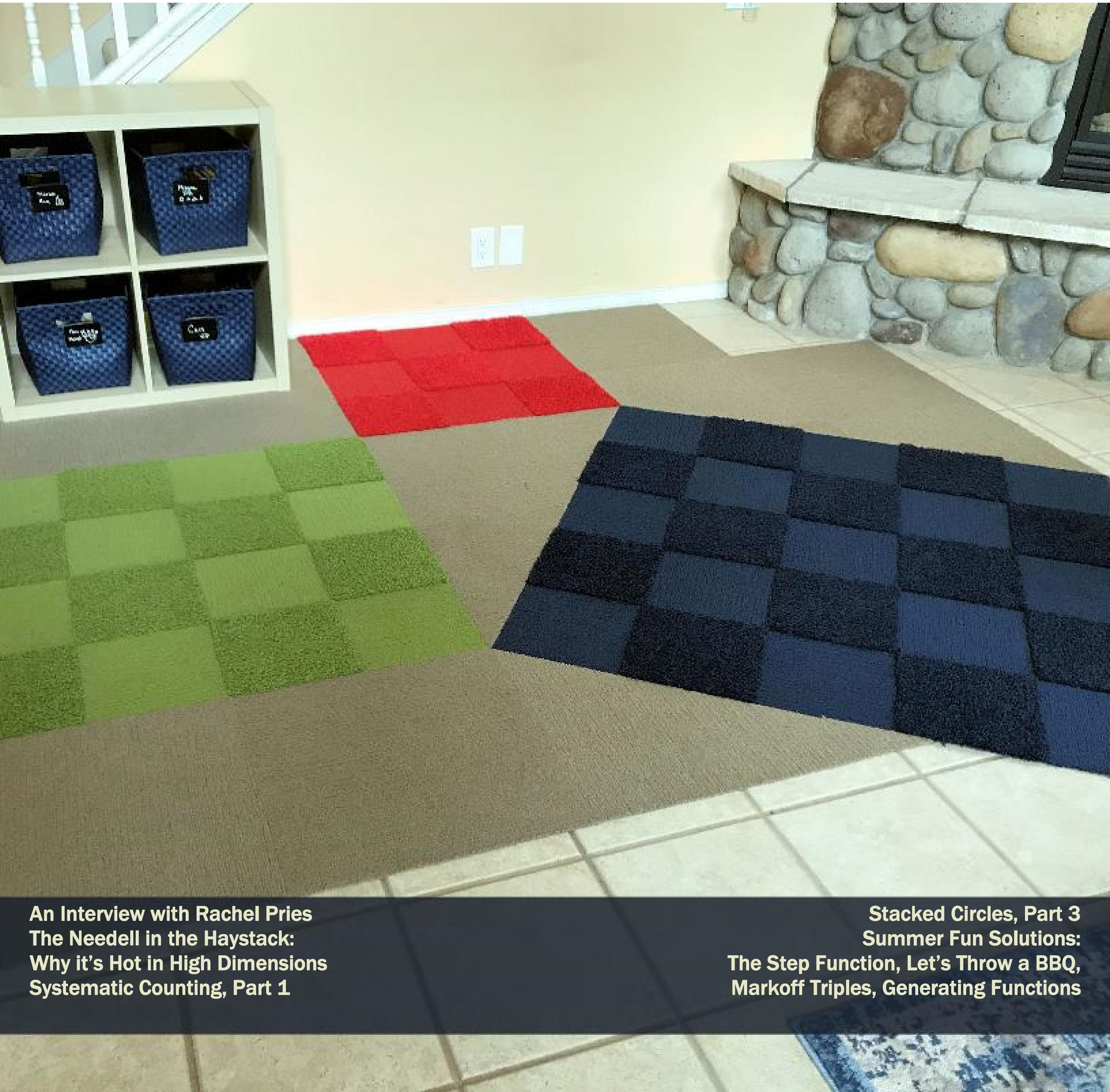
To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

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- Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Katy Cook nee Bold, our first Math In Your World columnist (see Vol. 2, No. 4 through Vol. 3, No. 5), made this amazing rug designed by her and Sara Eizen. What is its mathematical significance? Have you made a mathematical carpet? Would you like to share it? Send us a photo!

An Interview with Rachel Pries

…I work out a lot of examples and then look for what kinds of patterns are emerging. Sometimes it's helpful to try to explain the problem to colleagues – even if they don't understand it…

Rachel Pries is a Professor of Mathematics at Colorado State University. She is also a 2018 Fellow of the American Mathematical Society. She received her doctoral degree in mathematics from the University of Pennsylvania in 2000 under the supervision of David Harbater.

Ken: I usually begin interviews diving right into math, but I was looking over your personal website where you've put a tantalizing list of interests including treehouses, watercolors, social action theater, and … sunlight on brick row houses. Well, I've never heard "sunlight on brick row houses" explicitly mentioned as a personal interest, and, I admit that to make sure this wasn't some common interest that I'd missed out on, I googled the phrase in quotes, and you might be interested to know that that phrase uniquely identifies your personal website (as of this writing)! What is it about sunlight on brick row houses that interests you?

Rachel: During grad school, I lived in center city in Philadelphia. At the end of each work day, I would walk home just as the sun was low in the sky. The color of the brick row houses becomes warmer and deeper when the light hits it that way. Bricks look completely uniform from far away, but when you look closely, each brick has its own unique pattern.

There are no brick row houses where I live now, but I feel the same way about lichen on rocks. No one pays attention to lichen, but when you look closely, it grows in such a variety of colors and patterns, and gives clues about the moisture and orientation of the rock.

Ken: Intriguing! I feel like looking at lichen now. You also mention that an early mathematical highlight in your life was graphing $z = \sin(x + y)$ when you were a student at Cambridge Rindge and Latin (a school which is just down the street from some of our readers!). Could you please recreate the moment for us? What made it a highlight?

Rachel: It was a highlight for me because I thought up the question for myself. It took a long time to plot enough points and then the pattern of the wave slowly emerged in 3D. At that time, I hadn't seen any pictures of functions involving more than one variable.

Nowadays, with graphing software, it is easy to produce lots of these images. They look sharp and fancy. I recently found old school work from my parents' attic and I was struck by how rudimentary the hand-drawn graphs and tables look. A high school student today can investigate graphs much more quickly and produce documents that look professional. But I felt a deep connection to that graph, in part because it took so long to make.

Ken: What got you first interest in mathematics, and when did you start to think about mathematics as a potential career?

Rachel: At a high school summer program called MASP, I took a class about number theory and loved it. I had never imagined that you could learn so many different things about prime numbers. I didn't know it at the time, but number theory turned out to be the focus of my career in math. I don't know whether that's because I had a natural affinity for it or because of having an early exposure to it. There are a lot of beautiful

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Rachel Pries and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
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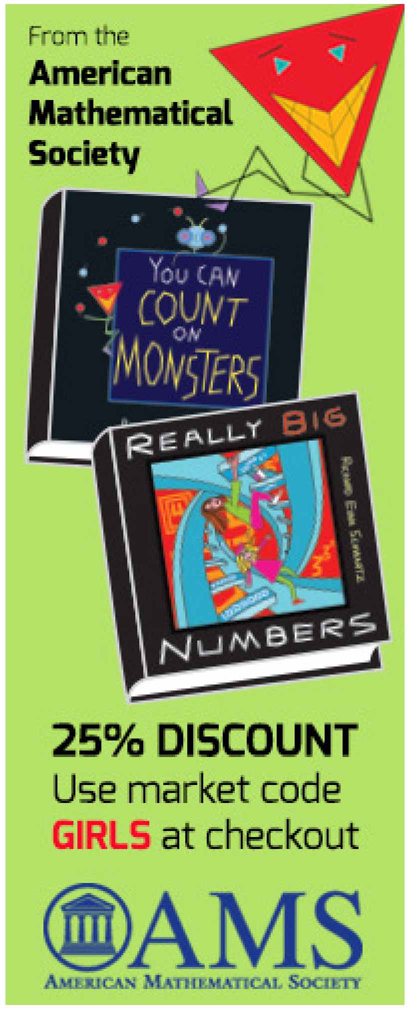

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The Needell in the Haystack

Why it's Hot in High Dimensions[1]

by Deanna Needell | edited by Jennifer Silva

For the last few issues, this column has been about various mathematical problems that have applications in several areas of data science and engineering. In most of these settings, the signal or data being studied has been extremely large-scale. The so-called "big data" phenomenon is in full swing these days, thus it is likely that you have already heard this phrase. Since the size of data is only continuing to grow, so is the need for sophisticated mathematical tools to acquire, store, and analyze these massive amounts of data.

Why is modern data so large-scale, and why is this a challenging and important problem to tackle? Technology has progressed very quickly in recent decades, thus the ability to efficiently and easily acquire large amounts of data has spread. Walk outside the airport in London and you are likely being caught on CCTV video surveillance. London, like some other cities, has a vast network of video cameras that capture and store live feed from public areas all over the city. Police officers in many cities in the Unites States must now wear body cameras that also capture live feed while they are active on duty. In fact, even robots on Mars send back constant streams of data to Earth. Besides this massive amount of streaming video data, environmental sensors across the world are consistently monitoring geophysical activity, web servers track internet usage, patient sensors and devices record medical data, and the list goes on and on. Essentially, if you have a favorite interest or activity, there is likely a large amount of data involved in one way or another. You would think that with this abundance of data would come a plethora of new information. In actuality, however, the opposite is often true. The data is so large-scale that it clogs modern analytical systems and brings new technological challenges along with it. This has sparked a critical need for mathematical innovations. Since we are focused on large-scale data, the mathematics involves studying objects in high dimensions. Here, we will touch on some of these high-dimensional ideas and reveal some beautiful surprises along the way.

Building geometric objects

In order to talk about geometric objects in arbitrary dimension, we will build these objects using various notions of distance. For example, suppose you draw two points a and b on a piece of gridded paper and want to know the distance between them. You may be inclined to draw a line between them and then a right triangle, so that the line between the two points is the hypotenuse and the legs are parallel to the grid lines; the grid lines may then be used as a ruler. The distance between the two points could be computed using the Pythagorean Theorem. If the coordinates of a and b are (a_1, a_2) and (b_1, b_2), respectively, then the distance between a and b is given by

$$\sqrt{(a_1-b_1)^2+(a_2-b_2)^2}\ .$$

This formula for distance is called the **Euclidean** distance or the $\mathbf{L^2}$ distance. If we think of a and b as position vectors, we can define the $\mathbf{L^2}$ **norm** of a by the formula

[1] This content supported in part by a grant from MathWorks.

$$||a||_2 = \sqrt{a_1^2 + a_2^2}\ .$$

The L^2 distance between a and b can then be expressed as $\| a - b \|_2$. The name "L^2" is not arbitrary; it corresponds to the powers of 2 (and 1/2) in this norm's definition. The L^2 distance can be generalized to arbitrary dimension. Let $a = (a_1, a_2, a_3, \ldots, a_n)$ and $b = (b_1, b_2, b_3, \ldots, b_n)$ be points in $\mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers. We define the L^2 norm of a by the formula

$$\|a\|_2 = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}\ .$$

We then define the L^2 distance between a and b by the formula $\|a - b\|_2$.

Using this L^2 distance, we can define a **sphere** in arbitrary dimension as the set of all points whose L^2 distance is a fixed distance from a given point, which is the center of the sphere. Thus, the sphere with radius 1 (i.e., the unit sphere) centered at the origin is given by

$$\{\ x \in \mathbb{R}^n \mid \|x\|_2 = 1\ \},$$

and we denote it by S_2^n.[2] Because we used the L^2 distance to define this sphere, we also call it an L^2-sphere. If we want to include the interior of the sphere, we refer to that as the L^2-ball. For example, the unit ball centered at the origin consists of the points $\{\ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\ \}$. Please check that S_2^2 corresponds to the unit circle centered at the origin of the coordinate plane and that S_2^3 corresponds to the usual unit sphere centered at the origin (which could be used to model the surface of the earth, for instance). It may be hard for us to visualize spheres in dimensions higher than 3, but they are well-defined mathematical objects.

We can use other ways of measuring distance to define new mathematical objects in arbitrary dimension. Consider taking a drive from the corner of 60th Street and 2nd Avenue to 82nd Street and 5th Avenue in New York City. You would no longer use L^2 distance since you are confined to taking streets along the grid. Instead, you would add the vertical and horizontal distances to measure the total distance of travel. This yields the so-called **taxicab distance** or **L^1 distance** between two points $a = (a_1, a_2, a_3, \ldots, a_n)$ and $b = (b_1, b_2, b_3, \ldots, b_n)$, which is defined in dimension n by the formula $\|a - b\|_1$, where $\|a\|_1 = |a_1| + |a_2| + |a_3| + \ldots + |a_n|$. Note the hidden powers of 1 in this definition, leading to the name L^1. We can now consider the L^1-unit ball centered at the origin: $B_1^n \equiv \{\ x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\ \}$. As an exercise, draw the shape of the L^1-ball in dimensions 2 and 3. (Spoiler alert: In dimension 2, it's a diamond, and in dimension 3, it's an octahedron.)

One can use this idea to define an L^p-ball for any positive real number p by defining distance in terms of the norm $\|(a_1, a_2, a_3, \ldots, a_n)\|_p = (a_1^p + a_2^p + a_3^p + \ldots + a_n^p)^{1/p}$. (So the L^p distance between $a = (a_1, a_2, a_3, \ldots, a_n)$ and $b = (b_1, b_2, b_3, \ldots, b_n)$ is given by $\|a - b\|_p$.) This gives us an infinite supply of fun exercises to draw these shapes in low dimensions!

[2] We point out that this is not the usual notation for the sphere; typically, the superscript corresponds to the dimension of the *surface* rather than the dimension of the ambient space.

The last important distance for our purposes is one that is defined *on* a sphere. For the moment, pretend that the earth is spherical and that you want to fly from one point on the globe to another using the quickest route. Since you cannot drill through the planet, a straight-line path is not feasible. Instead, you would travel along the shortest arc, which gives rise to the notion of the **geodesic** distance between two points on the sphere. A **great circle** on a sphere is a circle defined by the intersection of a plane that contains the center of the sphere and the surface of the sphere, such as lines of longitude or the equator. (The equator is the only line of latitude that is also a great circle. Planes that contain other lines of latitude do not pass through the center of the sphere.) The geodesic distance between two points on the surface of the sphere is equal to the shortest distance along a great circle that contains those two points. As before, we may consider the set of points on the sphere whose geodesic distance to the North Pole is less than or equal to r. On the two-dimensional sphere, this set of points looks like a **spherical cap**, which is how we will refer to this set: the spherical cap of radius r centered at the North Pole. Note that this definition also extends naturally to arbitrary dimension n, just like the previous geometric objects built from distances.

Geodesic extensions

The last geometric notion we will use here is that of the **geodesic extension** of a set on the sphere. Given a set T on the sphere S_2^n and a small positive number ε, we define its geodesic extension as the set of points that are (geodesic) distance ε or closer to some point of T. Since every point of T is zero distance away from itself, T is contained in its extension. In effect, the geodesic extension of T extends the boundaries of T to include more points. For example, if T is a spherical cap of radius r, then its geodesic extension will be a spherical cap of radius $r + \varepsilon$.

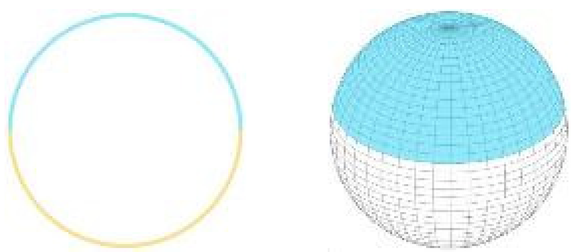

Figure 1. Left: A spherical cap in two dimensions (light blue). Right: A spherical cap in three dimensions (light blue).

One way of studying geometric objects is to compute certain quantities about them. For example, you may wish to gain understanding of the sphere (in three dimensions) by computing its surface area or volume. Here, we will be interested in studying *proportions* of spheres. For example, the light blue spherical cap in the left sphere of Figure 1 is a semicircle, thus we would say that its "measure" is 1/2, since the semicircle is half of the circle. As another example, the measure of land on Earth is roughly 0.29 since approximately 29% of the earth's surface is land.

Let's examine how the measure of a hemispherical cap changes when it is extended by ε, starting with the unit circle S_2^2. Consider shading the semicircle as in the left image of Figure 1. As mentioned, this cap has measure 1/2, since it is proportionately half of the circle. Now consider

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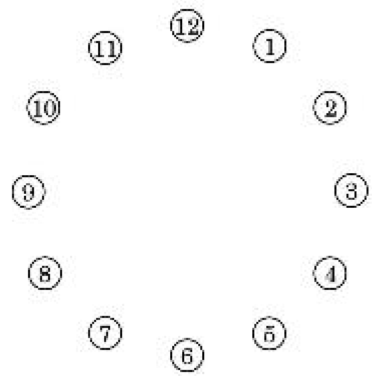
Systematic Counting, Part 1

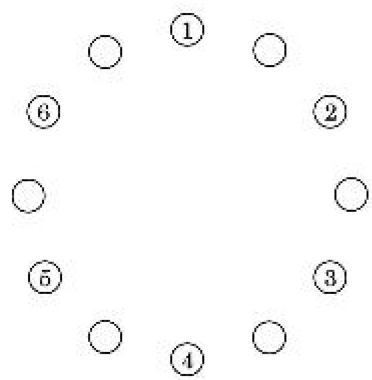
by Addie Summer | edited by Jennifer Silva

One thing I love about math is that you can do it almost anywhere, any time.

Last week, I found myself waiting for a bus with nothing to do. To pass the time, I drew a clock face:

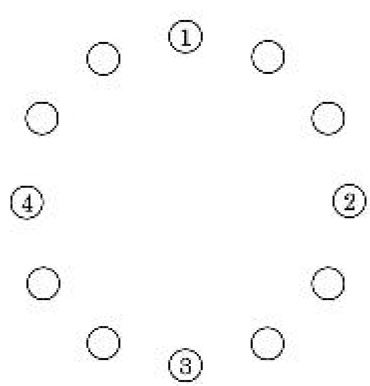


I wondered, "What would happen if I tried to count the 12 circles around the clock face by skipping 2 over each time instead of the standard 1 over," like this:

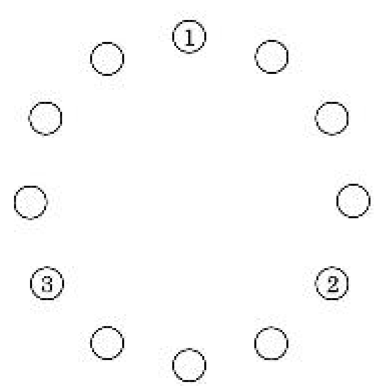


I discovered that you'd only count half of the circles before returning to the starting point.

The bus hadn't yet appeared, so I decided to see what would happen if I did the same thing, only skipping 3 over, 4 over, 5 over, and so on.



Skipping 3 over



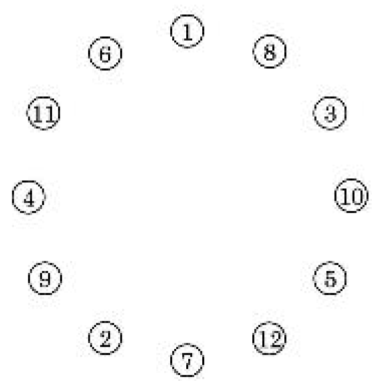
Skipping 4 over

I noticed that skipping 12 over would bring me right back to the starting point, so I'd only count one of the circles that way. Also, skipping 13 over is essentially the same thing as skipping 1 over. I can explain this by thinking of 13 as 12 + 1, meaning that skipping over 13 each time is the same as first skipping over 12 (which brings us right back to where we started), then skipping an additional 1.

In fact, by the same reasoning, skipping 14 over is the same as skipping 2 over, and skipping 15 over is the same as skipping 3 over. In general, if $n \geq 12$, then skipping n over is the same as skipping $n - 12$ over. And if $n - 12$ happens to be greater than or equal to 12, then we can subtract 12 again and say that skipping n over is the same as skipping $n - 24$ over. We can continue subtracting 12 until we obtain a number that is less than 12.

Upon further reflection, I realized that subtracting 12's like this is the same as finding the remainder that is left when we divide n by 12. That is, if r is the remainder we get by dividing n by 12, then skipping by n is the same as skipping by r. This means that to understand what happens with any skipping number, I only had to look at the skipping numbers 0 through 11.

To my surprise, I found that we would count all of the circles if we skipped not only 1 over, but also 5 over, 7 over, or 11 over. It made me wonder, what's special about the numbers 1, 5, 7, and 11 that allows us to count all 12 circles? For example, here's what happens when we skip by 7:



Then I noticed that skipping by 11 is the same as skipping by 1 counterclockwise, and that's one way to explain why skipping by 11 goes through all of the circles. As a matter of fact, skipping by n and skipping by $12 - n$ must hit the same number of circles since one looks just like the other, except that they go around the clock in opposite directions. This explains why skipping by 5 and 7 would hit the same number of circles; $5 = 12 - 7$. But it doesn't explain why skipping by 5 or 7 should hit *all* of the circles.

It seems like the bus is always late! But I didn't mind waiting this time, because I had the skipping number problem to think about. Eventually, I figured out why skipping by 5 or 7 results in hitting all of the circles. In fact, I managed to understand the situation for any skipping number and any number of circles. Given any number of circles c and any skipping number n, I can compute how many of the circles will be counted without having to actually draw all the circles and write down all the numbers. I can also say which circles will be hit, and I can quickly decide if a skipping number will hit all of the circles. For example, I can tell you that if you put 1,000,000 numbers in a circle and try to count them by skipping over 2018 each time, you'd end up hitting every other circle for a grand total of 500,000 of them.

I could explain this in detail, but it was so much fun to figure it out and I don't want to spoil the opportunity for you. See if you can figure it out for yourself. If you spend as much time waiting for buses as I did, you will surely succeed!

Stacked Circles, Part 3

by Ken Fan | edited by Jennifer Silva

Emily: I tend to think of harmonic sequences as reciprocals of arithmetic sequences.

Jasmine: So do I. If you have an arithmetic sequence that doesn't contain zero and you reciprocate every term, you end up with a harmonic sequence, and vice versa.

Emily: I wonder if we can find a curve that holds a stack of circles whose radii are in harmonic progression by somehow "reciprocating" a stack of circles whose radii are in arithmetic progression.

Jasmine: That's an intriguing thought! We'd need some kind of transformation of the plane that maps a circle of radius r to a circle of radius $1/r$.

Emily and Jasmine think in silence for several minutes.

Emily: Nothing comes to mind. I thought geometric inversion[1] might work, but it doesn't.

Jasmine: I thought of geometric inversion too, but you're right that it doesn't do what we want; if we imagine a stack of circles whose radii are in harmonic progression, the stack will extend without bound since the sum of an infinite harmonic progression isn't finite. But if we invert a stack of circles whose radii are in arithmetic progression, all but finitely many will be inverted to the interior of the circle of inversion, hence resulting in a bounded sequence of circles.

Emily: Nice argument. That makes it clear that inversion doesn't do what we want. Since we can't think of a transformation that will do the trick, why don't we try the same thing you did to discover the parabola for stacked circles whose radii are in arithmetic progression?

Jasmine: You mean form a stack of circles with radii in harmonic progression and see what that looks like?

Emily: Yes. And I suppose it couldn't hurt to start with the standard harmonic sequence 1/1, 1/2, 1/3, 1/4, 1/5, …. If we stack these along the y-axis with the first circle resting on the x-axis, the radius of the nth circle will be $1/n$, and the y-coordinate of its center will be

$$2(1/1 + 1/2 + 1/3 + \ldots + 1/(n-1)) + 1/n.$$

Jasmine: I agree, because that's the sum of all of the diameters of the circles below it, plus the length of its radius. Unfortunately, I don't think that expression can be written any more simply.

Emily: I know that as n gets larger and larger, the difference between the natural logarithm of n, which I denote by $\ln n$, and $1/1 + 1/2 + 1/3 + \ldots + 1/n$ gets closer and closer to a constant, called the Euler constant.

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[1] Geometric inversion in a circle is a transformation of the plane. Given a circle C in the plane, the point P is sent to the point P', where P and P' lie on the same ray pointing from the center of the circle and such that the product of the distances of P and P' to the center of C is equal to the square of the radius of C. Under this transformation, the center of the circle is sent to infinity. It can be thought of as a "reflection" in the circle instead of a line.

Jasmine puts on a puzzled expression.

Emily: Euler looked at the limit of the difference

$$\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\ln n$$

as n tends to infinity and showed that it converges to some constant, which is called Euler's constant.

Jasmine: I see. Well, in our problem, constant shifts aren't important. We can compensate for an up or down shift in the curve that contains the stack by shifting all of the circles up or down by the same amount. So we can ignore Euler's constant and check if circles that are stacked snugly inside the curve $(1/x, 2 \ln |x|)$ have radii in harmonic progression, since the rightmost point of the nth circle in our stack, up to Euler's constant, will be located approximately at $(1/n, 2 \ln |n|)$.

Emily: Why are you taking the absolute value of x? Oh, I see – you're just ensuring that we're stacking into something that has both "walls," so to speak.

Jasmine: Right. The natural logarithm of x isn't even defined for $x \leq 0$.

Emily: I'm skeptical that this will give us what we're looking for, since the logarithm is only an approximation; when we looked at arithmetic progressions, we found that a parabola *exactly* passes through the rightmost points of the circles in the stack.

Jasmine: You're probably right, but I'm curious to know how the radii behave if we do stack circles into the graph of … let's see, setting $x = 1/n$, the point $(1/n, 2 \ln |n|)$ becomes $(x, 2 \ln |1/x|)$ … into the graph of $y = 2 \ln |1/x| = -2 \ln |x|$.

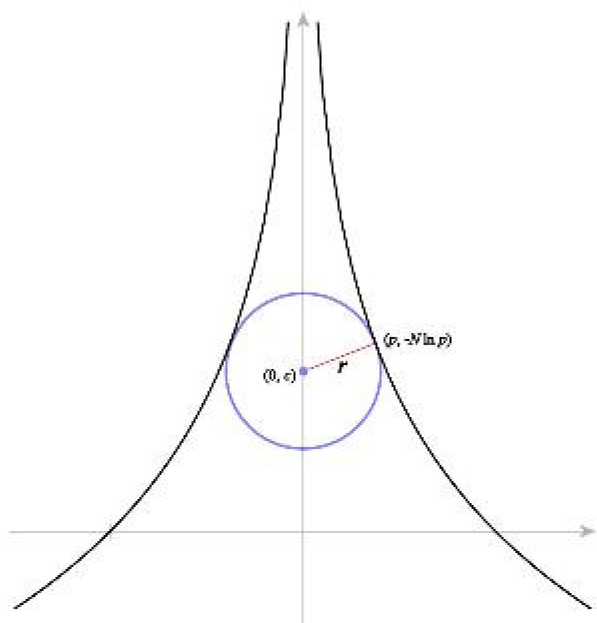
Emily: In the words of Francis Su, let's do it!

Jasmine: Let's begin by floating a circle of radius r into this upside-down, infinite funnel.

Emily: Sure, but first I suggest we work with the graph of $y = -N \ln |x|$, where N is any positive constant. The 2 in $-2 \ln |x|$ seems arbitrary, and I doubt replacing it with a general parameter will make things more difficult.

Jasmine: That's fine by me.

Emily: By symmetry, the center of the circle will be on the y-axis, so let's say the center has coordinates $(0, c)$.



Jasmine: And let's suppose the circle touches the graph of $y = -N \ln |x|$ at the point $(p, -N \ln |p|)$. By symmetry, we might as well assume that $p > 0$.

Emily: Now we need equations that relate r, the radius of the circle, c, the y-coordinate of the center of the circle, and p, the x-coordinate of the point on the graph that is tangent to the circle.

Jasmine: For one of the equations, we can express the fact that the distance from the center of the circle, $(0, c)$, to the point on the circle tangent to the graph, $(p, -N \ln |p|)$, is r. That gives us

$$p^2 + (c + N \ln p)^2 = r^2.$$

I dropped the absolute value since we're assuming that $p > 0$.

Emily: Since we want to find c and p in terms of r, we need at least one more equation.

Jasmine: What should that equation be?

Emily and Jasmine think.

Emily: We also have to express the fact that the radial line to the point of contact with the curve $y = -N \ln |x|$ is perpendicular to the tangent there.

Jasmine: Oh, right! We can get the slope of the tangent by taking the derivative of $-N \ln x$ with respect to x. I get $-N/x$. So the slope of that radial line has to be the negative reciprocal of $-N/p$, which is p/N.

Emily: So our second equation is

$$\frac{p}{N} = \frac{c - (-N \ln p)}{0 - p} = -\frac{c + N \ln p}{p}.$$

Jasmine: Hey! Both equations contain the expression $c + N \ln p$, and in both equations, the only occurrences of c are contained inside that expression. I think we should solve for $c + N \ln p$ in the second equation and substitute the result into the first to eliminate c, then solve the resulting equation for p in terms of r.

Emily: That sounds like a good plan! The second equation tells us that

$$c + N \ln p = -p^2/N.$$

Jasmine: Substituting $-p^2/N$ for $c + N \ln p$ in the first equation turns it into

$$p^2 + p^4/N^2 = r^2.$$

That's a quadratic in p^2, which we can rewrite as $(p^2)^2 + N^2p^2 - N^2r^2 = 0$.

Emily: Applying the quadratic formula, the solutions are

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Summer Fun!

In the previous issue, we presented the 2018 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Please refer to the previous issue for the problems.

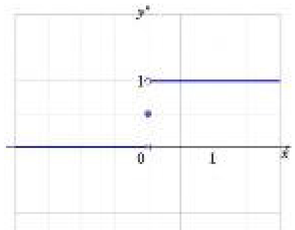
Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Summer Fun!

The Step Function

by Whitney Souery and Girls' Angle staff

1.



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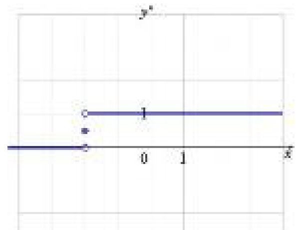

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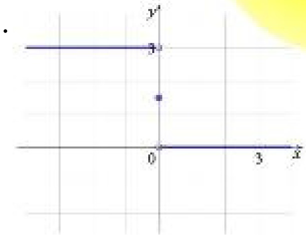

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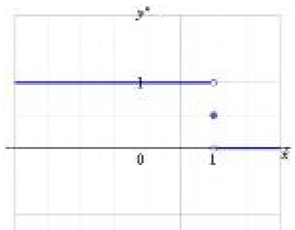

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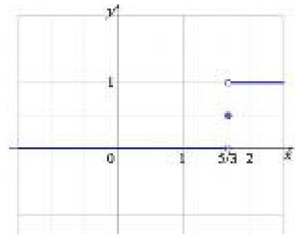
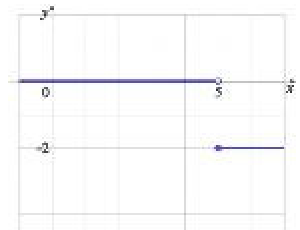


2E.



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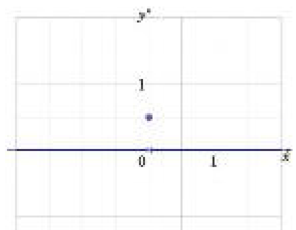
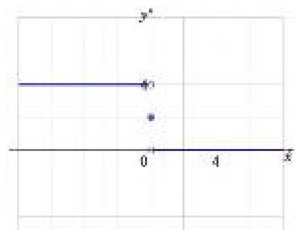
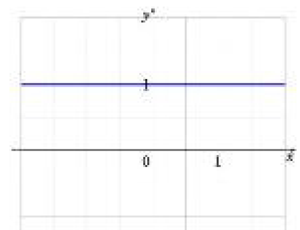


2G.



2H.



Step functions are examples of functions which are defined by cases. To prove identities that involve functions defined by cases, split the proof into cases.

3. If $x < 0$, then, by definition, $H(x) = 0$ and, therefore, $2H(x) - 1 = -1$ for $x < 0$. If $x = 0$, then, by definition, $H(x) = 1/2$ and, therefore, $2H(x) - 1 = 2(1/2) - 1 = 0$ for $x = 0$. If $x > 0$, then, by definition, $H(x) = 1$ and, therefore, $2H(x) - 1 = 1$ for $x > 0$. Hence, $2H(x) - 1$ is the sign function.

4. We must show that $u(x) - u(x - 1)$ is 0 when $x < 0$ or $x \geq 1$, and 1 when $0 \leq x < 1$. If $x < 0$, then x and $x - 1$ are both negative, hence $u(x) - u(x - 1) = 0 - 0 = 0$, as desired. If $x \geq 1$, then x and $x - 1$ are both positive, hence $u(x) - u(x - 1) = 1 - 1 = 0$, as desired. Finally, if $0 \leq x < 1$, then x is positive, but $x - 1$ is negative, hence $u(x) - u(x - 1) = 1 - 0 = 1$, as desired.

5. The characteristic function of the closed interval [0, 1] can be expressed as $u(x) + u(1 - x) - 1$.

6. The characteristic function of the open interval (0, 1) can be expressed as $1 - u(-x) - u(x - 1)$.

7. Let $f(x) = u(x) - 1 + \sum_{k=1}^{\infty}(u(x+k) + u(x-k) - 1)$. If $x = 0$, then $u(x + k) + u(x - k) - 1 = 0$ for all $k > 0$. Hence $f(x) = u(0) - 1 = 0$. For integer $x > 0$, $u(x + k) + u(x - k) - 1$ is 1 if $k \leq x$ and 0 if $k > x$. Hence $f(x) = u(x) - 1 + x = 1 - 1 + x = x$. For integer $x < 0$, $u(x + k) + u(x - k) - 1$ is -1 if $k < -x$ and 0 if $k \geq -x$. Hence,

$$f(x) = u(x) - 1 - (-x - 1) = x.$$

Summer Fun!

Now suppose x is not an integer. If $x > 0$, then $u(x + k) + u(x - k) - 1$ is 1 if $k < x$ and 0 if $k > x$. Hence $f(x) = 1 - 1 +$ (number of positive integers less than x) $= \lfloor x \rfloor$. Finally, if $x < 0$, then $u(x + k) + u(x - k) - 1$ is -1 if $k < |x|$ and 0 if $k > |x|$. Hence $f(x) = -1 + \lfloor |x| \rfloor = \lfloor x \rfloor$.

8. If $x < 0$, we have $x(u(x) - u(-x)) = x(0 - 1) = -x$. If $x = 0$, we have $x(u(x) - u(-x)) = 0$. Finally, if $x > 0$, we have $x(u(x) - u(-x)) = x(1 - 0) = x$. These agree with the absolute value of x.

The proof method for Problems 9-11 are the same, so we illustrate only with Problem 11.

11. If $x < 0$, then $u(x) = 0$ and $H(x)(3 - 2H(x)) = 0$. If $x = 0$, then $u(x) = 1$ and $H(x)(3 - 2H(x))$ evaluates to $(1/2)(3 - 2(1/2)) = 1$. If $x > 0$, then $u(x) = 1$ and $H(x)(3 - 2H(x)) = 1(3 - 2(1)) = 1$. Since the two sides agree for all values of x, they are identical.

12. Hint: As n tends to infinity, $(1/2)^n$ tends to 0.

13. The function $2(u(x - c) - H(x - c))$ is the characteristic function of $\{c\}$.

14. The function $4H(x - c)H(c - x)$ is the characteristic function of $\{c\}$.

15. The function $u(x - a)u(b - x)$ is the characteristic function of the interval $[a, b]$.

16. Hint: As n tends to infinity, $\arctan(nx)$ tends to $-\pi/2$, 0, or $\pi/2$, depending on whether $x < 0$, $x = 0$, or $x > 0$, respectively.

17. Given two finite sequences of real numbers $v_1, v_2, v_3, \ldots, v_n$ and $x_1 < x_2 < x_3 < \ldots < x_n$, if f is the associated mesa vista function defined by $f(x) = 0$ if $x < x_1$, $f(x) = v_k$ if $x_k \leq x < x_{k+1}$, for $0 < k < n$, and $f(x) = v_n$ if $x \geq x_n$, we will refer to the sequence $\{v_k\}$ as the "values," and the sequence $\{x_k\}$ as the "jump points" of the mesa vista function.

Here's a proof sketch. Note the D contains $u(x - a)$ for all real numbers a because it corresponds to the mesa vista function associated with the value sequence $v_1 = 1$ and jump point sequence $x_1 = a$. If f is a mesa vista function associated to the value sequence $v_1, v_2, v_3, \ldots, v_n$ and jump point sequence $x_1 < x_2 < x_3 < \ldots < x_n$, then af is the mesa vista function associated to the same jump point sequence, but with value sequence $av_1, av_2, av_3, \ldots, av_n$. Next, show that if f and g are mesa vista functions, then so are $f + g$ and fg. In other words, establish that D enjoys all the properties of C, and, therefore, C must be contained inside D.

On the other hand, let f be the mesa vista function associated to the value sequence $v_1, v_2, v_3, \ldots, v_n$ and jump point sequence $x_1 < x_2 < x_3 < \ldots < x_n$. For the convenience of writing the following expression for $f(x)$, let us extend $\{v_k\}$ by adding the term $v_0 = 0$ to its beginning. Then

$$f(x) = \sum_{k=1}^{n} (v_k - v_{k-1}) u(x - x_k).$$

This expression shows that f is in C. Therefore, C contains D.

We conclude that $C = D$.

Summer Fun!

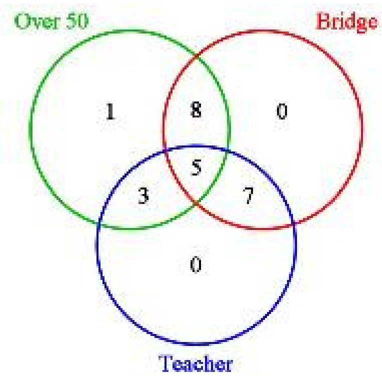
Let’s Throw a BBQ

by Vicky Xu | edited by Amanda Galtman

We will use the notation $P(A)$ to denote the probability that A occurs, where A is a subset of the possible outcomes. We will also write $P(A \mid B)$ for the probability that A occurs given that B has occurred. For example, if we are flipping a fair coin, the possible outcomes are heads and tails. Then $P(\{\text{heads}\}) = 1/2$ and $P(\{\text{heads, tails}\}) = 1$.

Note that in general, $P(A \mid B) = P(A \cap B)/P(B)$. To see this, imagine repeating the experiment a large number N of times. Of these, we expect B to occur $P(B)N$ times. Of these $P(B)N$ times that B occurs, we expect A to occur $P(A \cap B)N$ times. (Overall, A is expected to occur $P(A)N$ times, but to compute $P(A \mid B)$, we want to count only the times where A occurs given that B occurred, i.e., the number of times both A and B occur, and that is $P(A \cap B)N$.) We conclude that $P(A \mid B) = P(A \cap B)N/(P(B)N) = P(A \cap B)/P(B)$. The probability $P(A \mid B)$ is known as a **conditional probability** because it gives the probability of an event occurring *on the condition* that another event occur.

We denote by A^c the complement of A, i.e., all outcomes other than A. We also denote by ${}_nC_m$ the binomial coefficient n choose m.



1. We use a Venn diagram (shown at left) to sort out the given information and find the answer to be 24.

2. The answer is 36 people.

3. A. Let’s call the pets Cat 1, Cat 2, Dog 1, and Dog 2. Each of these four pets is equally likely to be first chosen. After the first pet is chosen, each of the three remaining pets is equally likely to be chosen next. This yields a total of 12 ways in which the pets can be chosen, all of them equally likely. These ways are summarized in the table below.

	1	2	3	4	5	6	7	8	9	10	11	12
1st Choice	Cat 1	Cat 1	Cat 1	Cat 2	Cat 2	Cat 2	Dog 1	Dog 1	Dog 1	Dog 2	Dog 2	Dog 2
2nd Choice	Cat 2	Dog 1	Dog 2	Cat 1	Dog 1	Dog 2	Cat 1	Cat 2	Dog 2	Cat 1	Cat 2	Dog 1

Of these 12 equally likely outcomes, exactly two see both cats being picked. Thus, the answer is 2/12 or 1/6.

B. Of the 12 outcomes, 10 include a cat, so the answer is 10/12 or 5/6.

C. Of the 12 outcomes, eight include both a cat and a dog, so the answer is 8/12 or 2/3.

D. Since we are given that the first choice was a dog, we eliminate outcomes 1 through 6 as possible outcomes. Of the six remaining outcomes, four have a cat as her second choice, so the answer is 4/6 or 2/3.

4. Since grandma brings two pets, let's assume that $C + D \geq 2$.

A. The probability of bringing two cats is $\dfrac{C(C-1)}{(C+D)(C+D-1)}$.

B. The probability is $1 - \dfrac{D(D-1)}{(C+D)(C+D-1)}$. C. The probability is $\dfrac{2CD}{(C+D)(C+D-1)}$.

D. The probability is $\dfrac{C}{C+D-1}$.

5. Let A be the event that your relatives choose to come. Let B be the event that Uncle John chooses to come. We compute

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c) = (1-(1-p)^2)(1/2) + p^2(1/2) = p.$$

Thus, your family and Emily have the same probability of coming!

6. (The wording of the problem was ambiguous. The first sentence, "Uncle John is never late," could be interpreted to mean that, since all three are driving together, and since Uncle John is never late, the probability that they're late is 0. However, what we meant was that Uncle John reaches the car on time, but whether he is late to the party depends on whether Aunt Sarah and Uncle Tom also reach the car on time. This solution uses this latter interpretation.) Let S be the event that Sarah is late and T be the event that Tom is late. We know $P(S) = 1/10$, $P(T) = 3/20$, and $P(S \cap T) = 1/20$. The probability that they arrive on time equals the probability that both Aunt Sarah and Uncle Tom are on time. We compute

$$P(S^c \cap T^c) = 1 - P(S \cup T) = 1 - (P(S) + P(T) - P(S \cap T)) = 1 - (1/10 + 3/20 - 1/20) = 4/5.$$

Thus, the probability that the car arrives late is 1/5.

7. A. You can give any number of ears of corn to each of the 10 guests, provided you give out 20 ears. In other words, you are distributing 20 ears into 10 piles, one pile for each guest. There are $_{29}C_9 = 10{,}015{,}005$ ways to do that. Imagine placing 29 ears of corn in a row, and then replacing nine with dividers like the ones you find in supermarket checkout lines. The dividers split the corn into 10 bins: one between each pair of consecutive dividers, one before the first divider, and one after the last divider. Note that there may be no ears in a particular bin.

B. After giving each person one ear of corn, there are 10 left over. These ears can be distributed as in the solution to 7A, so the answer is $_{19}C_9 = 92{,}378$.

C. You can ask each ear of corn, "Which person are you going to?" Each can answer independently that it is going to one of the 10 people, hence the answer is 10^{20}.

D. Label the 10 guests by the numbers 1 through 10. Let S_k be the set of ways the corn can be distributed so that guest k gets no corn. The set of ways to distribute corn so that each guest gets at least one ear is equal to the set of ways to distribute corn to 10 guests minus the union of S_k over all k from 1 to 10.

Summer Fun!

Using the principle of inclusion/exclusion, the size of the desired set is

$$ {}_{10}C_0 \cdot 10^{20} - {}_{10}C_1 \cdot 9^{20} + {}_{10}C_2 \cdot 8^{20} - {}_{10}C_3 \cdot 7^{20} + \ldots + {}_{10}C_8 \cdot 2^{20} - {}_{10}C_9 \cdot 1^{20} + {}_{10}C_{10} \cdot 0^{20}. $$

Using a computer, we find that the answer is 21,473,732,319,740,064,000.

8. A. The probability of that particular straight in the same suit is $4/{}_{52}C_6 = 1/5{,}089{,}630$.
B. The probability of all 6 cards being from the same suit is $4 \cdot {}_{13}C_6/{}_{52}C_6 = 66/195{,}755$.
C. The probability of getting 6 consecutive cards is $9 \cdot 4^6/{}_{52}C_6 = 4608/2{,}544{,}815$. (We are allowing both A-2-3-4-5-6 and 9-10-J-Q-K-A to qualify as six cards in consecutive order.)
D. The probability of getting at least one card of each suit is $(4 \cdot {}_{13}C_3 \cdot 13^3 + 6 \cdot {}_{13}C_2 \cdot 13^2)/{}_{52}C_6$, which equals 9,971/78,302.
E. The probability of getting three different pairs is ${}_{13}C_3 \cdot 6^3/{}_{52}C_6 = 594/195{,}755$.

9. This problem is a typical problem from a topic called **Markoff chains** (yes, the same Markoff in Matthew de Courcy-Ireland's Summer Fun problem set in this batch). Let's model Ruth's distance from the BBQ by a number line. Ruth is standing at the 110 yard mark. Each second, she moves either up 1 or down 1, with equal probability. We define a matrix M such that M_{ij} is the probability that Ruth will be at position i after one move if she starts at position j. Since all that matters is that she get within 10 yards of the BBQ within 5 minutes, we set $M_{ij} = 0$ if $i < 10$. We set $M_{10,j} = 1$ if $j = 10$, 1/2 if $j = 11$, and 0 otherwise. The problem asks us to determine the entry in row 10, column 110 of the matrix M^{300}. If you know how to multiply matrices, the problem is reduced to a long computation. A computer returns the answer

$$\frac{3042084424665716204375765672475533020280778381097970434115792291920378981492316377}{509258994083621521567111422102344540262867098416484062659035112338595324940834176545849344}$$

which is approximately six in a billion. Unfortunately for Ruth, it's not too likely that she'll get any food at this BBQ!

10. A time interval of 1 hour consists of six disjoint 10-minute time intervals. Let p be the probability that a patty gets burned in a time interval of 10 minutes. The probability that a patty gets burned in six disjoint 10-minute time intervals is $1 - (1 - p)^6$, which we are told is 0.7. Thus, $1 - (1 - p)^6 = 0.7$. Solving for p, we find that $p = 1 - \sqrt[6]{0.3} \approx 18.2\%$.

11. Let there be b beef patty requests from your guests. Let A be the event that the guest of interest wanted a veggie burger. Let B be the event that the person you asked wanted a veggie burger. We are interested in finding $P(A \mid B)$. Note that $P(A \mid B) = P(B \mid A)P(A)/P(B)$. If the guest of interest wanted a veggie burger, then three out of $3 + b$ people who submitted forms wanted a veggie burger. Thus, $P(B \mid A) = 3/(3 + b)$. We compute $P(B)\,) = P(B \mid A)P(A) + P(B \mid A^c)P(A^c) = \frac{1}{2}\frac{3}{3+b} + \frac{1}{2}\frac{2}{3+b} = \frac{5}{6+2b}$. Substituting this into the equation $P(A \mid B) = P(B \mid A)P(A)/P(B)$, we find that $P(A \mid B) = \frac{3}{3+b}\frac{1}{2}\frac{6+2b}{5} = \frac{3}{5}$.

Markoff Triples

by Matthew de Courcy-Ireland

The Markoff equation is

$$x^2 + y^2 + z^2 = 3xyz,$$

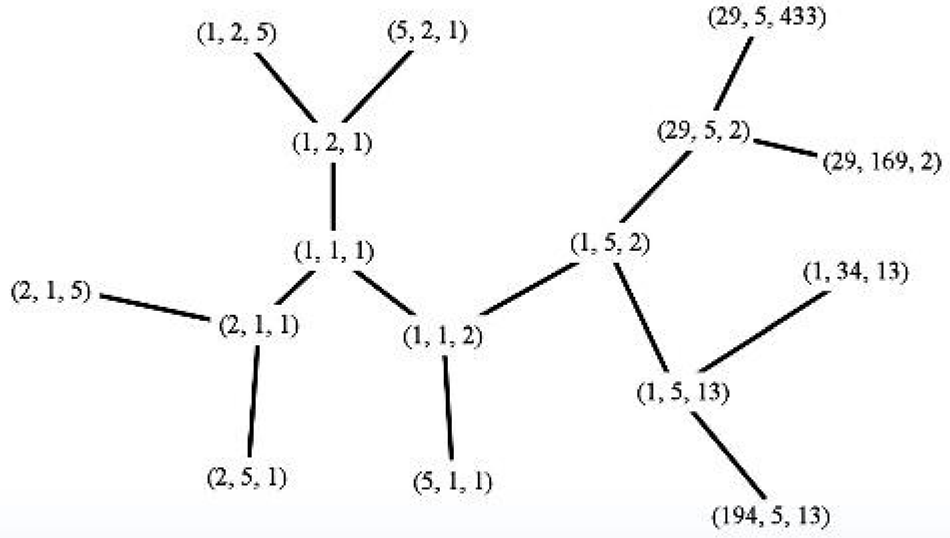
to be solved in positive integers x, y, and z. There is no general method for finding all the integer solutions of a cubic equation in three variables like this. However, this equation has a very special structure that can help us. Although it is a cubic equation because of the term xyz, it is only a quadratic equation when we consider one variable at a time.

1. Factor the quadratic as $(x - x_1)(x - x_2)$, where x_1 and x_2 are the two roots. Multiplying these gives $x^2 - (x_1 + x_2)x + x_1x_2$. For this to match $x^2 + bx + c$, we must have $x_1 + x_2 = -b$. As a bonus, we see that the product of the solutions, x_1x_2, is equal to c. An alternative way to prove these facts is to use the quadratic formula.

2. Suppose that (x, y, z) is a solution to the Markoff equation, i.e. $x^2 + y^2 + z^2 = 3xyz$. This can be rearranged to $x^2 - (3yz)x + y^2 + z^2 = 0$. This quadratic generally has two solutions, and from Problem 1, we know that the other solution must be $3yz - x$ because the middle coefficient is $-3yz$ and the sum of the two solutions is therefore equal to $3yz$. Therefore $(3yz - x, y, z)$ is also a solution to the Markoff equation.

3. Applying the same reasoning used for Problem 2 to the second and third coordinates shows that $(x, 3xz - y, z)$ and $(x, y, 3xy - z)$ are also solutions to the Markoff equation.

4. Here is a small part of the resulting tree-like picture:



Summer Fun!

5. We claim that doing the Markoff move on the largest coordinate must reduce its size. Since the equation $x^2 + y^2 + z^2 = 3xyz$ is symmetric under all orderings of x, y, and z, we may suppose that $0 < x \le y \le z$. Our goal is to show that $3xy - z$ is smaller than z. Suppose, to the contrary, that $3xy - z \ge z$. We will then show that $z < y$, contradicting our assumptions. We can find an expression for z by applying the quadratic formula to $z^2 - 3xyz + x^2 + y^2 = 0$:

$$z = \frac{3xy - \sqrt{(3xy)^2 - 4(x^2 + y^2)}}{2} = y \cdot \frac{3x - \sqrt{9x^2 - 4(x^2 + y^2)/y^2}}{2}.$$

(Note that we use the minus sign instead of the plus sign in the quadratic formula because we're assuming that $3xy - z \ge z$, i.e. z is not the bigger root.) This will be less than y provided that

$$3x - \sqrt{9x^2 - 4(x^2 + y^2)/y^2} < 2.$$

This rearranges to $9x^2 - 12x + 4 < 9x^2 - 4(x^2 + y^2)/y^2$. Multiplying by y^2 and isolating y, we find

$$y > \frac{2x}{\sqrt{12x - 8}}.$$

Since $x \le y$, this inequality holds if $2 > \sqrt{12x - 8}$, i.e, if $x > 1$, or, if $x = 1$, when $x < y$. In these cases, running the same steps backward show that $z < y$, contrary to hypothesis. In the case where $x = y = 1$, then the Markoff equation becomes the quadratic equation $2 + z^2 = 3z$. The two solutions to this quadratic are $z = 1$ and $z = 2$. If $z = 1$, we are already at our destination solution (1, 1, 1). If $z = 2$, when we apply a Markoff move to the 3rd coordinate of (1, 1, 2), we go from (1, 1, 2) to (1, 1, 1) and reach our desired destination.

This shows that doing a Markoff move on the largest coordinate must lower it, unless the coordinates are all equal. So if we continue applying Markoff moves to the largest coordinate, eventually, we will reach a point where all the coordinates are equal. But then we must be at the solution (1, 1, 1), because (1, 1, 1) is the only solution to $3x^2 = 3x^3$ with x a positive integer. Thus the tree you began drawing will eventually contain *all* solutions!

6. As hinted, examine the solutions (1, 1, 1), (1, 1, 2), (1, 2, 5), and (1, 5, 13). All these coordinates are Fibonacci numbers! If we let F_k be the Fibonacci sequence with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$, then in fact, the Markoff tree does contains the all the odd-indexed Fibonacci numbers F_{2n+1}. By applying the Fibonacci recursion three times, we arrive at

$$F_{2n+1} = F_{2n} + F_{2n-1} = 2F_{2n-1} + F_{2n-2} = 3F_{2n-1} - F_{2n-3}.$$

This means that applying a Markoff move to the 3rd coordinate of $(1, F_{2n-1}, F_{2n-3})$ yields the solution $(1, F_{2n-1}, F_{2n+1})$. Starting from the triple (1, 5, 2), this shows that all Fibonacci numbers with an odd label appear as solutions to the Markoff equation.

Summer Fun!

Generating Functions

by Laura Pierson | edited by Amanda Galtman

1. These definitions agree with what we would expect for polynomials because they generalize addition and multiplication of polynomials. When you add polynomials, the coefficient of x^n in the sum is the sum of the coefficients of x^n in the addends; that is exactly what the formula for addition of formal power series corresponds to. For multiplication of formal power series, note that the coefficient of x^n in the product depends only on the coefficients of x^k in the factors for $0 \le k \le n$. Multiplication corresponds to truncating the formal power series after x^n, regarding the truncation as a polynomial of degree n, and then multiplying as polynomials.

2. All of these facts are proven in a similar way, which is to carefully apply the definitions of addition and multiplication to both the left and right sides of the desired identity, then check that the results are the same. To illustrate, we show that multiplication is associative.

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$, and $h(x) = \sum_{k=0}^{\infty} c_k x^k$. We must check that $f(x)(g(x)h(x)) = (f(x)g(x))h(x)$. We compute:

Left Side

$$\begin{aligned} f(x)(g(x)h(x)) &= \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\left(\sum_{j=0}^{\infty} b_j x^j\right)\left(\sum_{i=0}^{\infty} c_i x^i\right)\right) \\ &= \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{m=0}^{\infty}\left(\sum_{p=0}^{m} b_p c_{m-p}\right)x^m\right) \\ &= \sum_{s=0}^{\infty}\left(\sum_{q=0}^{s}\left(a_q\left(\sum_{p=0}^{s-q} b_p c_{s-q-p}\right)\right)\right)x^s \\ &= \sum_{s=0}^{\infty}\left(\sum_{q=0}^{s}\sum_{p=0}^{s-q} a_q b_p c_{s-q-p}\right)x^s \end{aligned}$$

Right Side

$$\begin{aligned} (f(x)g(x))h(x) &= \left(\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)\right)\left(\sum_{i=0}^{\infty} c_i x^i\right) \\ &= \left(\sum_{m=0}^{\infty}\left(\sum_{q=0}^{m} a_q b_{m-q}\right)x^m\right)\left(\sum_{i=0}^{\infty} c_i x^i\right) \\ &= \sum_{s=0}^{\infty}\left(\sum_{p=0}^{s}\left(\left(\sum_{q=0}^{p} a_q b_{p-q}\right)c_{s-p}\right)\right)x^s \\ &= \sum_{s=0}^{\infty}\left(\sum_{p=0}^{s}\sum_{q=0}^{p} a_q b_{p-q} c_{s-p}\right)x^s \end{aligned}$$

Now observe that

$$\sum_{q=0}^{s}\sum_{p=0}^{s-q} a_q b_p c_{s-q-p} = \sum_{p+q=0}^{s}\sum_{k=0}^{p+q} a_k b_{p+q-k} c_{s-q-p} = \sum_{i=0}^{s}\sum_{k=0}^{i} a_k b_{i-k} c_{s-i} = \sum_{p=0}^{s}\sum_{q=0}^{p} a_q b_{p-q} c_{s-p},$$

which shows that the coefficients of x^s on the left and right sides are equal. (If you are finding it difficult to see this, pick specific values of s, such as $s = 3$ or $s = 5$, and explicitly write out that sum in full, without using summation notation. In the last equation, the first equality represents a change in the way the indices are traversed, the second equality substitutes the dummy variable i for $p + q$, and the last equality is a relabeling of the dummy variables.)

Another way to see equality of coefficients is to observe that on both sides, the coefficient of x^s is equal to the sum of all triples $a_p b_q c_r$ where $p + q + r = s$.

Summer Fun!

3. The key point is that $1/(1 - x^n) = 1 + x^n + x^{2n} + x^{3n} + x^{4n} + \ldots$, which follows from

$$(1 - x^n)(1 + x^n + x^{2n} + x^{3n} + x^{4n} + \ldots) = 1 + (1 - 1)x^n + (1 - 1)x^{2n} + (1 - 1)x^{2n} \ldots.$$

4. To choose n fruit, we have $n + 1$ options because we can pick between 0 and n apples, inclusive. Once we decide how many apples we want, we have no choice about the number of oranges since, together, they must total n fruit. Thus, the generating function is

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{k=0}^{\infty}(k+1)x^k .$$

We compute $\frac{1}{(1-x)^2} = (\sum_{k=0}^{\infty} x^k)(\sum_{j=0}^{\infty} x^j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x^k x^j = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x^{k+j} = \sum_{s=0}^{\infty}\sum_{p=0}^{s} x^s = \sum_{s=0}^{\infty}(s+1)x^s$, as desired. This makes sense because the x^n term in the product is made up by multiplying each x^k term in the first factor by the x^{n-k} term in the second factor. If the first factor is apples and the second oranges, we can think of k as the number of apples and $n - k$ as the number of oranges.

5. If we have apples, oranges, and bananas, the generating function will be

$$(1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) = 1/(1 - x)^3.$$

We can think of the first factor as representing the number of apples, the second oranges, and the third bananas. If we choose i apples, j oranges, and k bananas, this corresponds to picking x^i from the first factor, x^j from the second factor, and x^k from the third factor, and each such choice contributes x^{i+j+k} to the product. Therefore, the coefficient of x^n in the product corresponds to the number of ways we can pick i, j, and k so that $i + j + k = n$.

6. For an unlimited number of each of T different types of fruit, the generating function that gives the number of ways of picking n fruit is $1/(1 - x)^T$.

7. Since apples are only sold in pairs, we have to choose an even number of them, so our generating function for picking apples is $1 + x^2 + x^4 + x^6 + \ldots = 1/(1 - x^2)$. Our generating function for picking oranges is $1/(1 - x)$. Since there are only 5 bananas, the generating function for picking bananas is $1 + x^2 + x^3 + x^4 + x^5$, which can also be expressed as $(1 - x^6)/(1 - x)$. Thus, our generating function for choosing n of these fruit is:

$$\frac{1}{1-x^2}\cdot\frac{1}{1-x}\cdot\frac{1-x^6}{1-x} = 1+2x+4x^2+6x^3+9x^4+12x^5+15x^6+\ldots.$$

8. We compute $\underbrace{\frac{1}{1-x^2}}_{\text{(apples)}}\cdot\underbrace{\frac{1}{1-x^5}}_{\text{(bananas)}}\cdot\underbrace{\frac{1-x^5}{1-x}}_{\text{(oranges)}}\cdot\underbrace{(1+x)}_{\text{(pears)}} = \frac{1}{(1-x)^2}$. Thus, there should be exactly $n + 1$ ways to choose n fruit given these crazy constraints! Verify this directly for some small values of n.

9. To choose n items from among elements of set A and set B, we can choose k from set A and $n - k$ from set B, for some k. Our choice of k elements from set A does not affect our choice of $n - k$ elements from set B. Hence. the number of ways this can be done is the product of the number of ways to choose k items from set A and $n - k$ items from set B. When we add up these products for all possible values of k, from 0 to n, we get the coefficient of x^n in the product $A(x)B(x)$, by definition!

10. For a pair of standard dice, the generating function for sums is $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$. For Sicherman dice, the generating function is $(x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8)$. Check that these are equal.

11. You get the generating function for the Fibonacci numbers. To use the generating function to obtain a formula for the nth Fibonacci number, factor the denominator to express as a product of two generating functions of geometric sequences, then take the product of the power series.

12. The answer is $x(1 + x)/(1 - x)^3$. Hint: ${}_nC_2 + {}_{n+1}C_2 = n^2$, where ${}_nC_2$ is n choose 2.

13. For example, $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$, hence $\pi(5) = 7$.

14. To see why $P(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$, we can think of the first factor as counting the sum of the 1's in the partition, the second factor as counting the sum of the 2's, the third the sum of the 3's, and so on. These sums determine the partition, and vice versa.

15. We have $O(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}} = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \ldots$.

16. We have $N(x) = \prod_{k=1}^{\infty} (1+x^k) = (1 + x)(1 + x^2)(1 + x^3) \cdots = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + \ldots$.

17. Hint: Observe that $1 + x^k = (1 - x^{2k})/(1 - x^k)$.

18. For a proof without generating functions, we make a one-to-one correspondence between partitions with only odd parts and partitions with no repeated parts. Suppose we start from a partition with no repeated parts. Halve each even part, and iterate until all parts are odd. For instance, starting from the partition 4 + 3 of 7, split the 4 in half to get 3 + 2 + 2. Then, split each 2 in half to get 3 + 1 + 1 + 1 + 1, which has only odd parts.

Conversely, suppose we start from a partition with only odd parts. Combine a pair of repeated parts, and iterate until no repeated parts are left. For instance, starting from the partition 1 + 1 + 1 + 1 + 1 + 1 + 1 of 7, combine three of the pairs to get 2 + 2 + 2 + 1. Then, combine two of the 2's to get 4 + 2 + 1. Convince yourself that the two procedures are inverses of each other.

Summer Fun!

Calendar

Session 23: (all dates in 2018)

September	13	Start of the twenty-third session!
	20	
	27	
October	4	
	11	
	18	
	25	
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls’ Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don’t like math, what don’t you like? If you love math, what do you love? What would you like to get out of a Girls’ Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls’ Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors strive to get members to do math through inspiration and not assignment. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where are Girls' Angle meets held? Girls' Angle meets take place near Kendall Square in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________
(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ____________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________