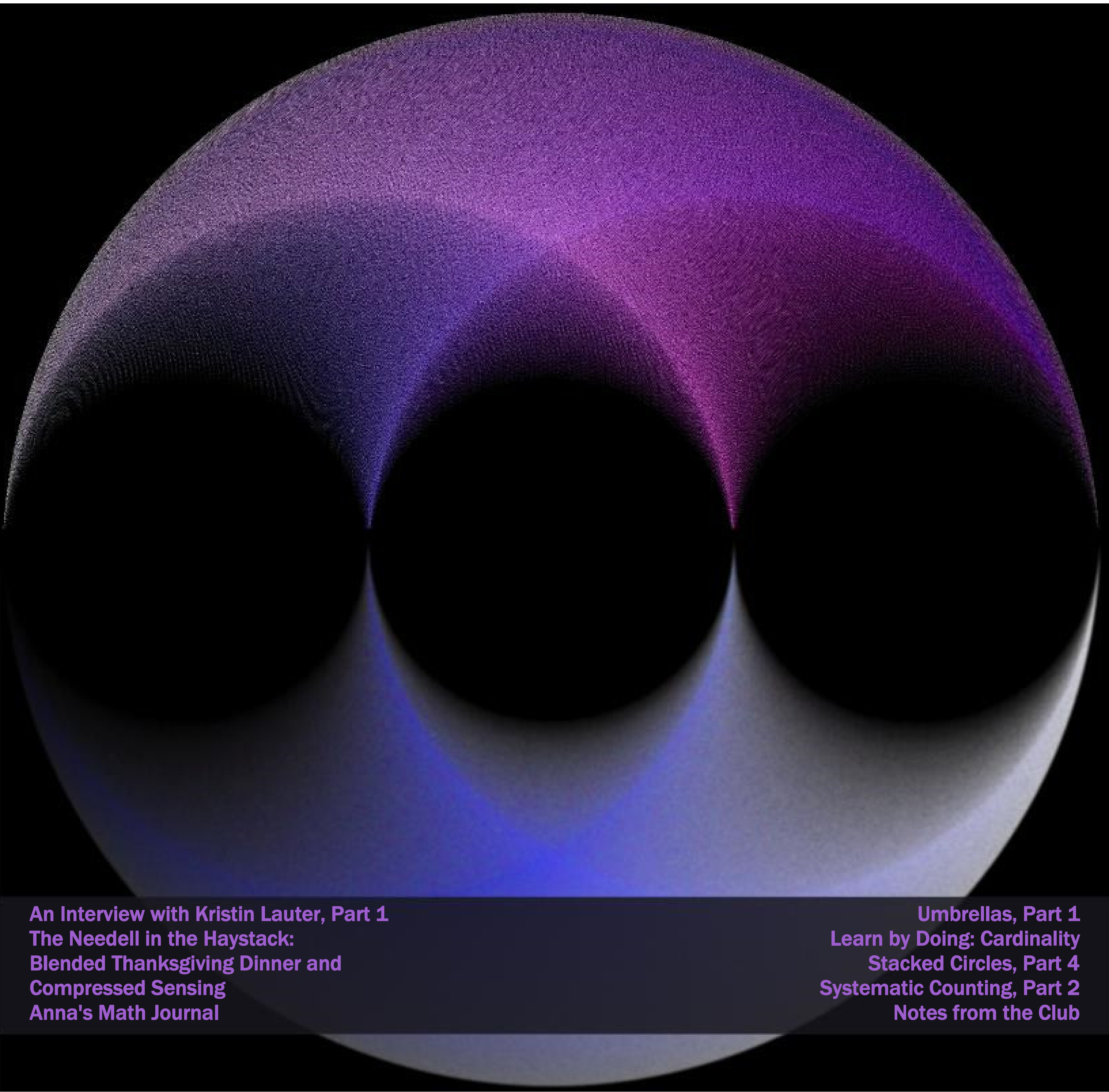
Girls' Angle Bulletin

October/November 2018 • Volume 12 • Number 1

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

We are thrilled to open this issue with the first of a multi-part interview with Dr. Kristin Lauter of Microsoft Research. Dr. Lauter and Microsoft Research have provided major financial support to Girls' Angle through the years and we are grateful. - Ken Fan, President and Founder

Girls' Angle Donors

Girls' Angle thanks the following for their generous contribution:

Individuals

Uma Achutha
Dana Albert
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Bill Bogstad
Ravi Boppana
Lauren Cipicchio
Merit Cudkowicz
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Kim Deltano
Eleanor Duckworth
Concetta Duval
Glenn and Sara Ellison
John Engstrom
Lena Gan
Courtney Gibbons
Vanessa Gould
Rishi Gupta
Larry Guth
Andrea Hawksley
Delia Cheung Hom and Eugene Shih
Mark and Lisel Macenka
Brian and Darlene Matthews
Toshia McCabe
Mary O'Keefe
Stephen Knight and Elizabeth Quattrocki Knight
Junyi Li
Alison and Catherine Miller
Beth O'Sullivan
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Christian Rudder
Craig and Sally Savelle
Eugene Shih
Eugene Sorets
Sasha Targ
Diana Taylor
Waldman and Romanelli Family
Marion Walter
Andrew Watson and Ritu Thamman
Brandy Wiegers
Brian Wilson and Annette Sassi
Lissa Winstanley
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathematical Sciences Research Institute
The Mathenaeum Foundation
Orlando Math Circle

Corporate Donors

Adobe
Akamai Technologies
Big George Ventures
John Hancock
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
Mathenaeum
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle

For Bulletin Sponsors, please visit girlsangle.org.

Girls' Angle Bulletin

The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Untitled* by C. Kenneth Fan. This image is based on the mathematics discussed in *Umbrellas* on page 14.

An Interview with Kristin Lauter, Part 1

Kristin Lauter is a mathematician and the head of the cryptography group at Microsoft Research[1]. She received her doctoral degree in mathematics from the University of Chicago. She is an Affiliate Professor at the University of Washington in Seattle.

This interview[2] was conducted in person at the University of Washington by University of Washington graduate student Ke Huang.

Ke Huang: How did you first become interested in math, and what was one of the first mathematical things that caught your interest? Did somebody make a big impact in your early math life?

Kristin: I actually really loved math from a very early age. I used to do story problems with my dad in the car when we were driving, and I always really liked it. What was great for me was that, in school, I had teachers who allowed me to work on my own at my own pace. So I zoomed ahead through years of math from elementary school and junior high school. That enabled me to graduate high school when I was 15. After high school, I went to the University of Chicago. The University of Chicago is a great place, especially for young students. They test and place all of their incoming students into the appropriate level of calculus or advanced math, and I was put into honors calculus, taught by Jill Pipher, and it was a great start.

Ke Huang: Wow, that's incredible. Can you tell us more about the story problems you did with your dad?

Kristin: When I was quite young, we had a summer home in the woods of Northern Wisconsin. It was about a two-hour drive to get there. During those rides, my dad would give me story problems. I enjoyed them so much. They were essentially algebra questions, like a grocer has 15 more apples than lemons and the number of lemons is two-fifths the number of apples, how many lemons does the grocer have? But he always said that I would answer the question almost before he finished asking it, which I think is really funny because that's not necessarily a good strategy, to answer before you know what the end of the question is, but I'd try to predict what he was asking me.

Ke Huang: It sounds like you got them right! Was your dad a mathematician?

Kristin: No, not at all. He was a Dean of Students at a college, so he really liked counseling and mentoring students. He was at Lawrence University in Wisconsin.

Ke Huang: What drew you to work on cryptography?

Kristin: So, when I was teaching at the University of Michigan[3], I taught courses on coding theory and number theory, and as part of those courses, I included an introduction to cryptography. The students loved that part of the class, so it was clear that it was something that captured their imaginations. Of course, I thought that it was a pretty neat application too, but one thing that really impressed me is that the graduate students in engineering that attended the class had a lot more knowledge about the real world than I did. They knew what was going on in industry. At that time, there was kind of a revolution going on in coding theory with the introduction of the

[1] Dr. Lauter and Microsoft Research have been a big supporter of Girls' Angle's Math Collaboration initiative and we thank them for their generosity and financial support over the years.

[2] Transcription by New England Transcript Services of Boston, Inc., with assistance from Peter Park.

[3] Kristin was a Hildebrandt Assistant Professor of Mathematics at the University of Michigan from 1996-1999.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

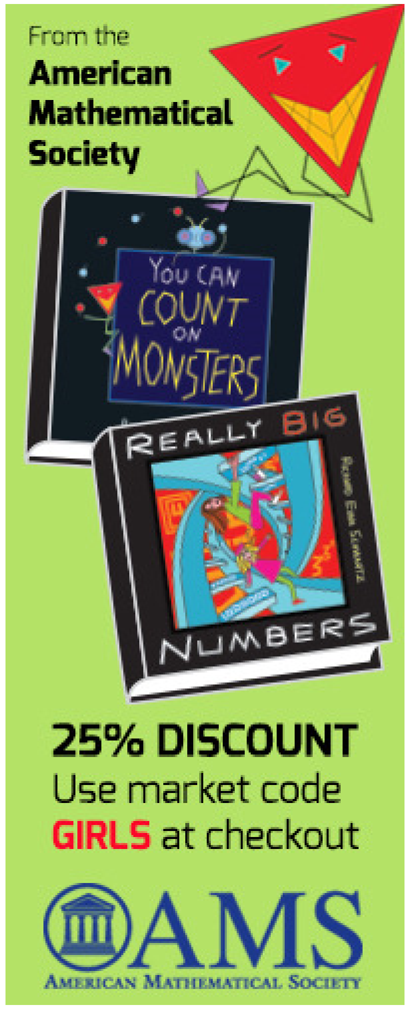
For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Kristin Lauter and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

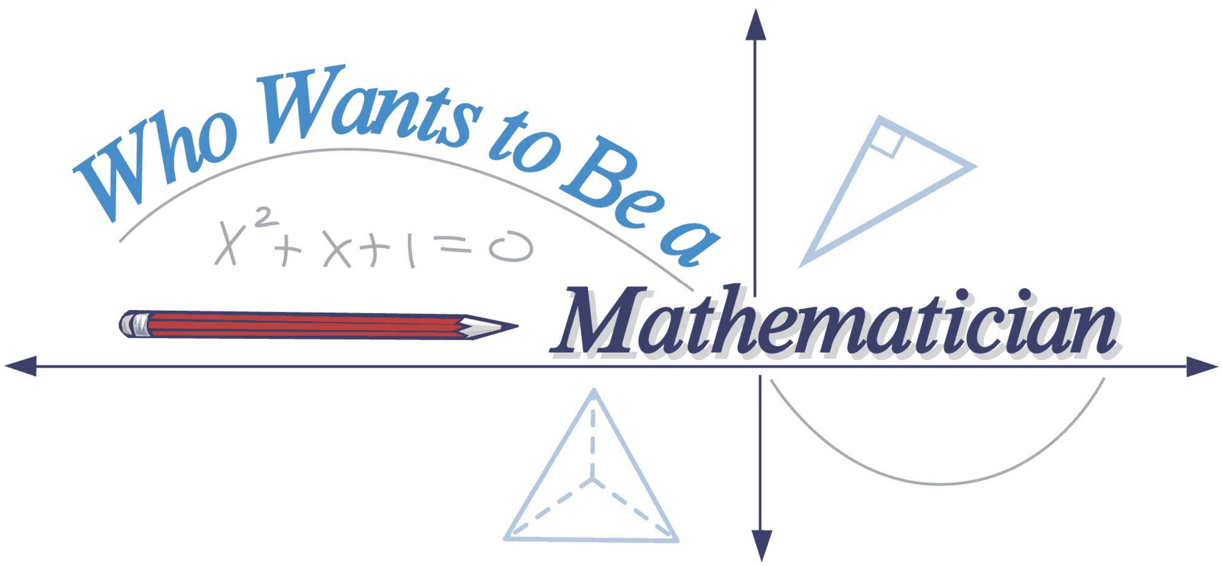
So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version



Content Removed from Electronic Version



America's Greatest Math Game: Who Wants to Be a Mathematician.

(advertisement)

The Needell in the Haystack

Blended Thanksgiving Dinner and Compressed Sensing[1]

by Deanna Needell | edited by Jennifer Silva

Thanksgiving is coming up, and in my family this is a big holiday. We spend two full days cooking everything from gluten-free stuffing and mashed potatoes to pecan pies and my favorite gelatin dessert, "silver slip," not to mention two large turkeys.[2] Picture this feast where painstaking effort has gone into cooking and preparing everything just right. Then, imagine Uncle Rick coming along, taking all of this perfectly prepared food, and throwing everything together into a large blender! How would you react? Perhaps, after some initial cringing, you might feel like it was a *waste* to have prepared the food so meticulously if it was just going to be blended all together in the end. You probably could have saved a lot of time if you had just thrown all the ingredients into the blender initially, right?

Metaphorically speaking, this is exactly what most modern digital cameras do. Take the camera in your phone, for example. When you take a photo, it measures the light intensity at every single "pixel" in the photographic region, only to discard most of that information when it compresses the image for storage. We will refer to this compressed version of the image as the *measurements*. This seemingly wasteful acquisition paradigm is what led to the field that is now known as *compressed sensing* or *compressive signal processing* [4, 5]. As the name might suggest, the main goal is to acquire the measurements directly in their compressed form, without the need to directly measure each pixel first. This eliminates wasteful time, energy and cost, potentially saving tremendous resources in many applications. For instance, in medical imaging it will lead to a significant reduction in scan time; this is especially important in situations such as magnetic resonance imaging (MRI), where the patient often has to remain perfectly still for up to 40 minutes per scan. In other types of imaging such as hyperspectral imaging – where "photos" are being taken using light outside the visible spectrum – this leads to a significant reduction in the number of photon diodes needed, which can reduce cost in some very expensive technologies.

So how do we do compressed sensing, and what does it entail? The first ingredient we need is a low-dimensional *model* that we can use to argue that compression without serious loss of information is even possible. One mathematical model that has gained a lot of recent attention is the use of *sparsity*. Sparsity captures the idea that high-dimensional signals often contain a very small amount of intrinsic information. Using this notion, one may design efficient low-dimensional representations of large-scale data, as well as robust reconstruction methods for those representations.

We will denote our signal of interest by $f \in \mathbb{C}^n$; f may be a large data vector, or the pixel intensities of a large image, etc. We say that f is *s-sparse* when f has at most s non-zero entries, written $\|f\|_0 \le s$. Sparsity plays an important role in compressive signal processing because compressible signals are those which are approximated well by sparse signals. In general, a signal can be sparse in this sense or it can be sparse with respect to some orthonormal or overcomplete basis, in which case $f = Dx$ for some matrix "dictionary" D and sparse vector x. Fortunately, many important signals in practical applications are known to have so-called sparsifying dictionaries. For example, natural images can be sparsely represented using a

[1] This content supported in part by a grant from MathWorks.

[2] My husband and his brother sometimes call these "dueling" turkeys and have a battle to see whose is better.

wavelet basis.[3] The details of these bases will not be important for this article; just knowing that they exist and give sparse representations for interesting signals is enough. For the rest of the article, we will simply assume that D is the identity matrix, so that $f = x$ is a sparse vector itself.

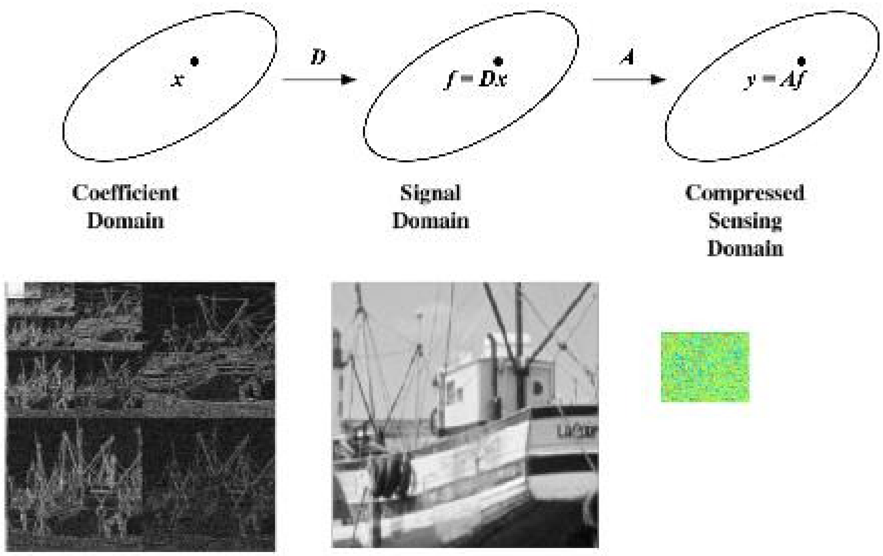


Figure 1. Summary of the compressive signal processing process. Here, D is a Haar wavelet transformation.

Given a compressible signal, we acquire the measurements by applying a wide sampling matrix $A \in \mathbb{C}^{m \times n}$, where $m \ll n$ (we want $m \ll n$ since m will be the size of the compression and n is the original dimension). The measurement vector can then be written as $y = Af + e \in \mathbb{C}^m$, where e is an arbitrary noise vector which usually cannot be avoided in practice. The compressive signal processing problem is to reconstruct an arbitrary compressible signal f from these noisy samples *using a computationally efficient algorithm*. That is, given knowledge of the measurements y and the measurement matrix A, we wish to (approximately) reconstruct the signal vector f.

This is challenging for several reasons. First, the noise vector e added to the measurements means we are unlikely to have any hope of ever exactly solving for f. But even ignoring the noise for a moment, the system $y = Af$ is a highly underdetermined system of m equations in n variables. Since $m \ll n$, the matrix A maps many vectors to the zero vector,[4] so there isn't a unique solution to the system; in fact, given y, there are infinitely many vectors x that satisfy $y = Ax$. However, all hope isn't lost! Remember, we aren't searching for just any solution. Rather, we seek a *sparse* solution. This means we seek a solution that will have many zero entries, although we don't know which entries are zero, nor do we know the magnitudes of the other entries. The overall process is illustrated in Figure 1, where we see the original image f (center), its sparse representation (left – black pixels represent zero in this image), and its compressed version (right).

[3] There are many kinds of wavelets, including Daubechies wavelets, pioneered by the amazing professor Ingrid Daubechies. I encourage the interested reader to google her name and watch one of her accessible and informative lectures. You can also read a 5-part interview with her in this *Bulletin*, Volume 1, Number 6 and Volume 2, Numbers 1-4.

[4] In the parlance of linear algebra, the matrix A has a large null space.

Methods in compressed sensing

Let us now turn to methods for solving the compressed sensing problem. Recall that the actual problem is to solve the noisy underdetermined linear system $y = Af + e$ for an s-sparse vector f. Two key questions must be asked: (i) what types of measurement matrices A can we use? and (ii) how do we actually solve the problem?

Let's begin by first ignoring the noise, that is, $e = 0$. Let's suppose we can get our hands on a matrix A that is one-to-one on all s-sparse signals, or equivalently, that there is no $2s$-sparse signal that A maps to the zero vector. Then, consider solving the problem by searching for the *sparsest* signal x that matches the measurements y. Mathematically, our solution is $\hat{x}$ defined by:

(1) $$\hat{x} \equiv \arg\min_x \|x\|_0 \text{ such that } Ax = y.$$

Recalling that f, the desired solution, is s-sparse, it follows immediately that this problem gives us what we want, namely that $\hat{x} = f$. Woohoo! So have we finished talking about solving the compressed sensing problem? Unfortunately, solving (1) in practice is NP-Hard, meaning that we do not know of any polynomial time algorithm to solve it. This makes it very impractical – often impossible – to use in practice. So although this idea gives us a nice result theoretically, we need a different approach if we want to use compressed sensing in practice.

So what shall we try next? We first remark that the reason (1) is hard to solve in practice is that the "L_0-norm"[5] is not convex. Indeed, draw the set of all 1-sparse vectors in two dimensions and note that this set is not convex (recall that a set is convex if the line connecting any of its two points is also completely contained in the set). It turns out that so-called convex programming methods are quite efficient for solving convex problems, so if we could find a convex formulation of our compressed sensing problem, this would yield a practical solution. You may recall from the previous installment of this series that we define the L_p-norm $\|\cdot\|_p$, for $0 < p \le \infty$, by $\|x\|_p \equiv \left(\sum_i x_i^p\right)^{1/p}$, and we define the L_p-ball as the set of all vectors whose L_p-norm is less than or equal to 1. Note that as $p \to 0$, the L_p-norm approaches the L_0-norm, giving it its name. Now try sketching the L_p-ball for $p = 1/2$ in dimension 2. Is it convex? Your answer should be no, as it looks like a diamond whose sides have been "pinched in." What about for $p = 2$? There your answer should be yes, as you get the circle, which is convex. So here is the penultimate question: What is the smallest value of p for which the L_p-ball is convex? After some experimenting, you might see that $p = 1$, which gives the diamond, is the smallest p for which we get a convex ball. So the ultimate question becomes this: Can we use the L_1-norm to solve the compressed sensing problem? In other words, would the small adjustment of (1) work:

(2) $$\hat{x} \equiv \arg\min_x \|x\|_1 \text{ such that } Ax = y.$$

Let's draw a few pictures to get some intuition. Since this article is printed on two-dimensional paper, we will begin by drawing in two dimensions. So, let's consider a 1-sparse vector f in two dimensions, for example the one that is drawn in Figure 2 (left). We know that f is a solution to $y = Af$, which means that it lies on the line described by $\{z : Ax = y\}$; see Figure 2 (right). The solution to (2) seeks the vector x that lives on this line and has the smallest L_1-norm. To find such a vector, we can draw (scaled) L_1-balls starting small and growing larger until one intersects the line. The point on the line with the smallest L_1-norm will be precisely

[5] The "L0-norm" is not actually a norm, but is commonly (yet inappropriately) called this in the literature.

Girls!

Learn Mathematics!

Make new Friends!

Meet Professional Women who use math in their work!

Improve how you Think and Dream!

Girls' Angle

A math club for ALL girls, grades 5-12.

girlsangle@gmail.com
girlsangle.org

Girls' Angle

little space for ice cream in the cone, but lots of space for your hand to move around).[6] This means that if we were to randomly orient the line (which in high dimensions becomes a hyperplane), the likelihood that we would obtain an intersection like that drawn in Figure 4 is *extremely small*. Therefore, if our signal f is sparse in high dimensions, and if we randomly design the measurement matrix A (thus, the hyperplane is random), then we can use (2) to recover the desired solution f with high probability. This claim is rigorously proven in several works [2, 3, 6, 1] that show, for example, that A can be taken to be a matrix with Gaussian entries and m on the order of $s \log(n)$. Notice that such values of m provide significant compression since the size of the compression, m, is only logarithmic in the dimension n.

This is a deep and non-trivial result. So while this is sinking in, let us settle perhaps the last remaining seed of doubt by the astute observer, namely that we have ignored the noise term e in the measurements $y = Af + e$. When we introduce noise, we no longer "know" the exact hyperplane $y = Af$, but we know that f lives in some "tube" whose radius is $r \equiv \|e\|_2$. This is visualized in Figure 5. However, we may play this same argument with the tube instead of the line; notice that although we may not "hit" the exact point f, we will hit some point nearby, since the tube and the L_1-ball have only a small region of intersection. This is also proved rigorously [2], and shows that with the same assumption on the matrix A, f can be reconstructed to within an accuracy that scales like $\|e\|_2$.

The takeaway message

Compressed sensing demonstrates that efficient methods exist that allow one to reconstruct a signal or image from compressed measurements. The applications are abundant and range from medical and hyperspectral imaging to wireless communications and environmental sensing. The underlying mathematics that makes this all possible is some of the beauty and surprise of high-dimensional geometry. So next time you want to get a shorter MRI or take a cheaper photograph, think of your friend, the spiky L1-ball ◇.

References

[1] E. J. Candès and Y. Plan, A probabilistic and RIPless theory of compressed sensing. *IEEE T. Info. Theory*, Vol. 57, Issue 11 (2010), 7235-7254.

[2] E. J. Candès, J. Romberg, and T. Tao, Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math*, Vol. 59, Issue 8 (2006), 1207-1223.

[3] E. J. Candès and T. Tao, Decoding by linear programming. *IEEE T. Info. Theory*, Vol. 51 (2005), 4203-4215.

[4] Y. C. Eldar and G. Kutyniok, *Compressed Sensing: Theory and Applications*. Cambridge University Press, 2012.

[5] S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*. Springer, 2013. In press.

[6] M. Rudelson and R. Vershynin, On sparse reconstruction from Fourier and Gaussian measurements. *Comm. Pure Appl. Math.*, Vol. 61 (2008), 1025-1045.

[6] Also, see Anna's Math Journal on page 12.

Anna's Math Journal

By Anna B.

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Anna convinces herself of a claim made in Deanna Needell's article on page 10.

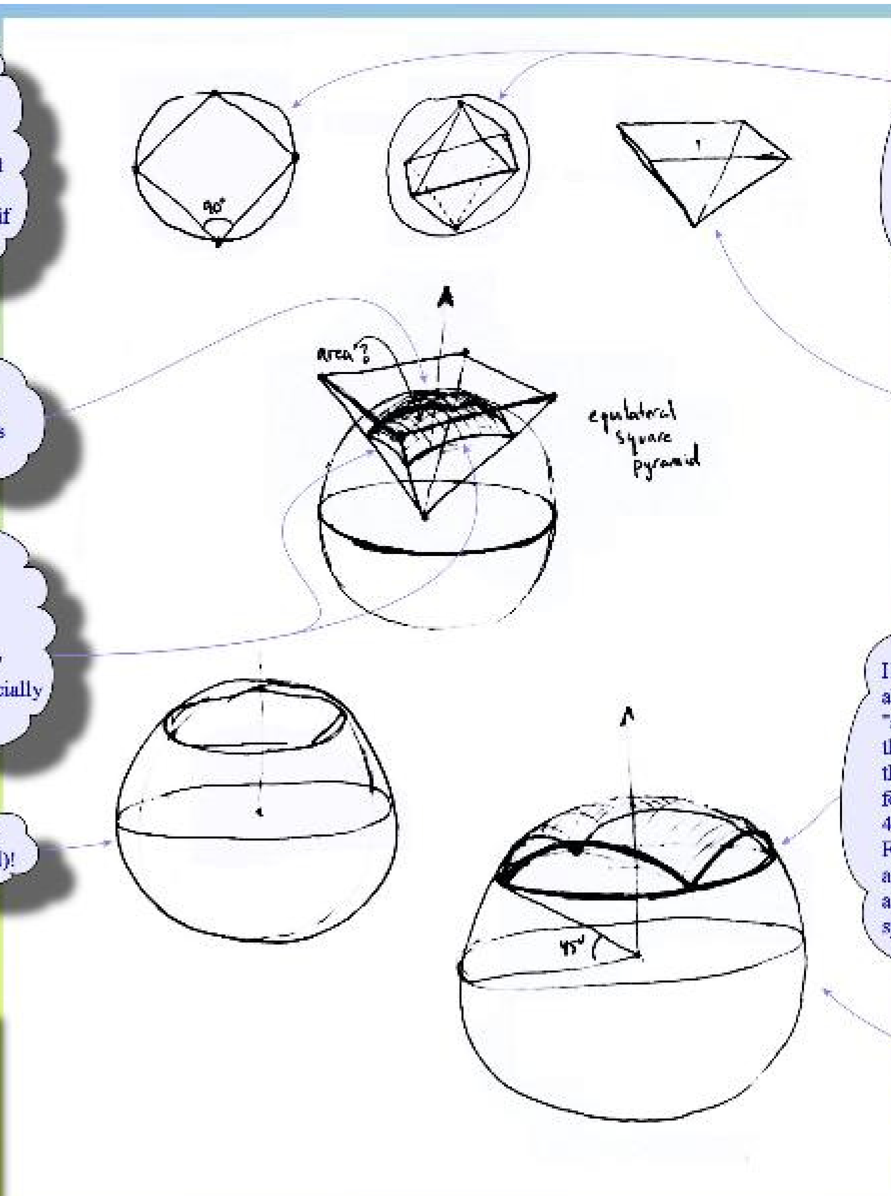
Deanna says the corners of an L_1-sphere have an angular measure that tends to zero as a fraction of the full spherical angle. This fact isn't clear to me, so I wanted to think about it and decide if I agree with it or not.

L_1-spheres are also known as hyperoctahedra. The 1D case is a square. The 2D case is the regular octahedron, and so on. I know the angle in the square is 90°, or a quarter of a full circle. But, I don't know what the spherical angle is at a vertex of a regular octahedron. I guess I'll start there.

Here's the bottom half of a regular octahedron.

What I want to know is, if I center a sphere at the vertex, what fraction of the sphere is cut out by the octahedron?

The sides of the octahedron intersect the sphere in great circle arcs. It feels a little unpleasant to determine the area of the "spherical square" these four arcs enclose, especially in higher dimensions.



Oops...bad drawing (because the sphere doesn't look round)!

Hey! Maybe I don't have to worry about the exact area of this "spherical square". It seem to me that the whole "square" sits above the latitude line that contains its four corners. That would be the 45° latitude line above the equator. From Deanna's previous article, as the dimension increases, more and more of the area of the hyper-sphere sits close to the equator.

That's my strategy! I'll try to show that a corner of a hyperoctahedron cuts off a region of the hypersphere whose points are all above 45° of latitude.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

Let's see. The vertices of the unit L_1-sphere are the points with all coordinates equal to zero, except in one coordinate, where the value is 1 or -1.

Hyper Octahedral vertices:
$$(\pm 1, 0, 0, \dots, 0)$$
$$(0, \pm 1, 0, \dots, 0)$$
$$(0, 0, \pm 1, 0, \dots, 0)$$
$$\vdots$$
$$(0, \dots, 0, \pm 1)$$

Points on the boundary satisfy that the L_1-norm is equal to 1.

Boundary: $(x_1, x_2, x_3, \dots, x_n)$
s.t. $|x_1| + |x_2| + |x_3| + \dots + |x_n| = 1.$

If I let v by the lower vertex, P be the origin, and Q be any point on the hyperoctahedron that is on the hyperplane that is the perpendicular bisector of the line segment connecting the lower and upper vertex, then the claim would be that angle PvQ is less than or equal to 45°.

Let $V = (0, 0, 0, \dots, 0, -1)$
$P = (0, 0, 0, \dots, 0)$ (origin)
Q = point on boundary in crossection by hyperplane $x_n = 0$, i.e. $(q_1, q_2, \dots, q_{n-1}, 0)$ s.t. $|q_1| + |q_2| + \dots + |q_{n-1}| = 1.$

Claim: $\theta = m\angle PVQ \le 45°$

Thinking of everything as position vectors, I can compute the angle using the dot product.

$$(P-V)\cdot(Q-V) = (0, \dots, 0, 1)\cdot(q_1, q_2, \dots, q_{n-1}, 1) = 1$$

$$(P-V)\cdot(Q-V) = |P-V|\,|Q-V| \cos\theta = 1 \cdot \sqrt{1+q_1^2+q_2^2+\dots+q_{n-1}^2}\,\cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{1+q_1^2+q_2^2+\dots+q_{n-1}^2}}$$

The cosine of 45 is 1 over root 2, so if I can show that this denominator is greater than or equal to root 2, that would establish my claim.

Hm. I'd like to be able to say that this is less than or equal to 2, but the sum of mixed products might have negative terms, and those would *increase* the value of this expression.

$$1 + q_1^2 + q_2^2 + \dots + q_{n-1}^2 = 1 + (q_1 + q_2 + \dots + q_{n-1})^2 - 2\sum_{1 \le i < j \le n-1} q_i q_j$$
$$\le 1 + (|q_1| + |q_2| + \dots + |q_{n-1}|)^2 - 2\sum_{1 \le i < j \le n-1} q_i q_j$$

Oh! I know what I can do. I can put in the absolute values right away because x squared is the absolute value of x, squared.

$$1 < 1 + q_1^2 + q_2^2 + \dots + q_{n-1}^2 = 1 + |q_1|^2 + |q_2|^2 + \dots + |q_{n-1}|^2$$
$$= 1 + (|q_1| + |q_2| + \dots + |q_{n-1}|)^2 - 2\sum_{1 \le i < j \le n-1} |q_i||q_j|$$
$$= 2 - 2\sum_{1 \le i < j \le n-1} |q_i||q_j| \le 2$$

$$\cos\theta \ge \frac{1}{\sqrt{2}} \Rightarrow \theta < 45°.$$

That worked out quite nicely! So Deanna's right, assuming what she said about most of a hypersphere being near its equator in her previous article.

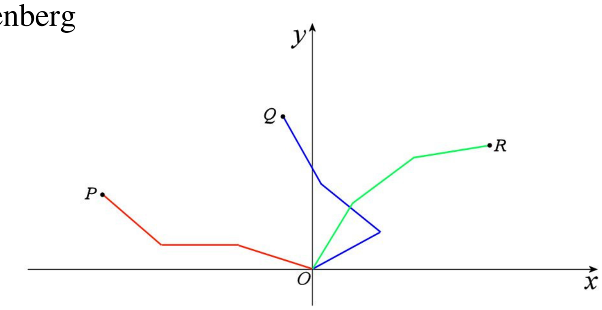
ABB 10/27/18

Key:
Anna's thoughts
Anna's afterthoughts
Editor's comments

Umbrellas, Part 1

by Ken Fan, Milena Harned, and Miriam Rittenberg

While doing math together, we stumbled upon the following question: In a coordinate plane, for fixed whole number n, what points can you reach if you start at the origin and take precisely n steps, where each step is of unit length and has a nonnegative vertical displacement?



Let's call the set of points reachable in n such steps U_n. The figure above shows three reachable points P, Q, and R in U_3, together with a way to get to each point from the origin $O \equiv (0, 0)$ using the allowed types of steps. Because every step must have nonnegative vertical displacement, it is impossible to reach any point in the lower half of the plane. And since every step is of unit length and the straight line path is the shortest between two points, U_n must be contained in the circle of radius n centered at O. Are all points in the upper semicircular disc of the circle of radius n and centered at O reachable? As we will show, the answer is no.

As an aside, if the restriction on the direction of the steps is dropped, so that we can take our steps in any direction we wish, then all points of the circular disc of radius n centered at O are reachable, except when $n = 1$. When $n = 1$, only the boundary of the unit circle is reachable.

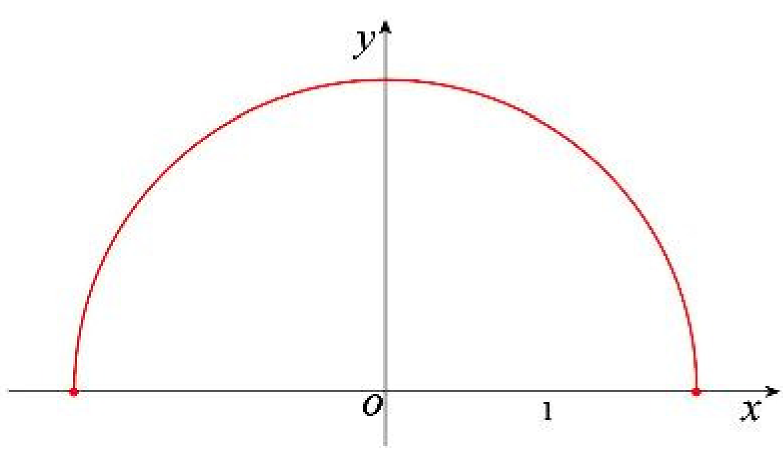
Suppose our kth step is a unit step in the direction α_k (given as an angle, in radians, as measured counterclockwise from the positive horizontal axis). Since we require that every step have nonnegative vertical component, we know that $0 \le \alpha_k \le \pi$ for all $k = 1, 2, 3, \ldots, n$. With the kth step, we translate by $(\cos \alpha_k, \sin \alpha_k)$. Therefore, U_n is the set of points in the coordinate plane of the form

$$(\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3 + \ldots + \cos \alpha_n, \sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3 + \ldots + \sin \alpha_n).$$

If you know about complex numbers, another way to describe U_n is as the set of points in the complex plane of the form $e^{i\alpha_1} + e^{i\alpha_2} + e^{i\alpha_3} + \ldots + e^{i\alpha_n}$, where $0 \le \alpha_k \le \pi$ for all $k = 1, 2, 3, \ldots, n$.

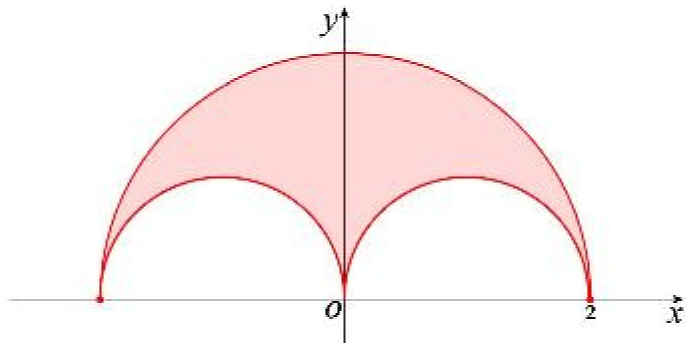
The case $n = 1$

When $n = 1$, you may only take one unit step from O, so U_1 is the upper half of the unit semicircle centered at O (including the endpoints (1, 0) and (-1, 0)). See the image at right. (When we say that a semicircle is centered at a point C, we mean that C is the center of the circle that has the semicircle as an arc.)



The case $n = 2$

Let's analyze the case $n = 2$ in detail. What are all the points within two steps of O?



After one step, we end up at some point P on U_1. From P, the points we can reach with our second step form the upper half of a unit semicircle centered at P. In other words, the points of U_2 are those swept out by a unit semicircle whose center follows the semicircular arc U_1. If you sketch this, you'll get a region that looks like the shaded region at right. It consists of a semicircular disc of radius 2 centered at O with two semicircular discs each of radius 1 centered at (-1, 0) and (1, 0), respectively, removed. More precisely, let

$$S_2 = \{ (x, y) \mid y \geq 0, x^2 + y^2 \leq 4, (x + 1)^2 + y^2 \geq 1, \text{ and } (x - 1)^2 + y^2 \geq 1 \}.$$

It appears that $U_2 = S_2$. Let's prove this rigorously.

First, suppose P is in U_2, so $P = (\cos\alpha + \cos\beta, \sin\alpha + \sin\beta)$, where $0 \leq \alpha, \beta \leq \pi$. We will verify that each of the defining inequalities for the set S_2 are satisfied in the order that we wrote them in the definition.

First, since $\sin\alpha$ and $\sin\beta$ are both nonnegative, so is the vertical coordinate of P.

Next, we compute

$$\begin{aligned}(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 &= \cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta \\ &= 2 + 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \\ &= 2 + 2\cos(\alpha - \beta) \leq 4.\end{aligned}$$

Also

$$\begin{aligned}(\cos\alpha + \cos\beta + 1)^2 + (\sin\alpha + \sin\beta)^2 &= 3 + 2\cos\alpha + 2\cos\beta + 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \\ &= 1 + 2(1 + \cos\alpha)(1 + \cos\beta) + 2\sin\alpha\sin\beta \geq 1,\end{aligned}$$

where the last inequality follows since $1+\cos\alpha$, $1 + \cos\beta$, $\sin\alpha$, and $\sin\beta$ are all nonnegative.

Similarly,

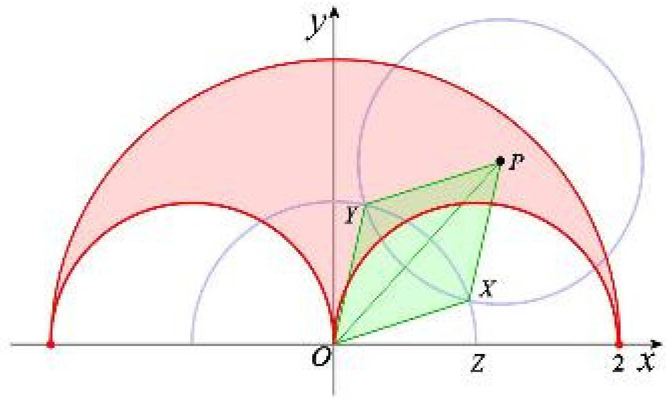
$$\begin{aligned}(\cos\alpha + \cos\beta - 1)^2 + (\sin\alpha + \sin\beta)^2 &= 3 - 2\cos\alpha - 2\cos\beta + 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \\ &= 1 + 2(1 - \cos\alpha)(1 - \cos\beta) + 2\sin\alpha\sin\beta \geq 1,\end{aligned}$$

where, this time, the last inequality follows because $1 - \cos\alpha$, $1 - \cos\beta$, $\sin\alpha$, and $\sin\beta$ are all nonnegative.

Since the four defining inequalities of S_2 are satisfied, we see that U_2 is contained in S_2.

Now suppose P is in S_2. We will show that P is in U_2. We'll think of coordinates as position vectors in this argument. Suppose $P = (a, b)$. We exploit the mirror symmetry (about the vertical axis) by assuming that a is nonnegative. If $P = O$, then we can see that P is in U_2 by expressing O as $(1, 0) + (-1, 0)$. So assume that P is not the origin. If the distance of P from the origin is 2 units, then we can see that P is in U_2 by writing $P = P/2 + P/2$. So assume further that P is strictly within the circle of radius 2 centered at O.

We first show that $P = X + Y$ where X and Y both have unit length. Because P is within 2 units of O, but not equal to O, we know that the unit circles centered at O and P intersect in exactly two points, which we'll label X and Y in such a way that the vertical coordinate of X does not exceed that of Y. Then O, X, Y, and P form the vertices of a rhombus with side length 1. Since all rhombi are parallelograms, we know that $P = X + Y$.



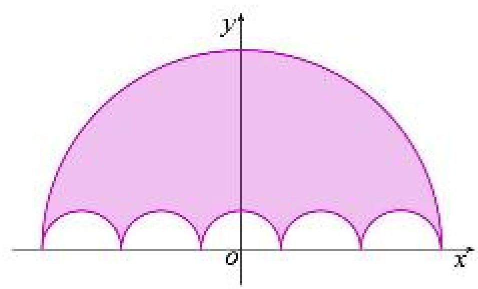
Next, we will show that the vertical coordinate of X is, in fact, nonnegative. This is equivalent to showing that $m\angle XOP \le m\angle POZ$, where $Z = (1, 0)$. Since P is in the first quadrant, this is, in turn, equivalent to showing that $\cos m\angle XOP \ge \cos m\angle POZ$. Observe that $\cos m\angle XOP$ is $\sqrt{a^2+b^2}/2$ (the diagonals of a rhombus are perpendicular bisectors) and $\cos m\angle POZ$ is $a/\sqrt{a^2+b^2}$. Therefore, we desire that $\sqrt{a^2+b^2}/2 \ge a/\sqrt{a^2+b^2}$. This can be rearranged to $a^2 + b^2 \ge 2a$, or $(a-1)^2 + b^2 \ge 1$, which is true for P in S_2. Therefore, P is in U_2, as desired.

A symmetric argument gives us the same conclusion if P is in the second quadrant.

Since S_2 is contained in U_2 and U_2 is contained in S_2, we must have $U_2 = S_2$.

The general case

In general, define S_n to be the region in the coordinate plane inside the upper half of the circle of radius n centered at the origin, but outside a row of unit circles centered at points along the horizontal axis. More precisely, define



$$S_n \equiv \{\ (x, y) \mid y \ge 0,\ x^2 + y^2 \le n^2, \text{ and } (x - (n + 1 - 2k))^2 + y^2 \ge 1 \text{ for } k = 1, 2, 3, \ldots, n\ \}.$$

The illustration shows S_5. The shape of S_n reminded us of the profile of an umbrella, hence the title.

In the sequel, we will prove that $U_n = S_n$ by induction on n. Can you prove this before the next issue appears?

Learn by Doing

Cardinality

by Michael Kielstra | edited by Amanda Galtman

What is the size of the biggest set?

By "size," I do not mean physical size. Instead, I'm referring to a measure of the quantity of elements that the set contains. For example, all sets with a single element are to be considered the same size, whether that element be an elephant or a point in space.

If we consider only finite sets, we can determine size by counting elements. For example, the set of capital letters in the alphabet has 26 elements, so we could say that it has size 26. Since every whole number can be increased by adding one to it, there is no "biggest" finite set. Any finite set can be made bigger by tossing a new element into it.

But what happens if we include infinite sets? After all, we frequently work with them. For example, consider the set of integers, the set of polynomials, the set of polygons, the set of dots-and-boxes starting configurations, the set of mathematical equations, and so on. Is there such a thing as a biggest infinite set? How can we even compare the size of one infinite set to another? Indeed, how can we even measure the *size* of an infinite set? Should we simply say that an infinite set has infinite size and be done with it?

Suppose S and T are infinite sets. We might be tempted to say that T is bigger than S if S is a proper subset of T, in other words, if every element of S is an element of T and there exist elements in T that are not in S. For example, if S is the set of even integers and T is the set of all integers, then S is a proper subset of T. Should we say T is bigger than S?

One objection to defining "bigger" in this way is that it is *not* compatible with our notion of "bigger" for finite sets. When we say that one finite set is bigger than another, we do not require that the smaller set be a subset of the bigger set. Instead, we count the number of elements in each set to determine its size, and then compare the sizes, not the sets. That is, our notion of size is not related to the nature of the elements in the set. Instead, we regard each element in a set as an equal contributor to the set's size. If, for instance, we were to swap out an element in a set for another, we would consider the resulting set to be the same size as the original.

So, how can we extend the notion of size to infinite sets? One frequently used approach to extending a notion to a new context is to redefine the notion in a way that *can* be applied in the new context.

1. We'll think of a finite set as a bag containing its elements. You're given two finite sets. Can you think of a way to decide which set is bigger or if they are the same size *without counting the number of elements in each set*? (That is, can you compare the sizes of the sets without having to concern yourself with keeping track of a number?)

(Spoiler Alert!) One way to solve Problem 1 is to reach into both bags, pull out one element from each, and repeat until one or both bags have no more elements to remove. If both bags empty at the same time, they had to be the same size. Otherwise, the one which emptied first is smaller.

Another way to put it is that if you can establish a one-to-one correspondence between the two sets, then the sets have the same size. (A one-to-one correspondence is a pairing of the elements of one set with the elements of the other in such a way that each element is paired with exactly one element of the other set. In the previous paragraph, elements removed at the same time form a pair.)

The beauty of using a one-to-one correspondence to establish that two sets are the same size is that this notion can be applied as is to infinite sets.

2. Show that the set of positive perfect squares is the same size as the set of positive integers. (!)

(Spoiler Alert!) The set of positive perfect squares and the set of positive integers are both infinite sets, and, in fact, one contains the other. But we've already seen that being a proper subset is not a good way to compare the sizes of sets. Instead, we can pair the positive integer n with the perfect square n^2 to establish a one-to-one correspondence between these two sets.

3. Let $\mathbb{Z}_{>0}$ be the set of positive integers and let $\mathbb{Z}$ be the set of integers. Show that $\mathbb{Z}_{>0}$ and $\mathbb{Z}$ are the same size.

Any set that can be put into one-to-one correspondence with the set of positive integers is said to be **countably infinite**. If a set can be put into one-to-one correspondence with a subset of the positive integers, it is said to be **countable**. You have just proven that the set of perfect squares and set of integers are both countably infinite.

We have declared that two sets have the same size if their elements can be put into one-to-one correspondence with each other. Two such sets are said to have the same **cardinality**. But for this notion to be a reasonable measure of the size of a set, we would want this measure of size to enjoy the following properties:

> Property 1. (Reflexivity) Every set has the same cardinality as itself.
>
> Property 2. (Symmetry) If set A has the same cardinality as set B, then set B has the same cardinality as set A.
>
> Property 3. (Transitivity) If set A has the same cardinality as set B, and set B has the same cardinality as set C, then set A has the same cardinality as set C.

(In other words, we want the relationship that two sets have the same cardinality to be an **equivalence relation**.)

4. Show that our definition of cardinality enjoys all three of these properties.

Are all infinite sets countable?

5. Let S be the set of infinite sequences $\{a_i\}_{i=1,2,3,\ldots}$ where a_i can be 0 or 1 only. Show that S is not countable. Hint: Suppose, to the contrary, that S is countable. Then we can establish a one-to-one correspondence between elements of S and the positive integers. In other words, we can form a sequence s_1, s_2, s_3, …, such that every element of S is equal to one of the s_k. (Here, each s_k is an infinite sequence of 0's and 1's.) For example, s_1 might be the sequence that begins with

a 1, but then every term after is 0, and s_2 might be the sequence that begins 0, 1, but then every term after is 0. Whatever the s_k specifically are, define a sequence $\{b_i\}_{i = 1, 2, 3, \ldots}$ by setting b_i to 1 if the ith term of s_i is 0, and 0 if the ith term of s_i is 1. Then $\{b_i\}_{i = 1, 2, 3, \ldots}$ is a sequence in S and must be equal to s_k for some k. Which s_k is equal to $\{b_i\}_{i = 1, 2, 3, \ldots}$?

The argument implied by the hint of Problem 5 is known as **Cantor's diagonalization argument**, after Georg Cantor. A set that is not countable is said to be **uncountable**.

Let $0 \le x < 1$ be a real number. If we express x in binary, we get a sequence of 0's and 1's by looking at its binary digits after the binary point. This is almost the situation of Problem 5. The difference is that some numbers do not have a unique binary expression. For example, as binary numbers, $0.0\overline{1} = 0.1$, whereas in Problem 5, the sequences 0, 1, 1, 1, … and 1, 0, 0, 0, … are distinct. Nevertheless, Cantor was able to address these technicalities and use his diagonalization argument to show that the set of real numbers is uncountable.

Try your hand at determining the cardinality of sets.

6. Is the set of rational numbers countable?

7. Is the set of quadratic polynomials $ax^2 + bx + c$, where a, b, and c are integers, countable?

8. Is the set of polynomials with rational coefficients countable?

9. Is the set of all roots of polynomials with rational coefficients countable?

10. Let P be the set of ordered pairs (x, y), where x and y are real numbers. Show that P has the same cardinality as the set of real numbers.

11. Tweak your solution to Problem 5 to show that the set of real numbers is uncountable.

12. Let S be a set. The **power set** of S is the set of all subsets of S. Show that the power set of S does not have the same cardinality as S.

13. Unlike Problem 12, given a countably infinite set S, let F be the set of all *finite* subsets of S. Show that F and S have the same cardinality.

14. Let A_k be a countably infinite set for each $k = 1, 2, 3, \ldots$. Let B be the union of all the A_k. Show that B is countable.

15. Construct a partition of $\mathbb{Z}_{>0}$ into infinitely many infinite sets. (A partition of a set S is a collection of subsets of S with the property that ever element of S is in precisely one of the subsets in the collection.)

We close with the Schröder–Bernstein theorem, which gives an often handy way to prove that two sets have the same cardinality.

16. (Schröder–Bernstein) Let S and T be two sets. An injective function is a function that maps distinct elements to distinct elements. Suppose that there is an injective function f from S to T and an injective function g from T to S. Prove that S and T have the same cardinality.

Stacked Circles, Part 4

by Ken Fan | edited by Jennifer Silva

Jasmine: This is painful!

Emily: I wonder why arithmetic and geometric sequences of circles worked out so nicely, but this harmonic sequence eludes us.

Jasmine: I can't think of another way to look at this problem, so I have no insight into why one works out and another doesn't, except to say that when stacking circles into an angle, similarity glues everything together.

Emily: Actually, this quest for a nice container for a stack of circles whose radii are in harmonic progression is a bit odd. Still, let's at the very least check to see if the logarithmic container does the job or not.

Jasmine: Okay. How do you propose we do that?

Emily: I guess we can write a computer program that determines the radius of the next circle in a stack. Though the program will have inherent error in its computations, I have a feeling that we will still be able to see that the sequence cannot be the desired harmonic sequence. You look sad, Jasmine!

Jasmine: Sorry. I *am* disappointed that we're contemplating writing a computer program to get an anticipated negative result. Plus, if we go through the process of writing a computer program – even if we do learn that the logarithm isn't the desired container – it probably won't give us further insight into the problem.

Emily: You're right, and I understand. But things feel stuck, and I don't think the computer program will take long to write. Besides, it will hopefully tell us that the logarithm doesn't work. I'm dying to know!

Jasmine: All right. Let's write the program.

Emily: Great! It helps us that the center of the circle and its radius are monotonically related. As the circle rises up the vertical axis, its radius shrinks.

Jasmine: Yes, that's good. It means that for each radius, there is a unique circle. But I think what we need is an algorithm that, given one circle in the stack, computes the next.

Emily: So how about this: Let r_0 be the radius of the given circle. We then halve r_0 repeatedly until we find a circle that is fully above the given circle. That would mean we've overshot the radius of the next circle in the stack. That is, we let $r_1 = r_0/2$ and check if the circle with radius r_1 is too high. If it isn't, we let $r_2 = r_1/2$ and see if the corresponding circle is too high. We keep going until we find r_n, the first circle with radius r_0 divided by a power of 2 that is too high. We then let r_{n+1} be the average of r_n and r_{n-1}. If the circle with radius r_{n+1} is too high, we let r_{n+2} be the average of r_{n-1} and r_{n+1}. Otherwise, we let r_{n+2} be the average of r_n and r_{n+1}. That is, we extend the sequence of radii by picking the next radius to be the average of the last two values that most tightly sandwich the desired radius. This way, we will double the accuracy of our estimate with each iteration, quickly exceeding the computational accuracy of the computer.

Jasmine: That sounds fine to me. We need the equation we found last time that related the radius of a circle to the vertical coordinate of its center. Here it is:

$$c = -\frac{-N+\sqrt{N^2+4r^2}}{2} - N\ln\sqrt{\frac{-N^2+N\sqrt{N^2+4r^2}}{2}},$$

where N is a fixed parameter that scales our logarithmic graph in the vertical direction, r is the radius of the circle, and c is the vertical coordinate of its center.

Emily: We can leave N as a parameter in the program. I'm thinking for the purposes of the program, it might make sense to express c like this:

$$\begin{aligned}
c &= -\frac{-N+\sqrt{N^2+4r^2}}{2} - N\ln\sqrt{\frac{-N^2+N\sqrt{N^2+4r^2}}{2}} \\
&= -\frac{-N+\sqrt{N^2+4r^2}}{2} - \frac{1}{2}N\ln\frac{-N^2+N\sqrt{N^2+4r^2}}{2} \\
&= -\frac{-N+\sqrt{N^2+4r^2}}{2} - \frac{1}{2}N\ln(N\frac{-N+\sqrt{N^2+4r^2}}{2}) \\
&= -\frac{-N+\sqrt{N^2+4r^2}}{2} - \frac{1}{2}N\left(\ln N + \ln\frac{-N+\sqrt{N^2+4r^2}}{2}\right) \\
&= -\frac{1}{2}\left(N\ln N + \left(-N+\sqrt{N^2+4r^2}\right) + N\ln\left(-N+\sqrt{N^2+4r^2}\right) - N\ln 2\right).
\end{aligned}$$

I'm just trying to reduce the number of times we have to extract roots. Also, we can compute the quantity $-N+\sqrt{N^2+4r^2}$ just once.

Jasmine: It might make sense to combine all the logarithms so that we only have to use the logarithm function once. So if we set $M = -N+\sqrt{N^2+4r^2}$, then

$$\begin{aligned}
c &= -\frac{1}{2}\left(N\ln N + \left(-N+\sqrt{N^2+4r^2}\right) + N\ln\left(-N+\sqrt{N^2+4r^2}\right) - N\ln 2\right) \\
&= -\frac{1}{2}\left(N\ln N + M + N\ln M - N\ln 2\right) \\
&= -\frac{1}{2}\left(M + N\ln\left(NM/2\right)\right).
\end{aligned}$$

Emily: That sounds fine to me. I'll start writing the code.

Emily pulls out her laptop and writes a computer program that implements the algorithm they just described. Jasmine peers over her shoulder trying to catch any errors.

Emily: Okay, that should do it.

Jasmine: Let's start with $N = 2$, since that's the value for N that should correspond most closely to the radii forming the harmonic sequence 1/1, 1/2, 1/3, 1/4, etc.

Emily types a few commands into the computer.

Emily: Okay, here's what I get for the approximate radii and centers of the first few circles, starting with the circle of radius 1:

Vertical Coordinate of Circle Center	**Radius of Circle**
−0.22598715591350	1
1.28393567366916	0.509922829582653
2.13318693070515	0.339328427453341
2.72632928824970	0.253813930091216
3.18275145862090	0.202608240279971
3.55391423079760	0.168554531896724
3.86675273712960	0.144283974435282
4.13715183469639	0.126115123131513
4.37527302547830	0.112006067650371
4.58801295337492	0.100733860246258
4.78026842497393	0.091521611352779

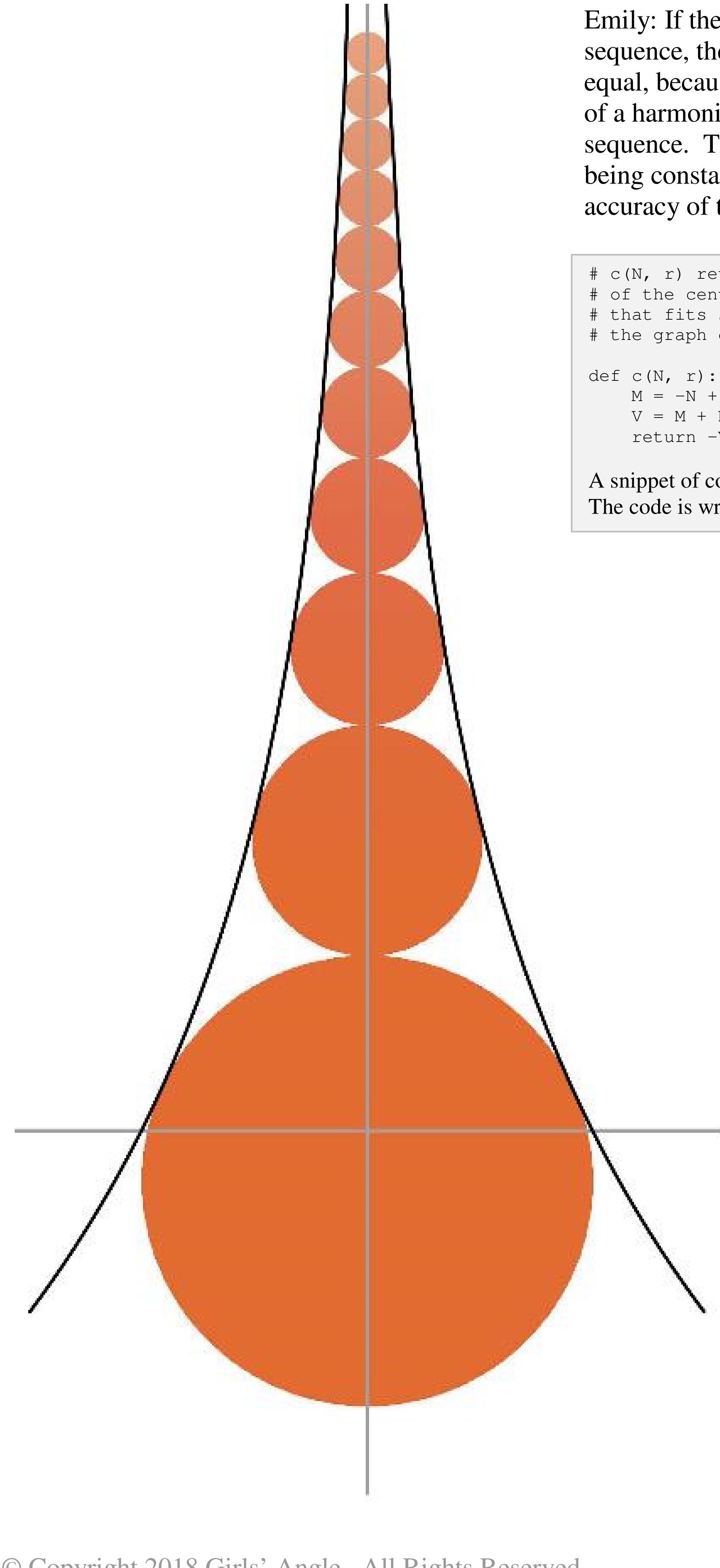
Jasmine: Wow, look at those radii! They are really close to the desired 1/1, 1/2, 1/3, 1/4, …, though not quite! Can you tell the computer to print out the differences between consecutive reciprocals of those radii?

Emily types at the computer.

Emily: Sure, here's the output.

```
0.96108105381054
0.98591636548910
0.99289665838404
0.99573933297210
0.99716465028699
0.99797858222867
0.99848651868743
0.99882458354363
0.99906085741453
0.99923244772416
```

Jasmine: Neat!



Emily: If these radii formed a harmonic sequence, then all those numbers should be equal, because the reciprocals of the terms of a harmonic sequence form an arithmetic sequence. This shows deviation from being constant that is well within the accuracy of the computer's computations.

```
# c(N, r) returns the vertical coordinate
# of the center of a circle of radius r
# that fits snugly into
# the graph of y = -N log x.

def c(N, r):
    M = -N + (N**2 + 4*r**2)**.5
    V = M + N*math.log(M*N/2)
    return -V/2
```

A snippet of code from Emily's program.
The code is written in Python.

Jasmine: So it's not a harmonic sequence, but it sure seems to approach one! At least, those differences appear to be approaching 1. I guess that's expected since $1/n$ approximates $\ln(1 + 1/n)$ better and better as n grows.

Emily: I'm not sure where to go from here.

Jasmine: We haven't found a nice container for a stack of circles whose radii form a harmonic sequence, and I have no idea how to find one. I suppose we could try to tweak the logarithm since that comes pretty close, but the math seems daunting.

Emily: Maybe it's time to give this problem a rest. We might as well synthesize our computations into one last pic, though.

Emily and Jasmine close the chapter on stacked circles with the image at left.

Systematic Counting, Part 2

by Addie Summer | edited by Jennifer Silva

The next day, I was waiting for the bus again. I decided to do more systematic counting, just for fun. Instead of a circle, I made a gridded rectangle – a 4 by 6 rectangle, to be exact:

Then, I decided I'd try to count the squares inside the rectangle by starting in the top left square and stepping through the squares in a southeasterly direction, wrapping around as necessary:

1		9		5	
	2		10		6
7		3		11	
	8		4		12

To my surprise, I wasn't able to count all 24 squares in the rectangle. Instead, the numbered squares formed a checker pattern and I only managed to count half of them. It made me want to try different grid dimensions to see what other patterns I might get.

Luckily for me, the bus was late as usual, so I had time to try the same thing with a 4 by 7 rectangle. This is what happened:

1	9	17	25	5	13	21
22	2	10	18	26	6	14
15	23	3	11	19	27	7
8	16	24	4	12	20	28

All squares counted!

Naturally, I wondered which rectangular grid dimensions would allow all of the squares to be counted. And, more generally, how many squares would be counted?

Content Removed from Electronic Version

With no bus in sight, I thought more about counting through rectangles. Instead of moving through the squares by going 1 to the right and 1 down, with wraparound, I wondered what would happen if I went a squares to the right and b squares down, with wraparound. For example, if $a = 2$ and $b = 1$, then stepping through the squares of a 3 by 5 rectangle would look like this:

1	4	7	10	13
11	14	2	5	8
6	9	12	15	3

And here's another example, with $a = 4$ and $b = 3$ in a 10 by 9 rectangle:

1	71	51	31	11	81	61	41	21
28	8	78	58	38	18	88	68	48
55	35	15	85	65	45	25	5	75
82	62	42	22	2	72	52	32	12
19	89	69	49	29	9	79	59	39
46	26	6	76	56	36	16	86	66
73	53	33	13	83	63	43	23	3
10	80	60	40	20	90	70	50	30
37	17	87	67	47	27	7	77	57
64	44	24	4	74	54	34	14	84

All of a sudden, a whole new set of questions arose: For a fixed n by m rectangle, for which a and b would you step through all the squares? In general, how many squares would be counted? If you fix a and b and always move a to the right and b down, with wraparound, for which rectangle dimensions would one count all of its squares? If the square in row r and column c is counted, when will it be counted? In what row and column is the k^{th} square that is counted?

By the time I managed to sort out the answers to these questions, my bus *still* hadn't arrived! What is with these buses? With more time to wait, I decided to move into the third dimension. Take an n by m by l block of cubes. Start in a corner and march through the cubes by moving 1 to the right, 1 down, and 1 forward, with wraparound. For what n, m, and l will all of the squares be counted? In general, how many squares will be counted?

But before I could think much about this, my bus finally appeared!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 23 - Meet 1 September 13, 2018	Mentors:	Anna Ellison, Alexandra Fehnel, Claire Lazar, Jennifer Matthews, Elise McCormack, Charity Midenyo, Kate Pearce, Laura Pierson, Gisela Redondo, Shohini Stout, Jane Wang, Josephine Yu, Jasmine Zou

We enjoyed a record number of new members.

When a girl arrives at Girls' Angle for the first time, one of the first things she'll do is an interview with one of our mentors. People are so diverse, and we want to know if our new member likes math or hates it, likes a challenge or not, likes to work alone or in groups, etc. We want to know which subjects she likes and which she loathes. We want to know what she hopes to get out of Girls' Angle and what her longer term goals are, especially with respect to math.

Math education is not a one-shoe-fits-all proposition. What works for one could easily fail for another. Based on what we learn from the interview, we begin the process of constructing a math project or activity that will resonate with her. This process continues for as long as the girl remains at Girls' Angle, though, over time, we aim to have the girl take more and more control over her own mathematical journey, so that, hopefully, when she leaves Girls' Angle, she knows how she best acquires knowledge and achieves understanding and has become her own best teacher.

Session 23 - Meet 2 September 20, 2018	Mentors:	Alexandra Fehnel, Claire Lazar, Kate Pearce, Laura Pierson, Gisela Redondo, Jane Wang, Josephine Yu, Jasmine Zou

An "Egyptian fraction" is a sum of reciprocals of distinct whole numbers. Every positive rational number can be expressed as an Egyptian fraction. For example, $1 = 1/2 + 1/3 + 1/6$. Here are some questions about Egyptian fractions that were contemplated at the club: For fixed n, what rational numbers are expressible as a sum of reciprocals of exactly n distinct whole numbers? What can be said about all the different ways of expressing a given rational number as an Egyptian fraction? What is the "sparsest" subset of the whole numbers that has the property that every positive rational number can be expressed as a sum of reciprocals of distinct elements in the subset?

Session 23 - Meet 3 September 27, 2018	Mentors:	Jacqueline Garrahan, Claire Lazar, Jennifer Matthews, Charity Midenyo, Kate Pearce, Laura Pierson, Gisela Redondo, Jane Wang, Josephine Yu, Jasmine Zou

Jacqueline gave us a fascinating account of her visit to Nepal this past summer. Her journey inspired the following meet challenge problem: If Mt. Everest, which is about 29,000 feet above sea level, were surrounded by ocean, how far would the horizon line be from the summit? The radius of the Earth is approximately 3,960 miles. (If you're having trouble solving this, check out *Haleakalā*, on page 8, Volume 4, Number 1 of this Bulletin.)

Session 23 - Meet 4 October 4, 2018	Mentors:	Grace Bryant, Jacqueline Garrahan, Claire Lazar, Elise McCormack, Kate Pearce, Laura Pierson, Gisela Redondo, Jane Wang

One is the only number with a single (positive) factor and prime numbers are the only numbers that have exactly two. What numbers have exactly three factors? Four factors? Five factors? Etc. Also, for each positive integer n, what is the longest string of consecutive numbers you can find that all have exactly n factors?

As a curiosity, the 7 consecutive numbers 171,893, 171,894, 171,895, 171,896, 171,897, 171,898, and 171,899 all have exactly 8 factors. Can you prove that there does not exist a string of 8 consecutive numbers that all have exactly 8 factors?

Session 23 - Meet 5 October 11, 2018	Mentors:	Grace Bryant, Claire Lazar, Elise McCormack, Charity Midenyo, Kate Pearce, Laura Pierson, Shohini stout, Jane Wang, Josephine Yu, Jasmine Zou

The concept of a variable is so fundamental to mathematics and to problem solving in general. Perhaps it should be regarded as one of the most important concepts of all. Some of our members are at that stage in life where they are on the cusp of grasping the concept. To get there, we try all kinds of things, including variants on 20 questions. See *It is a Variable!* by Timothy Chow on page 7 of Volume 7, Number 2 of this Bulletin.

Also, **Barry Allen** succeeded in finding a formula for the radius of the incircle of an equilateral triangle as a function of its side length. To highlight this, we made an equilateral triangle 6” on a side, then used her formula to compute the radius of its incircle. We made a circle with that radius, then, in front of the whole club, we slipped that circle into the triangle: a most satisfying *perfect* fit!

Session 23 - Meet 6 October 18, 2018	Mentors:	Grace Bryant, Neslly Estrada, Jacqueline Garrahan, Katie Gravel, Adeline Hillier, Claire Lazar, Elise McCormack, Kate Pearce, Jane Wang, Jasmine Zou

Suppose you want to flip a coin to choose between two options. You’d like each option to have an equal chance of being chosen, but, unfortunately, you don’t believe that the coin you have is fair. You’re pretty sure that it comes up heads a little more often than tails, as most coins do, but you don’t actually know the exact probabilities. Using this unfair coin, can you come up with a way to randomly pick between the two options which you can prove gives each an equal chance of being chosen?

Session 23 - Meet 7 October 25, 2018	Mentors:	Grace Bryant, Neslly Estrada, Jacqueline Garrahan, Katie Gravel, Adeline Hillier, Charity Midenyo, Laura Pierson, Shohini stout, Josephine Yu

Fix a positive integer n. Let $F(n)$ be the number of 3-term geometric sequences a, b, c, such that $a \le b \le c$ are integers and $c = n$. Here’s a table for the first few values of n:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$F(n)$	1	1	1	2	1	1	1	2	3	1	1	2	1	1	1	4	1	3	1	2

How can you compute $F(n)$ in general?

Calendar

Session 23: (all dates in 2018)

September	13	Start of the twenty-third session!
	20	
	27	
October	4	
	11	
	18	
	25	
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Session 24: (all dates in 2019)

January	31	Start of the twenty-fourth session!
February	7	
	14	
	21	No meet
	28	
March	7	
	14	
	21	
	28	No meet
April	4	
	11	
	18	No meet
	25	
May	2	
	9	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minute walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________