Girls' Angle Bulletin

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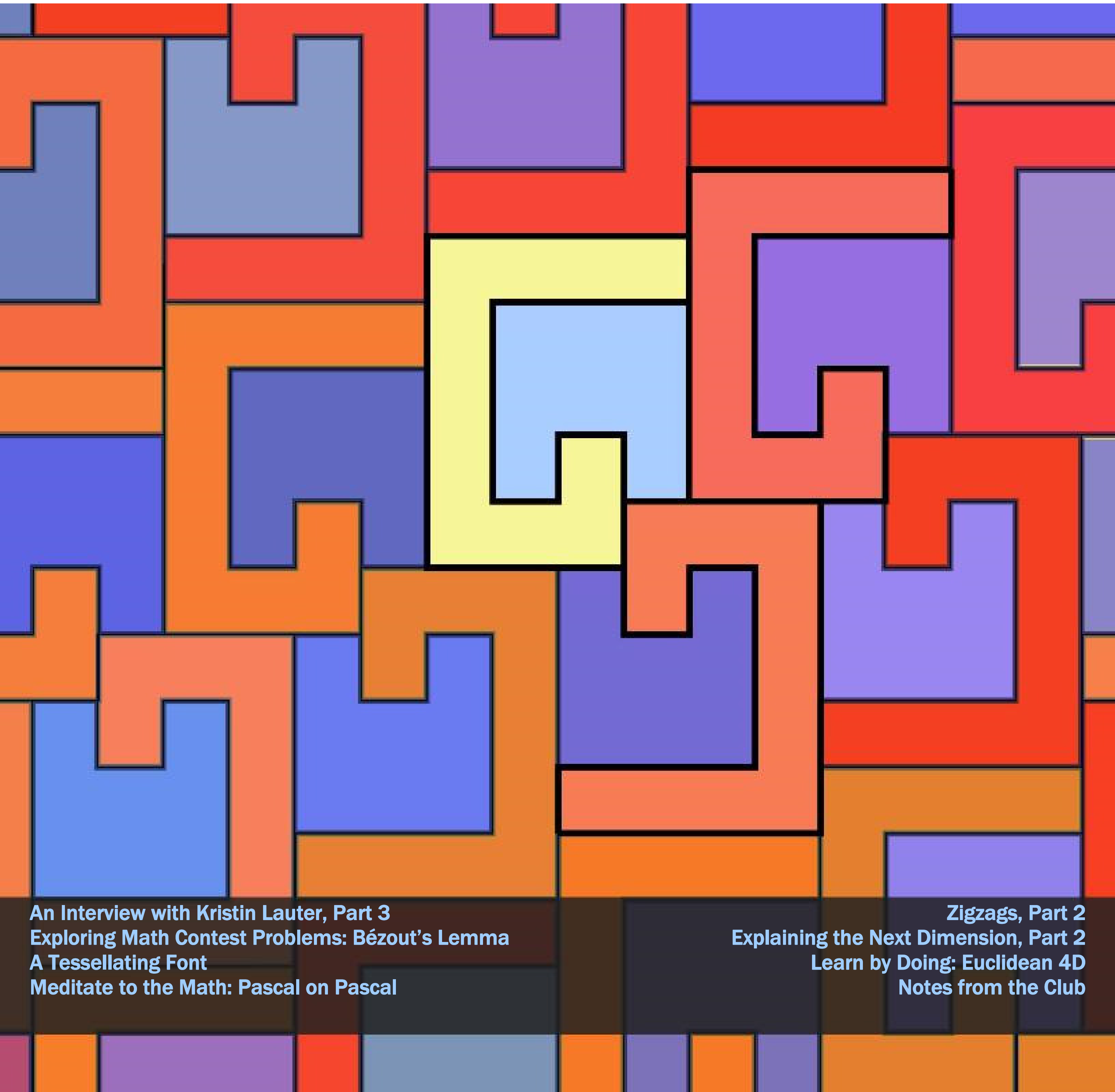
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

If you enjoy roller coaster drama, consider becoming a mathematician. The search for understanding is filled with emotional highs and lows so extreme that you just might find yourself someday jumping out of a bathtub shouting "Eureka!" - Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: G and A tessellation using tiles designed by **Katherine Dawson**. See page 10 for more information.

An Interview with Kristin Lauter, Part 3

This is part 3 of a multi-part interview with mathematician Kristin Lauter, which was conducted by Ke Huang at the University of Washington in April of 2018.

Ke Huang: Today you work at Microsoft Research, but you began your career in academia. What is the difference between academia and industry and what led to your decision to shift into industry?

Kristin: As I alluded to earlier, in the 90s, I enjoyed teaching engineering students who had real connections with industry, because I felt that they had more of a sense of how the mathematics was being used. I knew the mathematics and the foundational stuff, but I didn't know what questions were interesting and what research would be like in industry. And so, basically it was curiosity that led me to try it out. I first came to Microsoft Research as a Postdoctoral Researcher. It was a joint postdoc with MSRI, which is the NSF[1]-funded Mathematical Sciences Research Institute[2] in Berkeley. And so I was supposed to come to Microsoft Research for a year and then I was supposed to go to MSRI for a year. But when I came here I liked the research environment so much and got involved in elliptic curve cryptography right away, so that I never wanted to leave. So, I didn't go to MSRI.

Ke Huang: What was it about the research environment at Microsoft Research that proved so alluring?

Kristin: Well, what I really liked was that everybody had their doors open. Everybody had this expectation that they would be able to talk to anyone and explain their work to anyone, and there were several really concrete instances of that in my first couple of months that kind of surprised me. For example, one thing was that people would just get up and walk out of talks or meetings if the explanations were too esoteric. In academia, we tolerate very extreme levels of incoherence and bad presentation style and choices in talks that are not designed to communicate to a wide audience. But in industry, you simply don't have to listen to that. You just get up and leave. And it made me realize, hey, if I don't understand what the person is talking about and it doesn't have a clear relevance to something of interest or importance, then there's no reason to sit here and keep wasting my time listening. And that was a huge culture shift from academia to industry. And I was really impressed by that. And then a second thing was that I was asked to explain elliptic curve cryptography to lawyers, and I had never really been asked to explain very deep and difficult mathematics to, not just a non-mathematician, but a non-scientist. And the lawyers showed up and they had videotaping equipment and they had booked a couple of hours in a conference room and a whiteboard and everything. And they were really expecting me to explain it so that they could understand it. And I thought, "Oh, great!" I was really inspired by that.

Ke Huang: That's really interesting. I think that when I was younger it would've been really, really helpful for me to realize that I should speak up, and that if I didn't understand something, it might not have been my fault. It could have been the instructor's fault.

Kristin: Yeah, and it gave me the confidence to start asking a lot more questions in math talks at conferences, because I also had a mentor and a role model who was a very famous number

[1] National Science Foundation.

[2] Girls' Angle is grateful to MSRI for providing initial funding critical to enabling Girls' Angle's existence.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Kristin Lauter and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
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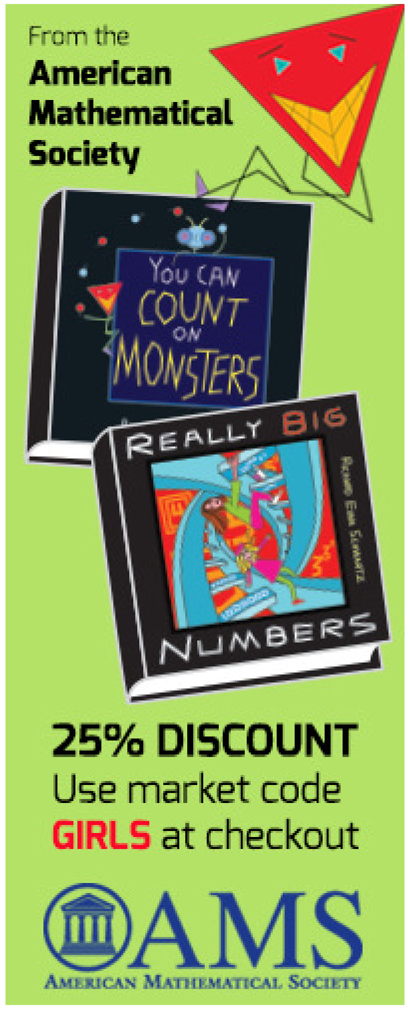

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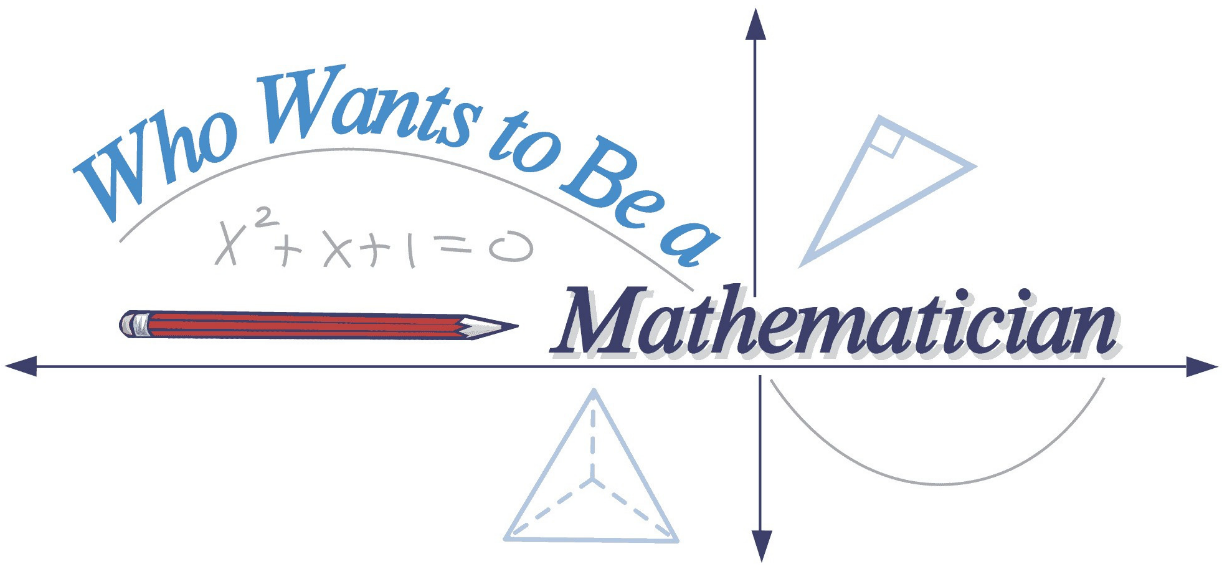

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Exploring Math Contest Problems

Bézout's Lemma

by Marlie J. Kass | edited by Amanda Galtman

Let's take a look at Problem 14 from the 2006 AMC 12A math competition, which is created by the Mathematical Association of America. For your convenience, here is the problem:

> Two farmers agree that pigs are worth \$300 and that goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

To start, let's create an algebraic equation representing the problem. Let x be the smallest positive debt that can be resolved as in the problem statement. Since x can be resolved, it means that it is possible to exchange some pigs and goats so that the net change is x. Let p be the number of pigs and g be the number of goats needed to resolve this debt. We'll take negative values of p to mean that we receive pigs instead of giving them away, and, similarly for negative values of g. We then have

$$x = 300p + 210g.$$

Notice that the greatest common factor of 300 and 210 is 30, so we can change the equation to:

$$x = 30(10p + 7g).$$

Since p and g are integers, so is $10p + 7g$. Hence, the above equation shows that x must be a multiple of 30. If we can find p and g so that $10p + 7g = 1$, then x is 30.

Going one by one through the multiples of 7, we find that $7 \times 7 = 49$, which is one away from 10×5 or 50. Therefore, if we let $p = 5$ and $g = -7$, we find that

$$x = 30(10p + 7g) = 30(10(5) + 7(-7)) = 30(50 - 49) = 30.$$

Therefore, the smallest amount of debt is \$30, which can be paid by exchanging 5 pigs for 7 goats.

What Does This Problem Teach Us?

This problem is an application of an important mathematical result known as Bézout's lemma. Bézout's lemma states the following:

> Suppose X and Y are integers. Let d be the greatest common factor of X and Y. Then there exist integers a and b such that $d = aX + bY$.

The AMC problem can be generalized like this: Suppose that pigs cost X dollars and goats cost Y dollars, where X and Y are positive integers. What is the smallest positive integer d such that d can be expressed as $aX + bY$ for integers a and b?

Let d be the greatest common factor of X and Y. Then we can write

$$aX+ bY = d(a(X/d) + b(Y/d)).$$

Note that both X/d and Y/d are integers because d is, by definition, the greatest common factor of X and Y. Therefore, $a(X/d) + b(Y/d)$ is also an integer, and so the smallest positive amount that can be resolved with pigs and goats must be a multiple of d. On the other hand, by Bézout's lemma, there exist integers a and b such that $d = aX + bY$. Hence, the smallest positive debt that can be resolved is d, the greatest common factor of X and Y. The contest problem is the special case where $X = 300$ and $Y = 210$.

Proof of Bézout's Lemma

Bézout's lemma is so important that we give a proof of it here. We shall give a proof that uses the Euclidean algorithm.

Let X and Y be positive integers. Without loss of generality, assume $X \geq Y$. We can divide X by Y to get a quotient q_1 and a remainder r_1. That is,

$$X = Yq_1 + r_1.$$

Notice that $0 \leq r_1 < Y$. (We cannot have $r_1 \geq Y$ because when one divides X by Y, one finds the biggest multiple of Y less than or equal to X and subtracts this from X to get the remainder. If $r_1 \geq Y$, it means that there are bigger multiples of Y that are less than or equal to X.)

If $r_1 = 0$, then Y divides X, and their greatest common factor is Y. In this case, we can prove Bézout's lemma by observing that $Y = X - Y(q_1 - 1)$. So assume $r_1 > 0$. We can then divide Y by r_1 to get a quotient q_2 and a remainder r_2:

$$Y = r_1q_2 + r_2.$$

Notice that $0 \leq r_2 < r_1$.

If $r_2 > 0$, we can divide r_1 by r_2 to get a quotient q_3 and remainder r_3:

$$r_1 = r_2q_3 + r_3.$$

Notice that $0 \leq r_3 < r_2$.

So long as we get a remainder $r_k > 0$, we can continue dividing r_{k-1} by r_k to get a quotient q_{k+1} and remainder r_{k+1} such that $0 \leq r_{k+1} < r_k$.

The remainders form a strictly decreasing sequence of nonnegative integers. Since whenever a remainder is positive, we can divide again, we must eventually reach a remainder that is equal to 0. Define n to be the index of the last nonzero remainder. That is, our last equation in this sequence of divisions is $r_{n-1} = r_nq_{n+1} + 0$. (For notational convenience, we set $r_0 = Y$ and $r_{-1} = X$.)

We claim that r_n can be expressed as a linear combination[1] of X and Y with integer coefficients and that r_n is the greatest common factor of X and Y.

[1] A linear combination of X and Y is an expression of the form $aX + bY$. A linear combination of X and Y with integer coefficients is an expression of the form $aX + bY$ where a and b are integers.

We'll first show that r_n is a linear combination of X and Y with integer coefficients. To see this, we repeatedly use substitution in the sequence of equations we obtained by repeatedly dividing. First, we rewrite the first equation

$$X = Yq_1 + r_1$$

as

$$r_1 = X - Yq_1,$$

which shows that r_1 is a linear combination of X and Y with integer coefficients. We substitute this into the next equation

$$Y = r_1q_2 + r_2$$

to obtain

$$Y = (X - Yq_1)q_2 + r_2,$$

or

$$r_2 = -q_2X + (1 + q_1q_2)Y,$$

which shows that r_2 is a linear combination of X and Y with integer coefficients. We can then substitute these expressions for r_1 and r_2 into the next equation, $r_1 = r_2q_3 + r_3$, to see that r_3 is also a linear combination of X and Y with integer coefficients. By continuing this process, we see that each r_k is a linear combination of X and Y with integer coefficients. In particular, r_n is a linear combination of X and Y with integer coefficients, which is what we wanted to show.

Now we will prove that r_n is the greatest common factor of X and Y. From the equation $r_{n-1} = r_nq_{n+1}$, we know that r_n divides r_{n-1}. Also, since r_n is the largest factor of itself, r_n is the greatest common factor of r_{n-1} and r_n. Now let's examine one of the equations in the sequence of equations we obtained by repeated division, say,

$$r_k = r_{k+1}q_{k+2} + r_{k+2}.$$

We claim that the greatest common factor of r_k and r_{k+1} is equal to the greatest common factor of r_{k+1} and r_{k+2}. To see this, let D be the greatest common factor of r_k and r_{k+1}, and let d be the greatest common factor of r_{k+1} and r_{k+2}. Since D divides r_k and r_{k+1}, we see from the equation that D must also divide r_{k+2}. Hence D divides r_{k+1} and r_{k+2}, so D must also divide their greatest common factor, which is d. Likewise, we know that d divides r_{k+1} and r_{k+2}, and from the equation, we deduce that d must also divide r_k. Since d divides both r_k and r_{k+1}, d must divide their greatest common factor, which is D. Since d divides D and D divides d, it must be the case that $d = D$, which is what we wanted to show.

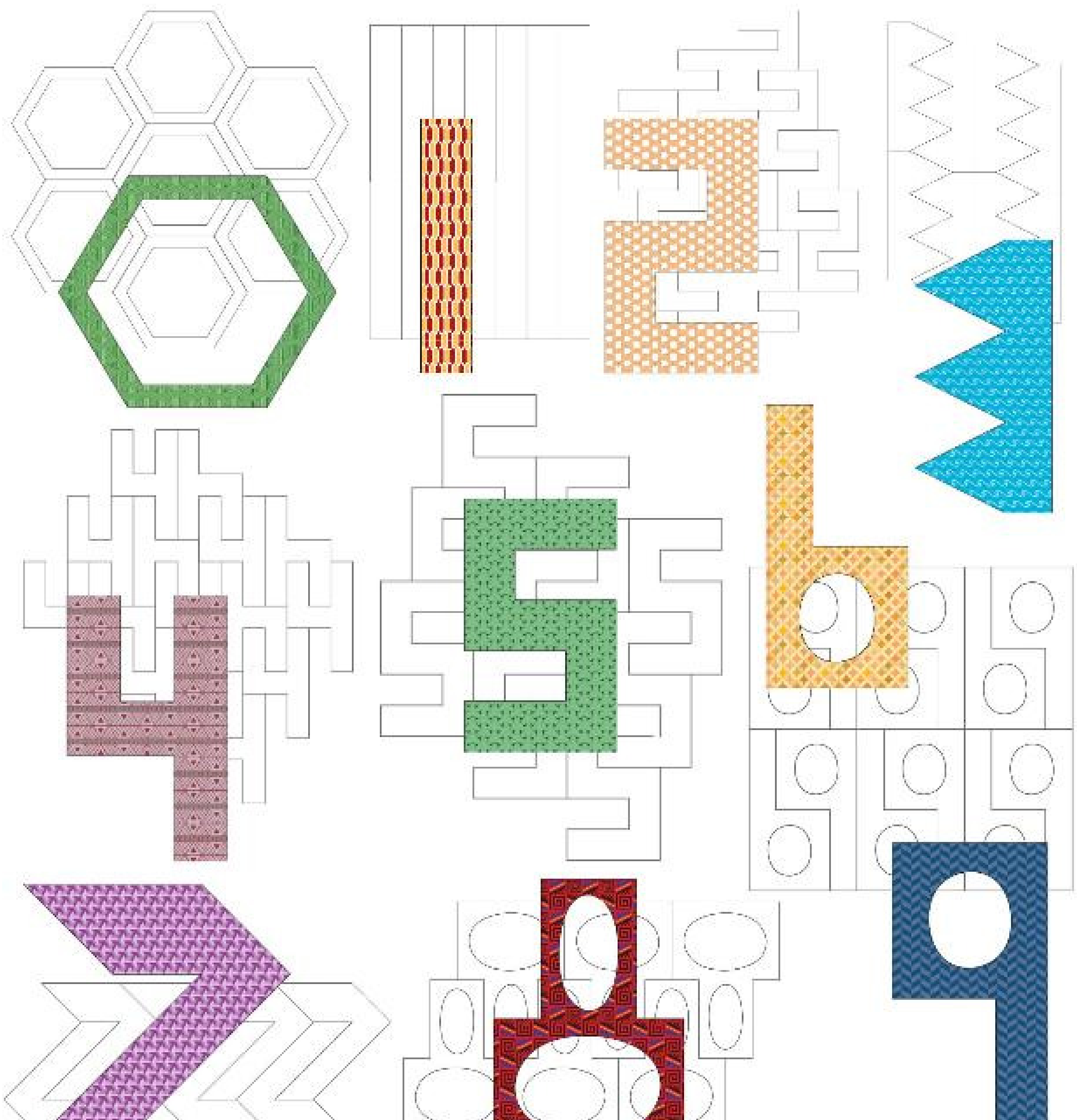
Using this fact repeatedly, we see that the greatest common factor of r_{n-1} and r_n is the greatest common factor of r_{n-2} and r_{n-1}, which is the greatest common factor of r_{n-3} and r_{n-2}, and so on. Continuing this argument, we find that all these greatest common factors are equal to the greatest common factor of X and Y.

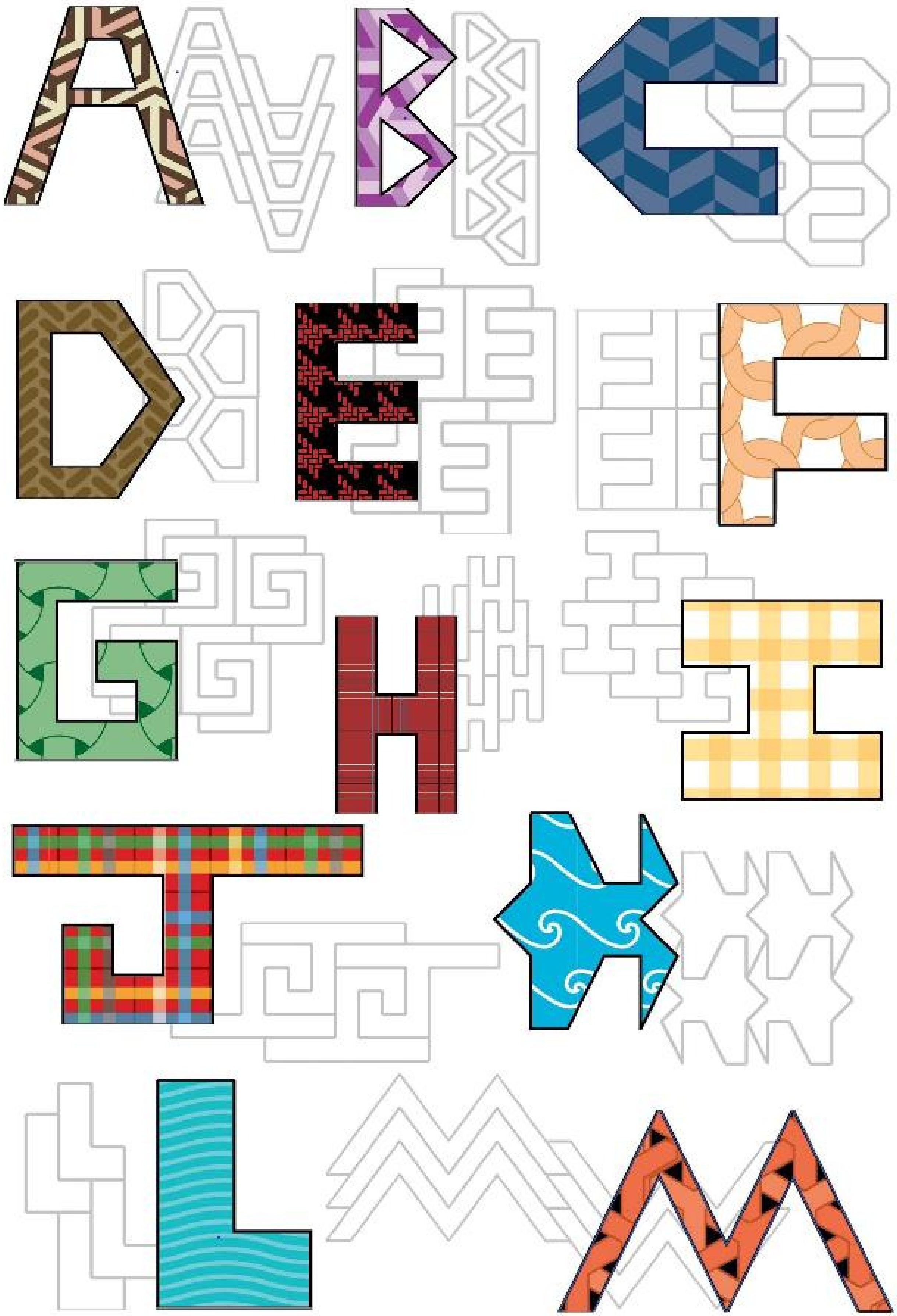
Since we have shown that r_n is the greatest common factor of X and Y and is also a linear combination of X and Y with integer coefficients, we have proven Bézout's lemma.

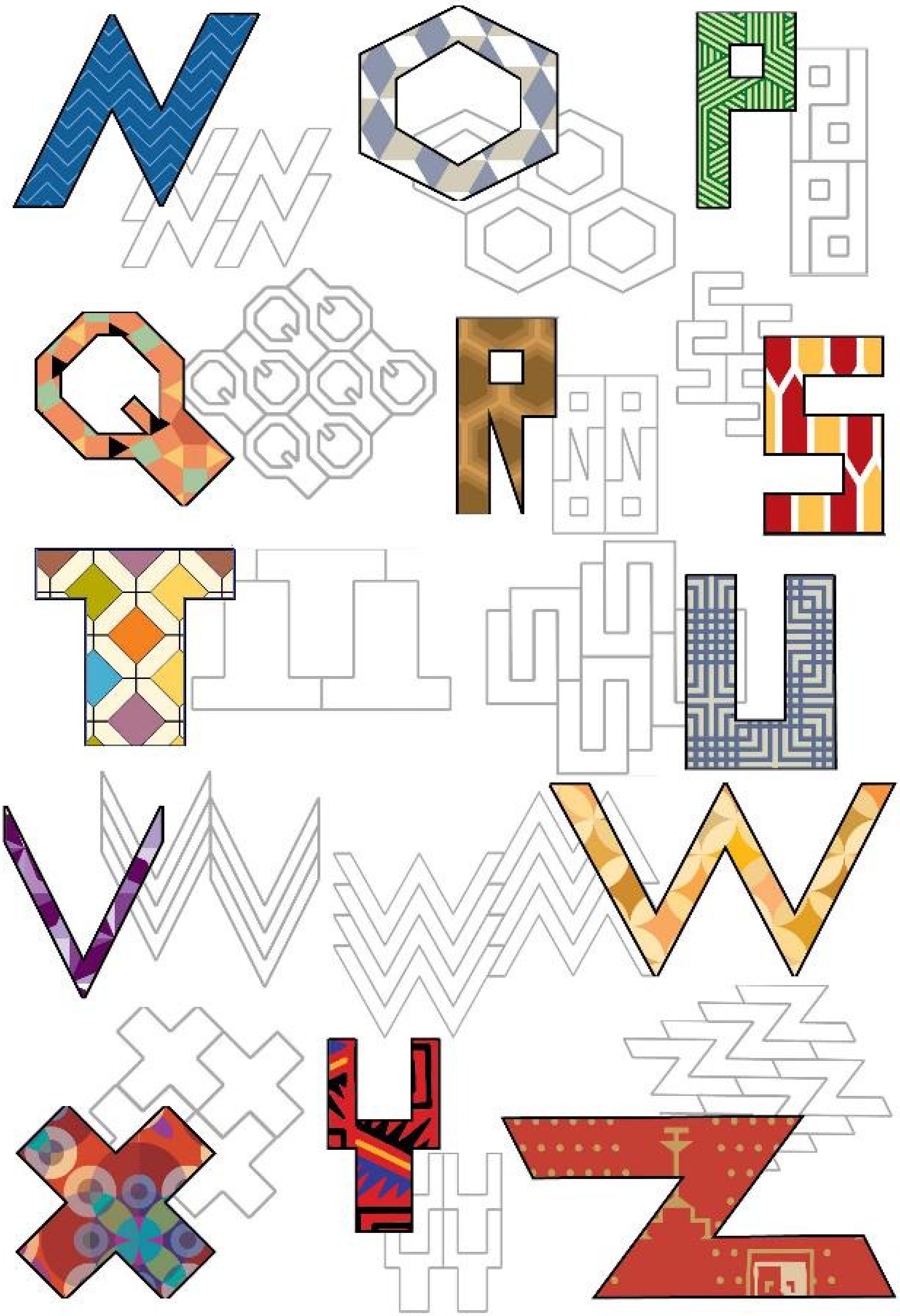
A TESSELLATING FONT

By **Katherine Dawson**
Computer graphics by Amy Fang, Adeline Hillier, and Elise McCormack-Kuhman

Katherine Dawson, one of our fifth grade members, really took to tessellations and designed an entire tessellating font for all the letters and digits. Every one of these characters can be used to tile the plane. To make full use of **Katherine Dawson**'s accomplishment, we wanted to create a cover for this issue using her "G" and "A" for "Girls' Angle". However, her "G" and "A" do not allow for a simultaneous tiling of the plane, so we contacted **Katherine Dawson**'s mom and hours later, **Katherine Dawson** came back to us with a new "G" and "A" that nicely do the job! See the cover for her "G" and "A" tiles that can be used simultaneously to tile the plane.







Meditate$^{\text{Math}}$

Pascal on Pascal

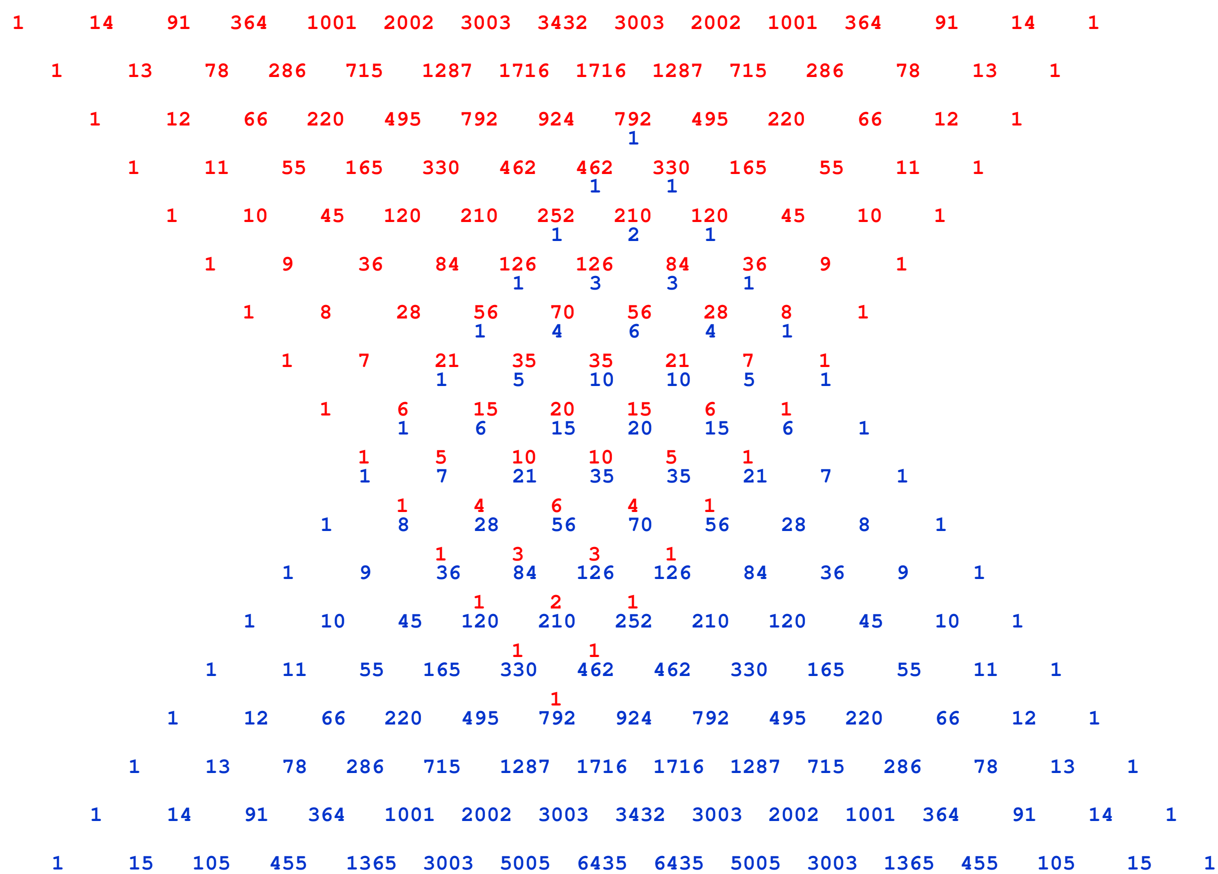
by Addie Summer | edited by Jennifer Silva

Make Pascal's triangle.[1] Now take another copy of Pascal's triangle, flip it over, then place its apex over an entry of the first Pascal's triangle.

Find a row where the two triangles overlap and add up the products of overlapping entries in that row. No matter which row you pick, you will always get the same answer, provided that there is some overlap in that row.

In this installment of Meditate to the Math, your first task is to sit back, relax, think, and figure out why.

Here's an illustration with the apex of the inverted red triangle placed over $\binom{12}{5}$ in the upright blue triangle. The products of the overlapping entries in any row that has them will always add up to 792.



[1]Pascal's triangle is a triangular array of numbers that has a 1 at the very top and each entry below is the sum of the two nearest numbers just above the entry. For example, the leftmost 66 in the blue Pascal's triangle above is equal to the sum of the 11 and 55 above it. We can imagine infinitely many unwritten zeros that extend each row to the left and right.

These identities are known as Vandermonde's identities. Let's take special note of the case where we place the apex of the inverted triangle over $\binom{2n}{n}$ and add up the products of the overlapping entries in the nth row. We obtain the identity

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

For your next task, we again overlap two copies of Pascal's triangle, although we arrange one triangle so that it is left-justified (i.e., so that all entries $\binom{n}{k}$ for fixed k run down a single column), and another triangle so that each row of Pascal's triangle is written as a column to the left of the previous column. We then place the second one so that its apex falls anywhere on or to the right of the apex of the first one, as illustrated below for the case where the apex of the second triangle (in red) is placed ten entries to the right of the apex of the first triangle (in blue).

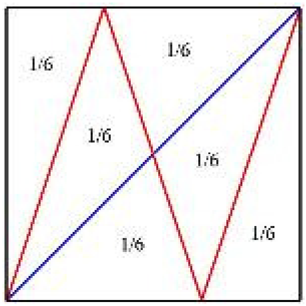
1	1	1	1	1	1	1	1	1	1	1					
1															
10	9	8	7	6	5	4	3	2	1						
1	1														
45	36	28	21	15	10	6	3	1							
1	2	1													
120	84	56	35	20	10	4	1								
1	3	3	1												
210	126	70	35	15	5	1									
1	4	6	4	1											
252	126	56	21	6	1										
1	5	10	10	5	1										
210	84	28	7	1											
1	6	15	20	15	6	1									
120	36	8	1												
1	7	21	35	35	21	7	1								
45	9	1													
1	8	28	56	70	56	28	8	1							
10	1														
1	9	36	84	126	126	84	36	9	1						
1															
1	10	45	120	210	252	210	120	45	10	1					
1	11	55	165	330	462	462	330	165	55	11	1				
1	12	66	220	495	792	924	792	495	220	66	12	1			
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1		
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1	
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1

Now pick any column that contains overlapping entries. As you go down that column, alternately add and subtract the products of the overlapping entries, starting with the first place where entries overlap. For example, if we were to pick the third column from the left, we would compute $28 \times 1 - 56 \times 3 + 70 \times 6 - 56 \times 10 + 28 \times 15 - 8 \times 21 + 1 \times 28$. Amazingly, you will always get zero, unless you happen to pick the column where the only overlapping entries are two ones.

Sit back, relax, meditate on this, and figure out why.

Zigzags, Part 2

by Ken Fan | edited by Jennifer Silva



Jasmine: The 1-zigzag, 3-zigzag case is amusing! I never knew that you could cut a rectangular cake into sixths that way. We're going to have to tell Mr. ChemCake the next time we go to Cake Country.

Emily: I don't see much rhyme or reason in the fractions we're getting, nor in the shapes the cake gets split into, either. Come to think of it, we've only seen triangles and quadrilaterals so far.

Jasmine: I have a hunch that we'll only get triangles and quadrilaterals. Maybe we can try to prove just that for now.

Emily: Okay. I'm inclined to add the two zigzags in one at a time; with just one zigzag, we can see that there will be two congruent right triangles at the ends and some number of congruent isosceles triangles in between.

Jasmine: I agree. Specifically, an n-zigzag will split the square into two right triangles and $n - 1$ isosceles triangles. The isosceles triangles alternate in orientation, pointing up then down, then up then down, and so on.

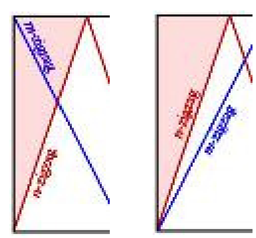
Emily: Now we have to understand what happens to those triangles if we add in the m-zigzag. How will they be sliced? We understand the $n = m = 1$ case, so let's assume that n and m are relatively prime and unequal. By doing so, we'll never have to worry about the zigzags sharing a line segment, because the linear segments of the zigzags will have different slopes.

Jasmine: I'm with you. Please keep going.

Emily: Actually, it seems like it could get a bit complicated. Suppose that m is much larger than n, for example. Then in each of the triangles produced by the n-zigzag, the m-zigzag will zig and zag back and forth many times, chopping the triangle up into lots of pieces.

Jasmine: Or we can assume, without loss of generality, that $m < n$, since there's no harm in assuming that we've drawn the high frequency zigzag first. That might make our analysis easier.

Emily: Yes, let's assume that! We can travel along the m-zigzag and see how it slices through the triangles created by the n-zigzag, using the assumption that $n > m$. We might as well start on the left edge of the square and travel toward the right. It seems that there are two cases. Either the m-zigzag starts in the right-angled vertex of the leftmost right triangle, or it starts in one of its acute-angled vertices. If the m-zigzag starts in the right-angle vertex, it will slice through the right triangle and exit it somewhere along its hypotenuse, never returning to that right triangle again. So the right triangle is split into two triangles.



Jasmine: And if the m-zigzag starts in an acute-angled vertex of the right triangle, it won't even cut through the triangle since the slope of the m-zigzag is shallower than that of the n-zigzag.

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Explaining the Next Dimension, Part 2

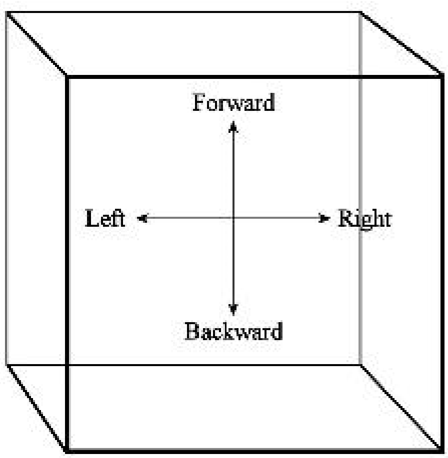
by C. Kenneth Fan | edited by Jennifer Silva

In Part 2, we solidify our understanding of the next dimension by observing an ant.

You: Great! To solidify your growing understanding of the 3rd dimension, let's imagine a little ant crawling around the surface of the cube. To get oriented, let me show you movements in the various possible directions.

Skate: Okay.

You: Here's movement in the forward, backward, left, and right directions:



Skate: Those are the directions I know well.

You: I cannot really show you the up and down directions because when you move in those directions, you essentially come out of your 2D world. But I think these directions will be clearer once we place our ant on the cube and have her walk about a bit. You will see the projection of the ant moving about the projection of the cube.

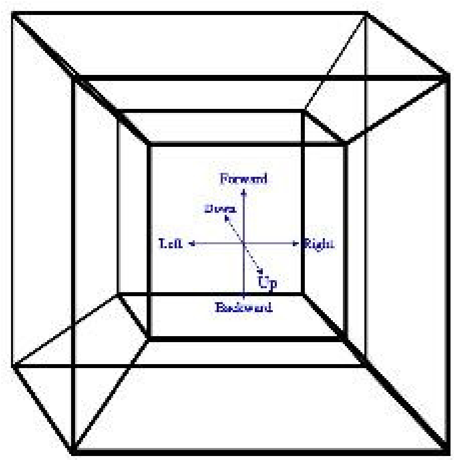
Skate: Okay.

You: Let's have the ant start in a corner of the cube.

Hyperia: Great! To solidify your growing understanding of the 4th dimension, let's imagine a little ant crawling around the surface of the hypercube. To get oriented, let me show you movements in the various possible directions.

You: Okay.

Hyperia: Here's movement in the forward, backward, left, right, up, and down directions:

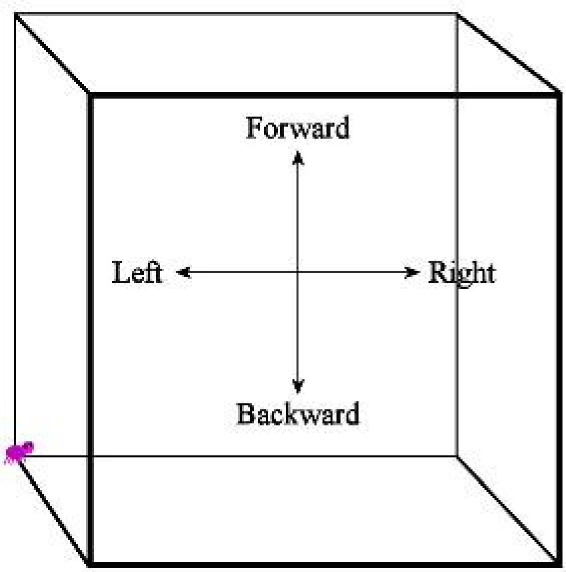


You: Those are the directions I know well.

Hyperia: I cannot really show you the ana and kata directions because when you move in those directions, you essentially come out of your 3D world. But I think these directions will be clearer once we place our ant on the hypercube and have her walk about a bit. You will see the projection of the ant moving about the projection of the hypercube.

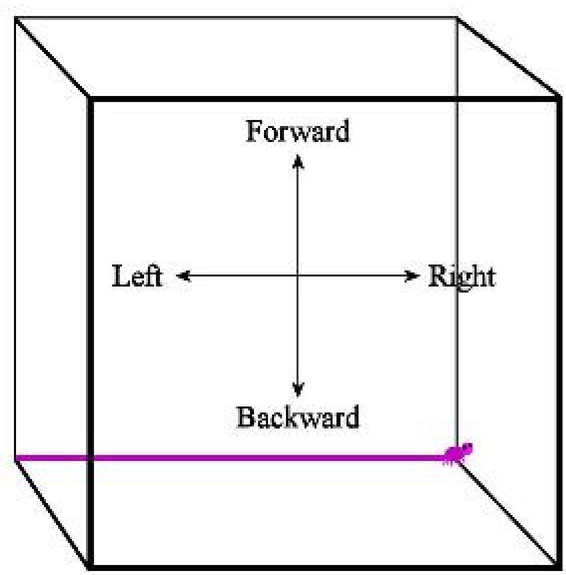
You: Okay.

Hyperia: Let's have the ant start in a corner of the hypercube.

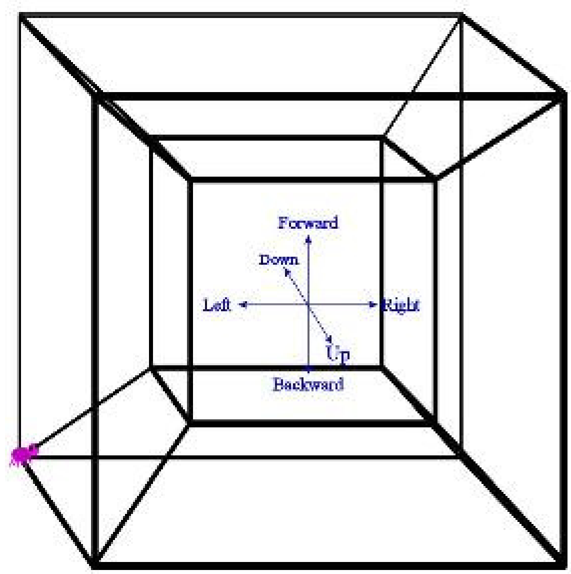


Skate: I've never seen a pink ant before.

You: The ant has graciously agreed to be dyed pink to make her easier to find. Now watch what happens when the ant walks as far as she can to the right on this cube.

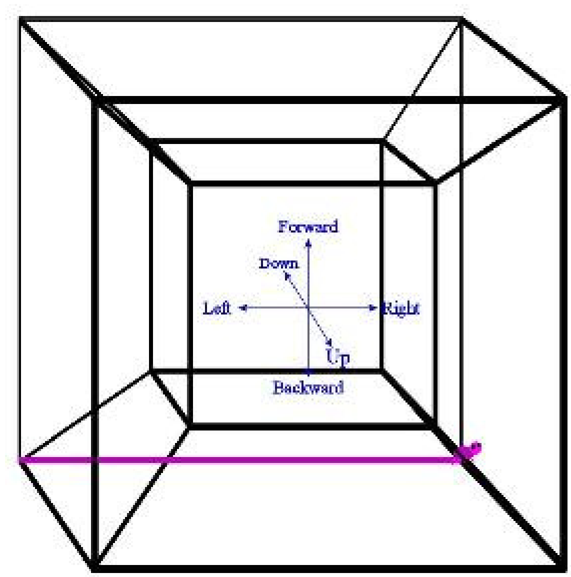


Skate: That makes sense.



You: I've never seen a pink ant before.

Hyperia: The ant has graciously agreed to be dyed pink to make her easier to find. Now watch what happens when the ant walks as far as she can to the right on this hypercube.



You: That makes sense.

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Learn by Doing

Euclidean 4D

by Ken Fan | edited by Amanda Galtman

Let's get to know the fourth dimension by exploring some 4D objects.

Because it's hard to visualize the fourth dimension, it's extra valuable to learn how to feel one's way around using mathematics. Mathematics provides us with tools that we can use to inform us about the geometry of objects without ever having to see the objects.

In these problems, we will be working in Euclidean space, which means that the distance between the points $(x_1, x_2, x_3, \ldots, x_n)$ and $(y_1, y_2, y_3, \ldots, y_n)$ is given by

$$\sqrt{\sum_{k=1}^{n}(x_k - y_k)^2} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + \ldots + (x_n - y_n)^2}\,.$$

We assume that you have some experience with analytic geometry and are familiar with the equations for lines and planes as well as basic trigonometry. This Learn by Doing is more computationally intensive than average, so please have an extra thick pile of paper on hand.

Let's warm up with a few exercises in 2D and 3D.

1. Consider the points (1, 3, 8), (3, 8, 1), and (8, 1, 3) in 3D space. They are the vertices of a triangle. Exactly what kind of triangle is it?

2. Consider the three points (12, 7), (20, 10), and (6, 23) in the coordinate plane. Without using pictures, determine whether these points are the vertices of an acute, right, or obtuse triangle.

(Spoiler Alert!) Using the distance formula, we can compute the lengths of the sides of this triangle. We find the lengths to be $\sqrt{73}$, $\sqrt{292}$, and $\sqrt{365}$. These three lengths satisfy the Pythagorean relationship since 73 + 292 = 365. Therefore, the triangle is a right triangle.

Another way to solve this problem is to observe that the line passing through (12, 7) and (20, 10) has slope 3/8 and the line passing through (12, 7) and (6, 23) has slope -16/6 = -8/3. Since -8/3 is the negative reciprocal of 3/8, these lines are perpendicular to each other. As a consequence, the triangle has two sides that are perpendicular to each other and is therefore a right triangle.

By the way, note that you can't use pictures to solve Problem 2. A nice picture might suggest the type of triangle, but from the drawing alone, you can never be absolutely sure if the triangle has a right angle or one that is only very nearly a right angle.

3. Let ABC be a triangle. Prove that $AB^2 = CA^2 + CB^2 - 2 \cdot CA \cdot CB \cdot \cos \angle ACB$. This is known as the Law of Cosines and can be used with the distance formula to compute the measure of angles formed by two intersecting lines in any dimension.

4. Let $A = (3, 5, 10)$, $B = (10, 3, 5)$, and $C = (12, 8, -2)$. What is $m\angle BAC$?

For problems 5 and 6, let $S = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), \{3, 2, 1)\}$.

5. Show that the points of S are coplanar.

6. By Problem 5, the convex hull of S is a polygon. Show that this polygon is a regular hexagon.

7. Let $g = \frac{1+\sqrt{5}}{2}$. Consider the 12 points $(\pm g, \pm 1, 0)$, $(0, \pm g, \pm 1)$, and $(\pm 1, 0, \pm g)$, taking all possible combinations of signs. Show that these points are the vertices of a **regular icosahedron**. (A regular icosahedron is a polyhedron with 12 vertices, 20 equilateral triangular faces, and 30 edges.)

Now let's move into the fourth dimension. The first problem will be about an object that lives in 4D space but is actually a 3D polyhedron. Throughout, we label the 4 axes by x, y, z, and w.

8. Let S consist of the point (1, 2, 3, 4), together with all points whose coordinates are a permutation of the coordinates of (1, 2, 3, 4). Since there are $4! = 24$ permutations of 4 coordinates, there are 24 points in S. Notice that all the points in S sit in the hyperplane consisting of points (x, y, z, w) such that $x + y + z + w = 10$. For this reason, we know that S lives inside a 3D hyperplane in 4D space. Let P be the convex hull of the points in S. How many faces does P have? What are the shapes of its faces? How many faces of each shape does P have?

(Spoiler Alert!) The polyhedron P has 14 faces, 6 of which are squares and 12 of which are regular hexagons.

The objects examined in Problems 5-6 and 8 are examples of "**permutohedrons**."

9. A hyperplane in 4D space is given by the set of points (x, y, z, w) that satisfy an equation of the form $ax + by + cz + dw = e$, where a, b, c, d, and e are constants and not all four of the constants a, b, c, and d are equal to 0. Show that a hyperplane divides 4D space into two disconnected pieces.

Notice, by the way, that while a 2D plane splits 3D space into two disconnected pieces, a 2D plane in 4D does not split 4D space into disconnected pieces.

10. Let P be the plane in 4D space consisting of the points $(0, 0, z, w)$ where z and w can be any real numbers. Find a continuous path that connects the point (-1, 0, 0, 0) to the point (1, 0, 0, 0) without intersecting P. (Note that the straight line path between the two points intersects P at the origin, so you'll have to come up with a different path.)

11. Let S be the set of points $(\pm 1, \pm 1, \pm 1, \pm 1)$, taking all possible combinations of plus and minus signs. (S consists of 16 points.) Let T be the convex hull of the points in S. Then T is a 4D polyhedron known as a **hypercube**.

> A. Notice that all 8 points $(\pm 1, \pm 1, \pm 1, 0)$ sit inside the hyperplane consisting of points (x, y, z, w) such that $w = 0$. Let P be this hyperplane. Explain why the intersection of P with T is the convex hull of the 8 points $(\pm 1, \pm 1, \pm 1, 0)$, which is a cube. In other words,

the cross-section of T by the hyperplane P is a cube. However, explain why this cross-section is *not* a 3D face of T.

B. What are the 3D faces of T?

(Spoiler Alert!) The hypercube has 8 faces, all cubes. These 8 faces are cross-sections by the hyperplanes $x = 1$, $x = -1$, $y = 1$, $y = -1$, $z = 1$, $z = -1$, $w = 1$, and $w = -1$.

C. Show that each cubic 3D face of T is adjacent to 6 other cubic 3D faces of T.

D. Let D be the hyperplane consisting of the points (x, y, z, w) such that $x + y + z + w = 0$. Let C be the cross-section of T by D. Identify the polyhedron C.

12. Let's augment the set S from Problem 11 by including the 8 points obtained by permuting the coordinates of $(\pm 2, 0, 0, 0)$ in all possible ways and taking both sign choices. Let O be the convex hull of the points in our new set, S.

A. Show that the points in S are exactly the vertices of O.

B. Show that O has 24 3D faces, all congruent to the same regular octahedron.

C. How many edges emanate from each vertex of O? How many edges does O have in total?

D. Let S' be the set of 24 points obtained by taking the 24 centers of each 3D face. Show that the convex hull of the points in S' is similar to O. What is the scale factor?

E. The dihedral angle (i.e., the angle between vectors that are perpendicular to the 3D faces) between adjacent 3D faces of a hypercube is 90°. What is the dihedral angle between adjacent 3D faces of O?

One could spend days exploring O. We'll instead move on to an even bigger 4D object.

13. Let's further increase the set S from Problem 12 by including the 96 points given by all combinations of sign and **even permutations** of the coordinates of $(\pm g, \pm 1, \pm 1/g, 0)$, where $g = \frac{1+\sqrt{5}}{2}$. (An even permutation is a permutation that can be expressed as a product of an even number of transpositions. There are 12 even permutations of 4 elements. For each of these 12 even permutations, there are 8 combinations of sign, yielding $12 \times 8 = 96$ points in total.) Let I be the convex hull of the 120 points in S.

A. Show that I has 600 3D faces, all congruent to a regular tetrahedron.

B. How many edges does I have? How many edges emanate from each of its vertices?

C. Show that I has 1200 equilateral triangular faces.

D. Let I' be the convex hull of the 600 points that are the centers of each of the 600 3D faces of I. Show that the faces of I' are all regular dodecahedra.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 24 - Meet 1 January 31, 2019	Mentors:	Mandy Cheung, Anna Ellison, Neslly Estrada, Adeline Hillier, Elise McCormack, Rebecca Nelson, Laura Pierson, Melissa Sherman-Bennett, Shohini Stout, Jane Wang, Josephine Yu

Some members played "goal-based" origami. In goal-based origami, either a member or a mentor would suggest a shape and then participants would do their best to create that shape by folding a piece of origami paper.

Related to goal-based origami, one can ask: What are all possible isosceles triangles that can be made using at most two folds, starting with an origami square? Or, what are all possible quadrilaterals the can be made using at most two folds, starting with an origami square? Etc.

Session 24 - Meet 2 February 7, 2019	Mentors:	Grace Bryant, Samantha D'Alonzo, Anna Ellison, Neslly Estrada, Amy Fang, Katie Gravel, Adeline Hillier, Claire Lazar, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Melissa Sherman-Bennett, Shohini Stout, Afura Taylor, Jane Wang, Rebecca Whitman, Jasmine Zou

Some members tried to determine all trapezoids whose four sides have integer lengths. This would be a generalization of the Pythagorean theorem.

Session 24 - Meet 3 February 14, 2019	Mentors:	Grace Bryant, Anna Ellison, Neslly Estrada, Amy Fang, Katie Gravel, Adeline Hillier, Rebecca Nelson, Kate Pearce, Laura Pierson, Afura Taylor, Savannah Tynan, Jane Wang, Rebecca Whitman, Jasmine Zou

Some members worked through our "anti-calculator" worksheet. The anti-calculator worksheet consists of problems which may not seem doable in the head, but can be solved with a minimum of computation by using a more conceptual approach.

Session 24 - Meet 4 February 28, 2019	Mentors:	Grace Bryant, Kelly Chen, Amy Fang, Adeline Hillier, Claire Lazar, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Melissa Sherman-Bennett, Rebecca Whitman, Josephine Yu

Can mathematics be applied to the concept of fairness? Some members explored ways to use mathematics to both define what is fair and develop protocols for achieving fairness.

Calendar

Session 23: (all dates in 2018)

September	13	Start of the twenty-third session!
	20	
	27	
October	4	
	11	
	18	
	25	
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Session 24: (all dates in 2019)

January	31	Start of the twenty-fourth session!
February	7	
	14	
	21	No meet
	28	
March	7	
	14	
	21	
	28	No meet
April	4	
	11	
	18	No meet
	25	
May	2	
	9	

SUMIT 2019 is scheduled for April 6 and 7, 2019. Registration is now open and conducted on a first-come-first-served basis. We're especially interested in including more 10th and 11th graders. Visit http://girlsangle.org/page/SUMIT/SUMIT.html for updates.

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minute walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: _____________________________________