Girls' Angle Bulletin

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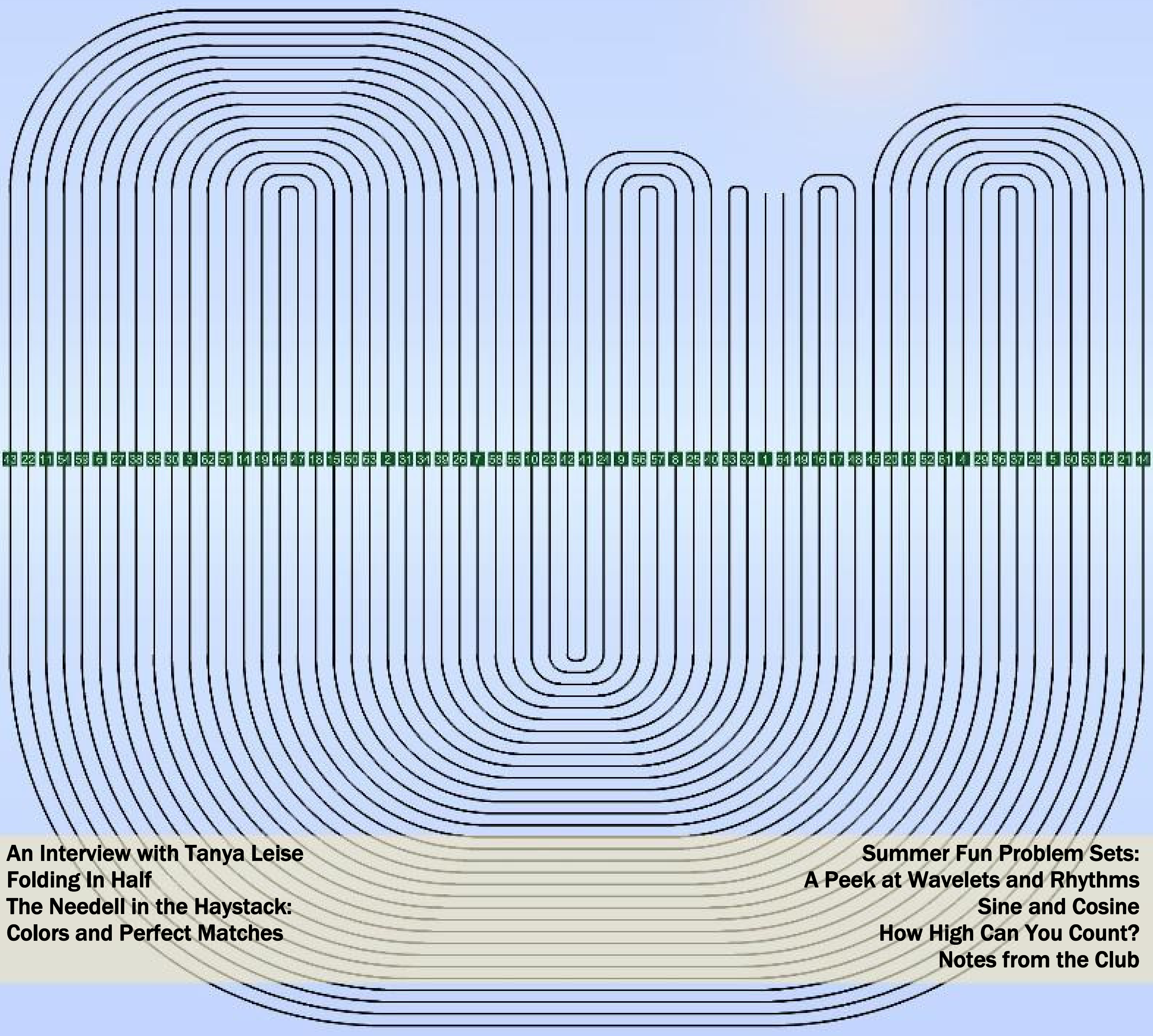
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

From day one, we aimed to hire a woman with a PhD in math who cares very much about girls' math education at the K12 level as Head Mentor. Thanks to the Mathenaeum Foundation, this goal is a reality. Welcome to our new Head Mentor, Grace Work! - Ken Fan, President and Founder

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Girls' Angle welcomes submissions that pertain to mathematics.

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Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Illuminating Half Folds* by C. Kenneth Fan. An example of a folded model considered in *Folding In Half* on page 7.

An Interview with Tanya Leise

Give kids lots of experiences, both indoor academics and outdoor sports and nature camps, so they can discover what they like best.

Tanya Leise is a Professor of Mathematics in the Mathematics and Statistics Department at Amherst College. She received her doctoral degree from Texas A&M University.

Ken: You study something that pertains to every single one of us: circadian rhythms. What got you interested in this topic and how does it involve mathematics?

Tanya: I've always enjoyed applying math to intriguing problems. As an undergrad, I worked on a project on capillary surfaces in zero gravity. If you put a straw into water, what shape will the surface of the water take inside the straw? If you try this experiment here on Earth, you'll see that it curves, with the water appearing to climb up the side of the straw. The curve in zero gravity is a bit different. We tested the model for capillary surfaces in zero g using a specially designed container that went up on the space shuttle. That was very exciting, and the model predictions turned out to be quite accurate. Later, in graduate school, I turned to modeling how cracks propagate through materials like metals and polymers, which involves a challenging mixed boundary value problem and a lot of tricky computations.

When I moved to Amherst, I wanted to find a new project to work on with someone nearby, so we could work in person together. I talked to people working on a variety of projects in areas like physics and biology, and decided to jump into circadian rhythms after I met Mary Harrington in the Neuroscience Program at Smith College. She's been a great mentor and colleague, and we've done quite a bit of joint work involving undergrads from both of our colleges, which has been really fun. The circadian rhythms field in general has been very welcoming to mathematicians, and the biologists are eager to see more mathematical modeling and time-frequency analysis to help them more fully understand their experimental data.

Ken: What kinds of questions are you interested in answering about biological rhythms? Could you please describe some of the big mysteries of that topic?

Tanya: I focus on circadian rhythms, which are the 24 hour patterns observed in most organisms, whether bacteria, plants, or animals, which persist even in the absence of any time cues. They are generated by feedback loops of "clock genes" in our cells. The big question is how the expression of these clock genes, which happens at the time scale of seconds, can lead to a 24 hour rhythm, several orders of magnitude longer. Mathematical modeling has been crucial in deducing how the intricate dance of interacting genes and their proteins can lead to the observed rhythms. The interplay between experiments and modeling has been fantastic, and the studies I most admire have been by teams of biologists and mathematicians working together to creatively combine their approaches.

Ken: Would you please describe one of your discoveries that you are most proud of? How did you discover it?

Tanya: My work applies existing methods to new areas, rather than creating new mathematics. A major contribution I've made to the analysis of circadian rhythms is

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Tanya Leise and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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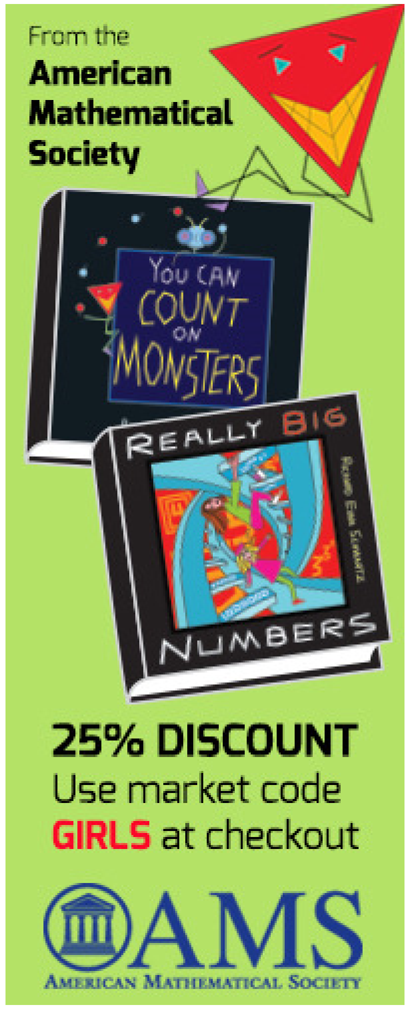

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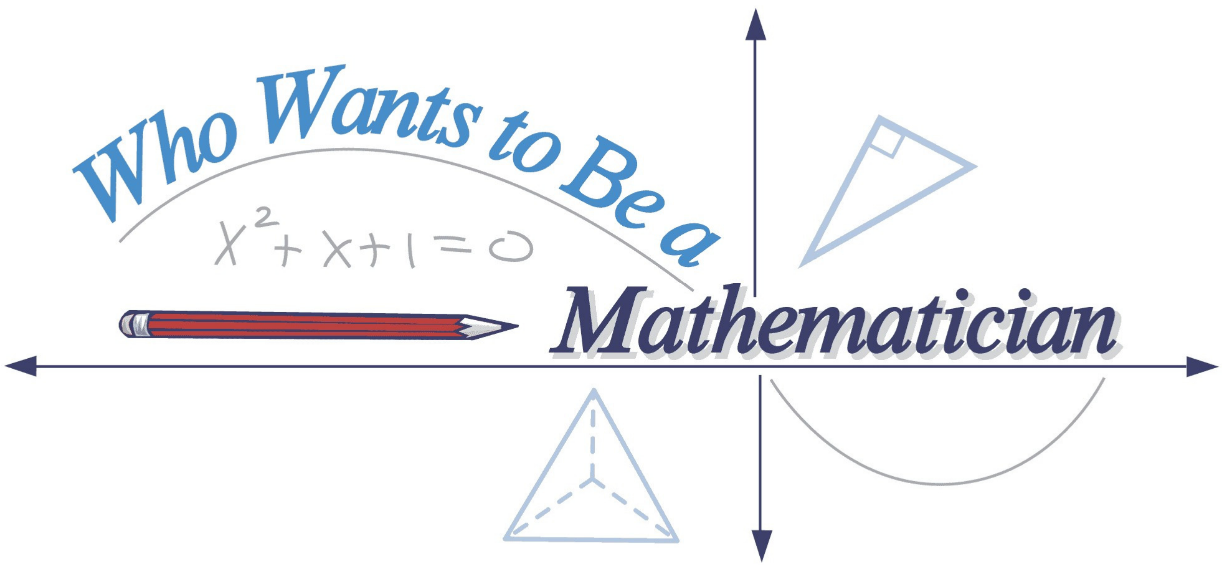

America's Greatest Math Game: Who Wants to Be a Mathematician.

Folding In Half[1]

by Jade Buckwalter, Milena Harned, Martina Maximovich, and Miriam Rittenberg
edited by Amanda Galtman

Introduction

We consider the repeated folding of a rectangular strip of paper and the order of the resulting layers. In this problem, there are two methods of folding paper: right folds and left folds. A **right fold** is a fold in which the right side of the paper is folded over to the left side. Similarly, a **left fold** is a fold in which the left side of the paper is folded over to the right side (see Figure 1).

A right fold A left fold

Figure 1. Right folds versus left folds.

Denote a right fold by R and a left fold by L. A **fold sequence** F_n is a sequence $A_1A_2\ldots A_n$ where each A_k is an R or an L. Performing the right and left folds in order of the sequence results in a **model** (i.e., a folded strip of paper). Notice that a sequence of n folds creates creases that split the paper into 2^n sections (see Figure 2). From left to right (in the unfolded rectangle), label these sections consecutively from 1 to 2^n.

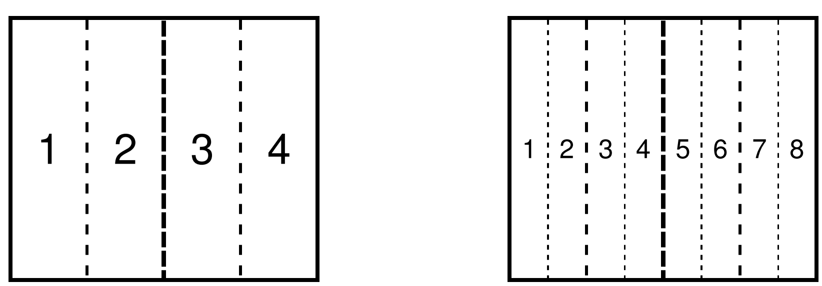

Figure 2. How we are splitting the paper.

Performing the fold sequence F_n creates a model consisting of 2^n layers, each layer consisting of a single one of the 2^n labeled sections. The numbers labeling the sections, read from bottom to top, form a sequence $a_1, a_2, a_3, \ldots, a_{2^n}$ of section numbers. Let $\boldsymbol{S}$ be the function from fold sequences to sequences of section numbers such that

$$\boldsymbol{S}(A_1A_2A_3\ldots A_n) = a_1, a_2, a_3, \ldots, a_{2^n}.$$

[1] To the best of our knowledge, these are new results. Please let us know if you know otherwise. Thank you!

For example, $\boldsymbol{S}(RL)$ = 2, 3, 4, 1. We also define the function $\boldsymbol{L}$ from fold sequences to sequences of layer numbers: $\boldsymbol{L}(A_1A_2A_3\ldots A_n) = b_1, b_2, b_3, \ldots, b_{2^n}$, where b_k is the layer number (counting from bottom to top) of section number k. In our previous example, $\boldsymbol{L}(RL)$ = 4, 1, 2, 3. We will refer to sequences $\boldsymbol{S}(F_n)$ as **section sequences**, and sequences $\boldsymbol{L}(F_n)$ as **layer sequences**.

In this paper, we will prove some properties of section and layer sequences, and show how to use these properties to construct the layer sequence resulting from a given fold sequence. We will then give sufficient conditions for a permutation of the numbers 1 through 2^n to be a layer sequence.

Most of our conjectures are based on direct observation, using data in the table below.

Data

Fold Sequence F_n	$\boldsymbol{S}(F_n)$	$\boldsymbol{L}(F_n)$
R	1, 2	1, 2
L	2, 1	2, 1
RR	1, 4, 3, 2	1, 4, 3, 2
RL	2, 3, 4, 1	4, 1, 2, 3
LR	3, 2, 1, 4	3, 2, 1, 4
LL	4, 1, 2, 3	2, 3, 4, 1
RRR	1, 8, 5, 4, 3, 6, 7, 2	1, 8, 5, 4, 3, 6, 7, 2
RRL	2, 7, 6, 3, 4, 5, 8, 1	8, 1, 4, 5, 6, 3, 2, 7
RLR	3, 6, 7, 2, 1, 8, 5, 4	5, 4, 1, 8, 7, 2, 3, 6
RLL	4, 5, 8, 1, 2, 7, 6, 3	4, 5, 8, 1, 2, 7, 6, 3
LRR	5, 4, 1, 8, 7, 2, 3, 6	3, 6, 7, 2, 1, 8, 5, 4
LRL	6, 3, 2, 7, 8, 1, 4, 5	6, 3, 2, 7, 8, 1, 4, 5
LLR	7, 2, 3, 6, 5, 4, 1, 8	7, 2, 3, 6, 5, 4, 1, 8
LLL	8, 1, 4, 5, 6, 3, 2, 7	2, 7, 6, 3, 4, 5, 8, 1
RRRR	1, 16, 9, 8, 5, 12, 13, 4, 3, 14, 11, 6, 7, 10, 15, 2	1, 16, 9, 8, 5, 12, 13, 4, 3, 14, 11, 6, 7, 10, 15, 2
RRRL	2, 15, 10, 7, 6, 11, 14, 3, 4, 13, 12, 5, 8, 9, 16, 1	16, 1, 8, 9, 12, 5, 4, 13, 14, 3, 6, 11, 10, 7, 2, 15
RRLR	3, 14, 11, 6, 7, 10, 15, 2, 1, 16, 9, 8, 5, 12, 13, 4	9, 8, 1, 16, 13, 4, 5, 22, 11, 6, 3, 14, 15, 2, 7, 10
RRLL	4, 13, 12, 5, 8, 9, 16, 1, 2, 15, 10, 7, 6, 11, 14, 3	8, 9, 16, 1, 4, 13, 12, 5, 6, 11, 14, 3, 2, 15, 10, 7
RLRR	5, 12, 13, 4, 1, 16, 9, 8, 7, 10, 15, 2, 3, 14, 11, 6	5, 12, 13, 4, 1, 16, 9, 8, 7, 10, 15, 2, 3, 14, 11, 6
RLRL	6, 11, 14, 3, 2, 15, 10, 7, 8, 9, 16, 1, 4, 13, 12, 5	12, 5, 4, 13, 16, 1, 8, 9, 10, 7, 2, 15, 14, 3, 6, 11
RLLR	7, 10, 15, 2, 3, 14, 11, 6, 5, 12, 13, 4, 1, 16, 9, 8	13, 4, 5, 12, 9, 8, 1, 16, 15, 2, 7, 10, 11, 6, 3, 14
RLLL	8, 9, 16, 1, 4, 13, 12, 5, 6, 11, 14, 3, 2, 15, 10, 7	4, 13, 12, 5, 8, 9, 16, 1, 2, 15, 10, 7, 6, 11, 14, 3
LRRR	9, 8, 1, 16, 13, 4, 5, 12, 11, 6, 3, 14, 15, 2, 7, 10	3, 14, 11, 6, 7, 10, 15, 2, 1, 16, 9, 8, 5, 12, 13, 4
LRRL	10, 7, 2, 15, 14, 3, 6, 11, 12, 5, 4, 13, 16, 1, 8, 9	14, 3, 6, 11, 10, 7, 2, 15, 16, 1, 8, 9, 12, 5, 4, 13
LRLR	11, 6, 3, 14, 15, 2, 7, 10, 9, 8, 1, 16, 13, 4, 5, 12	11, 6, 3, 14, 15, 2, 7, 10, 9, 8, 1, 16, 13, 4, 5, 12
LRLL	12, 5, 4, 13, 16, 1, 8, 9, 10, 7, 2, 15, 14, 3, 6, 11	6, 11, 14, 3, 2, 15, 10, 7, 8, 9, 16, 1, 4, 13, 12, 5
LLRR	13, 4, 5, 12, 9, 8, 1, 16, 15, 2, 7, 10, 11, 6, 3, 14	7, 10, 15, 2, 3, 14, 11, 6, 5, 12, 13, 4, 1, 16, 9, 8
LLRL	14, 3, 6, 11, 10, 7, 2, 15, 16, 1, 8, 9, 12, 5, 4, 13	10, 7, 2, 15, 14, 3, 6, 11, 12, 5, 4, 13, 16, 1, 8, 9
LLLR	15, 2, 7, 10, 11, 6, 3, 14, 13, 4, 5, 12, 9, 8, 1, 16	15, 2, 7, 10, 11, 6, 3, 14, 13, 4, 5, 12, 9, 8, 1, 16
LLLL	16, 1, 8, 9, 12, 5, 4, 13, 14, 3, 6, 11, 10, 7, 2, 15	2, 15, 10, 7, 6, 11, 14, 3, 4, 13, 12, 5, 8, 9, 16, 1

Lemmas

Note: In lemmas 1-5, $\{A_k\}$ is a fold sequence, $\{a_k\} = \boldsymbol{S}(\{A_k\})$, and $\{b_k\} = \boldsymbol{L}(\{A_k\})$.

Lemma 1. We have $b_{a_k} = k$ and $a_{b_k} = k$. In other words, the permutations $k \mapsto a_k$ and $k \mapsto b_k$ are inverse to each other.

Proof. First, b_k is the layer where we find section k, so b_{a_k} is the layer where we find section a_k. By definition, a_k is the section number in the kth layer. Hence, $b_{a_k} = k$. Second, a_k is the section number sitting in the kth layer, so a_{b_k} is the section number in layer b_k. By definition, k is the section number in layer b_k, hence $a_{b_k} = k$. □

Lemma 2. Let $\{A_k\}$ be a sequence of n folds, and let $\boldsymbol{S}(\{A_k\}) = a_1, a_2, a_3, \ldots, a_{2^n}$. Then, for any integer $1 \le k \le 2^n$, there exists an integer i such that $\{a_k, a_{2^n+1-k}\} = \{2i - 1, 2i\}$ (as sets). Conversely, for any integer $1 \le i \le 2^{n-1}$, there exists k such that $\{a_k, a_{2^n+1-k}\} = \{2i - 1, 2i\}$.

Proof. Start with a strip with 2^{n-1} sections that has been folded into a model according to the first $n - 1$ terms of the fold sequence $\{A_k\}$. We can form the desired model from this model by first subdividing the section on each layer into two sections, and then performing the fold A_n. For example, we divide the layer labeled "1" into sections labeled "1" and "2". In general, the layer with section i is divided into sections labeled $2i - 1$ and $2i$. (Sections 1 through $i - 1$ must be relabeled with the section numbers 1 through $2i - 2$, and section i is relabeled with the next two section numbers, which are $2i - 1$ and $2i$.) After we perform the fold A_n, the sections labeled $2i - 1$ and $2i$ will make up the layers k and $2^n + 1 - k$ in some order, for some integer k. Conversely, every pair of layers k and $2^n + 1 - k$ consists of sections formed by folding in half a single layer that was relabeled with section numbers $2i - 1$ and $2i$ for some integer i. □

Lemma 3. For any integer $1 \le i \le 2^{n-1}$, we have $b_{2i-1} + b_{2i} = 2^n + 1$.

Proof. By Lemma 2, there exists k such that $\{a_k, a_{2^n+1-k}\} = \{2i - 1, 2i\}$ (as sets). Since $b_{a_k} = k$, this implies that $\{b_{2i-1}, b_{2i}\} = \{k, 2^n + 1 - k\}$, and the lemma follows. □

Lemma 4. Let A_{n+1} be a fold and let $\{b_k\} = \boldsymbol{L}(A_1A_2A_3\ldots A_n)$ and $\{b_k'\} = \boldsymbol{L}(A_1A_2A_3\ldots A_nA_{n+1})$. Then the numbers 1, 2, 3, …, 2^n appear in the same order in both sequences $\{b_k\}$ and $\{b_k'\}$.

For example, the fold sequence *RR* results in the layer sequence 1, 4, 3, 2. Meanwhile, *RRL* results in the layer sequence 8, 1, 4, 5, 6, 3, 2, 7, in which the numbers 1, 2, 3, and 4 still appear in the order 1, 4, 3, and then 2.

Proof. Let $\{a_k\} = \boldsymbol{S}(A_1A_2A_3\ldots A_n)$ and $\{a_k'\} = \boldsymbol{S}(A_1A_2A_3\ldots A_nA_{n+1})$.

The order in which we encounter the numbers 1 through 2^n in the sequence $\{b_k\}$ corresponds to the order in which we pass through layers 1 through 2^n in the model as we traverse the strip of paper from its original left side to its original right side. Suppose we pass through layer p before we pass through layer q, that is, $a_p < a_q$. Now perform the fold A_{n+1}. As we saw in the proof of Lemma 2, the effect of performing this fold is to split the layer labeled i into two sections labeled $2i - 1$ and $2i$. One of these halves is folded into layers $2^n + 1$ through 2^{n+1}, while the other half remains where it is. Therefore, a_p' becomes either $2a_p - 1$ or $2a_p$, while a_q' becomes either $2a_q - 1$ or $2a_q$. Since $a_p < a_q$ implies $2a_p - 1 < 2a_p < 2a_q - 1 < 2a_q$, we see that irrespective of which half is folded up to the top half of the model, we have $a_p' < a_q'$. This inequality means that when we perform the fold A_{n+1} and then traverse the model from the original left side to the original right side of the strip of paper, we pass through layer p before layer q. Since p and q were arbitrary, the lemma follows. □

Lemma 5. Let $b_1, b_2, b_3, \ldots, b_{2^n}$ be the layer sequence $\boldsymbol{L}(\{A_k\})$, where $n > 1$. If $b_{2i-1} < b_{2i}$, then $b_{2i+1} > b_{2i+2}$. If $b_{2i-1} > b_{2i}$, then $b_{2i+1} < b_{2i+2}$.

Proof. Fix i. Let's observe how the model forms from the unfolded rectangular strip with 2^n sections labeled 1 through 2^n, paying particular attention to the sections $2i - 1$, $2i$, $2i + 1$, and $2i + 2$. At the beginning, the section numbers $2i - 1$, $2i$, $2i + 1$, and $2i + 2$ are all in the same layer because there is only one layer. Suppose that sections $2i - 1$ and $2i$ end up on a different layer from the sections $2i + 1$ and $2i + 2$ for the first time after performing the jth fold. (Note that until the final fold is performed, sections $2i - 1$ and $2i$ are on the same layer, and sections $2i + 1$ and $2i + 2$ are on the same layer.) Prior to the jth fold, the sections $2i - 1$, $2i$, $2i + 1$, and $2i + 2$ are on the same layer and in that order either from left to right (of the model) or from right to left. The jth fold forms a crease between sections $2i$ and $2i + 1$, and after the jth fold is done, one of the two section pairs $\{2i - 1, 2i\}$ and $\{2i + 1, 2i + 2\}$ increases from left to right, while the other increases from right to left (in their respective layers). Additionally, either both pairs abut the left edge of their respective layers, or both pairs abut the right edge (because of the common crease between sections $2i$ and $2i + 1$). Therefore, any further folds before the last will either fold both section pairs $\{2i - 1, 2i\}$ and $\{2i + 1, 2i + 2\}$ over or leave both in place. In either case, there is still one pair increasing from left to right and one pair increasing from right to left, and they are still either both on the leftmost side of their layers, or both on the rightmost side. When the last fold occurs, it either puts $2i - 1$ and $2i + 2$ in the top half and $2i$ and $2i + 1$ in the bottom half, or vice versa. In other words, of the four numbers b_{2i-1}, b_{2i}, b_{2i+1}, and b_{2i+2}, the numbers b_{2i-1} and b_{2i+2} are either the two largest or the two smallest. □

Sequence Construction

We can now use these lemmas to predict what happens when we add a fold, i.e., construct a layer sequence $\{b_k'\} = \boldsymbol{L}(A_1A_2A_3\ldots A_nA_{n+1})$ from $\{b_k\} = \boldsymbol{L}(A_1A_2A_3\ldots A_n)$. By Lemma 3, the sequence $\{b_k'\}$ consists of consecutive pairs of numbers that add up to $2^{n+1} + 1$. By Lemma 4, these pairs are positioned in the same order that their smaller elements took in the sequence $\{b_k\}$. By Lemma 5, the numbers within each pair must alternate whether the higher or lower number comes first. The only thing left undetermined is the order of the first pair, which depends on the nth fold.

By definition, b_1' is the position of the layer that contains section 1 in the folded model. If section 1 is in the top half of the model, then $b_1' > b_2'$. If section 1 is in the bottom half, then $b_1' < b_2'$. If we start with an unfolded strip of paper divided into 2^{n+1} sections and if $n > 1$, then after $n - 1$ folds, we have a layer composed of the four sections 1, 2, 3, and 4 in either increasing or decreasing order (from left to right in the model). If they are in increasing order, if the nth fold is an L, we now have a layer of the sections 2 and 1 in that order; therefore, another L puts 1 in the bottom half and an R puts 1 in the top half. If the nth fold is an R, we have sections 1 and 2 in that order; therefore, a subsequent L puts 1 in the top half, whereas an R puts 1 in the bottom half. If the sections are in the order 4, 3, 2, 1 instead, an L again results in a layer with sections 2 and 1 in that order, whereas an R results in a layer with sections 1 and 2 in that order. So, if fold $n + 1$ is the same type of fold as the nth fold, then $(b_1', b_2') = (b_1, 2^{n+1} + 1 - b_1)$. Otherwise, $(b_1', b_2') = (2^{n+1} + 1 - b_1, b_1)$. Once we know the order of the first pair, we can fill in everything else using the information above.

Examples

We will now give some examples of finding the layer sequence resulting from adding a fold.

Suppose we start with the fold sequence *RL*, and want to add an *L*.

$$\begin{aligned} \boldsymbol{L}(RL) &= 4, 1, 2, 3 \\ \boldsymbol{L}(RLL) &= ? \end{aligned}$$

Because the new fold matches the last fold of the starting sequence, the first number in $\boldsymbol{L}(RLL)$ is the same as the first number in $\boldsymbol{L}(RL)$. We insert a new layer next to each layer in $\boldsymbol{L}(RL)$, alternating whether the new layer comes first or second:

$$\begin{aligned} \boldsymbol{L}(RL) &= 4, 1, 2, 3 \\ \boldsymbol{L}(RLL) &= 4, \_\_, \_\_, 1, 2, \_\_, \_\_, 3 \end{aligned}$$

We can fill in the blanks using the rule that $b_{2i-1} + b_{2i} = 2^n + 1$:

$$\begin{aligned} \boldsymbol{L}(RL) &= 4, 1, 2, 3 \\ \boldsymbol{L}(RLL) &= 4, \mathbf{5}, \mathbf{8}, 1, 2, \mathbf{7}, \mathbf{6}, 3 \end{aligned}$$

If we had instead added an R to the original sequence, the first number in $\boldsymbol{L}(RL)$ would be the second number in $\boldsymbol{L}(RLR)$:

$$\begin{aligned} \boldsymbol{L}(RL) &= 4, 1, 2, 3 \\ \boldsymbol{L}(RLR) &= \_\_, 4, 1, \_\_, \_\_, 2, 3\ \_\_ \end{aligned}$$

We again fill in the blanks using the rule that $b_{2i-1} + b_{2i} = 2^n + 1$:

$$\begin{aligned} \boldsymbol{L}(RL) &= 4, 1, 2, 3 \\ \boldsymbol{L}(RLL) &= \mathbf{5}, 4, 1, \mathbf{8}, \mathbf{7}, 2, 3, \mathbf{6} \end{aligned}$$

The two sequences $\boldsymbol{L}(RLL)$ and $\boldsymbol{L}(RLR)$ are very similar, but the order of the two numbers in each pair (b_{2i-1}, b_{2i}) is switched.

Sequence Deconstruction

Now we will consider how to tell if a given ordering of the numbers 1 through 2^n is equal to $\boldsymbol{L}(A_1A_2A_3\ldots A_n)$ for some fold sequence $\{A_k\}$.

Theorem. Let $b_1, b_2, b_3, \ldots, b_{2^n}$ be a permutation of the numbers 1 through 2^n. There exists a fold sequence $A_1A_2A_3\ldots A_n$ such that $\{b_k\} = \boldsymbol{L}(A_1A_2A_3\ldots A_n)$ if and only if the following three conditions hold:

A. For all $1 \le i \le 2^{n-1}$, we have $b_{2i-1} + b_{2i} = 2^n + 1$.
B. For all $1 \le i < 2^{n-1}$, if $b_{2i-1} < b_{2i}$, then $b_{2i+1} > b_{2i+2}$, and if $b_{2i-1} > b_{2i}$, then $b_{2i+1} < b_{2i+2}$.
C. If $n > 1$ and we remove the numbers $2^{n-1} + 1$ through 2^n from the sequence $\{b_k\}$, we are left with a sequence of length 2^{n-1} that satisfies all three of these conditions.

Proof. We will prove that these three conditions are sufficient by showing that there are exactly 2^n sequences of length 2^n that satisfy the conditions. Because there are also 2^n possible fold sequences, and every layer sequence satisfies all three conditions, this will show that every permutation that satisfies the conditions can be produced by some sequence of folds. The proof will be by induction on n. For the base case, there are only two permutations of length two: 1, 2 and 2, 1. These sequences are produced by the fold sequences R and L, respectively. Now suppose there are exactly 2^{n-1} sequences of length 2^{n-1} that satisfy all three conditions. Then, if a permutation $\{b_k\}$ of length 2^n satisfies condition C, the order of the numbers $1, 2, 3, \ldots, 2^{n-1}$ within $\{b_k\}$ must be one of the 2^{n-1} fold sequences. By condition A, each of these numbers i is paired with the number $2^n + 1 - i$. The first pair can go in either of two orders, after which the order within each pair is determined by condition B. Therefore, each of the 2^{n-1} sequences of length 2^{n-1} results in two sequences of length 2^n, so there are 2^n sequences total.

Two different fold sequences can never produce the same permutation, because the choice of fold k determines the order of the first two numbers in the subsequence of length 2^k, so there are exactly 2^n permutations produced by fold sequences. That conditions A, B, and C are satisfied follows from lemmas 3, 4, and 5, so every layer sequence satisfies them. □

If we were to continue studying this topic, we would look at layer sequences resulting from types of folds other than R or L, and try to find methods for predicting the resulting section and layer sequences.

Acknowledgements

We thank Iris Liebman and Oriana McKanan for useful comments.

We also thank Claire Lazar and Melissa Sherman-Bennett for their guidance on this project.

The Needell in the Haystack[1]

Colors and Perfect Matches

by Deanna Needell | edited by Jennifer Silva

In the last article, we learned a bit about graph theory and how it helped my father George navigate the globe in an optimal fashion. Indeed, we discussed the "traveling salesperson" problem, which asks for the most efficient path through a graph that visits each node exactly once. Graph theory is such a rich and broad subject that I wanted to write a follow-up article discussing other interesting graph theoretic problems motivated by real applications. For example, George uses lots of maps when he travels and utilizes different colors to label various cities and attractions. When coloring a map, one typically tries to avoid coloring adjacent regions the same color (otherwise, it's harder to distinguish them from one another). This leads to the **graph coloring** problem. As another example, when George and his partner Signe are in a large group of people, George may want to match people with their friends. This type of goal can be formulated as a **graph matching** problem.

Let's quickly review mathematical graphs and their notation, as in the last article. A **graph** is a discrete object consisting of a set of **vertices** V and a set of **edges** E. The vertex set V is simply a set of objects; in the above examples, the vertices could be the regions on a map, or the people in the group. Each edge in E then corresponds to a connection between two vertices; in the above examples the edges could represent two regions being adjacent on the map, or two people who like one another. The skeptic asks, "What if one person likes the other but not vice versa?" In an **undirected graph**, edges are simply pairs of vertices, with no direction information; in a **directed graph**, these edges contain directional information, so can be written as ordered pairs. For example, in a directed graph, the edge (George, Signe) might indicate that George likes Signe, while the edge (Signe, George) indicates that Signe likes George. We may abuse notation slightly and refer to edges in an undirected graph as ordered pairs as well, with the understanding that the information $(u, v) \in E$ is the same as $(v, u) \in E$. While a graph can be defined by listing its vertex and edge sets, it is often easier to visualize them by drawing a diagram with vertices drawn as circles and edges drawn as arcs between them. See Figure 1 for an example.

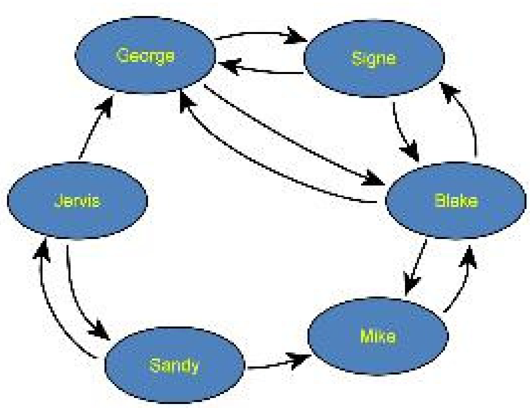

Figure 1. Example of a directed graph with people as vertices and directed connections as edges.

[1] This content supported in part by a grant from MathWorks.

Graph Colorings

Let us now turn to the problem of graph colorings. As mentioned, the problem is motivated by the need to color regions on a map so that no two neighboring regions are the same color. To state this as a graph problem, we create a vertex for each region, and put an edge between two vertices if they are neighbors on the map. We'll call two vertices "adjacent" if they are connected by an edge. Note that there is no need for edges to have a direction because if region A abuts region B, then region B also abuts region A. So our graph will be undirected. The graph coloring problem is the problem of labeling each vertex with a color so that no two adjacent vertices are labeled with the same color. We refer to this type of color assignment as a **coloring,** and say that the graph is **colored**.

Let us pause here and note that if there are $|V| = n$ vertices then we can always color the graph using n colors and painting each vertex a different color. So a more interesting question would be this: can we color the graph with fewer than n colors? For a given number of colors k, a coloring of the graph using only these k colors is called a ***k*-coloring** of the graph. (More formally, given a graph $G = (V, E)$, we define a k-coloring of G to be a function $f_k : V \to \{1, 2, 3, \ldots, k\}$ such that $f_k(v) \neq f_k(u)$ whenever $(u, v) \in E$.) We may also ask what the *minimal* number of colors is to color the graph in this way. In other words, what is the smallest k so that the graph has a k-coloring? The answer to this latter question for a graph G is called the graph's **chromatic number**, and is often denoted $\chi(G)$.

There are other interesting things we can associate to a graph. For example, the **chromatic polynomial**, denoted $P(G, k)$, gives the number of different ways the graph G can be colored using at most k colors. It is an interesting fact that the chromatic polynomial is a polynomial in the input k. We can define the chromatic number using the chromatic polynomial: $\chi(G) \equiv \min\{k : P(G, k) > 0\}$.

In our motivating example of map coloring, the chromatic number of the associated graph is the minimum number of colors needed to color the map so that no two neighboring regions are the same color. But graph colorings appear in numerous other applications as well. For example, they appear in scheduling applications, where events need to be allocated to time slots in such a way that events requiring the same type of equipment do not occur during the same time. In fact, even the now famous puzzle of Sudoku can be phrased as a graph coloring problem. For the reader familiar with Sudoku, she is encouraged to construct a graph with 81 vertices so that a 9-coloring of the graph corresponds to a solved 9×9 Sudoku puzzle. But we'll focus on smaller graphs that are easier to draw on paper. (It is worth a shout-out here to the **planarity problem**, which asks whether a given graph can be drawn on paper without any two edges crossing – another fun and related graph problem).

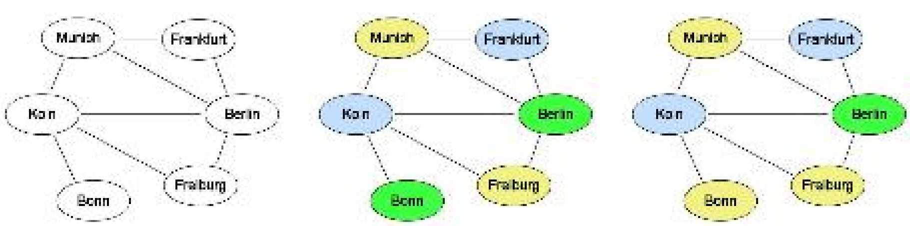

Figure 2. Examples of a graph and colorings.

Look at the graph on the left of Figure 2. It has 6 vertices and 8 edges. One example of a 3-coloring of this graph is shown in the center of the same figure. Since there is a 3-coloring, we know that for this graph, $\chi(G) \leq 3$. So here is the question: can it be colored using fewer than 3 colors? Given that some cities are connected by an edge, the graph cannot be 1-colored. So we

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Summer Fun!

The best way to learn math is to do math. Here are the 2019 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems on your own.

Some problems are quite a challenge and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can feel frustrating to work on problems that take weeks to solve. But there are things about the journey that are enjoyable. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

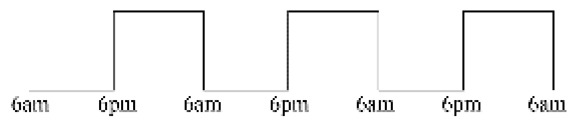
This is Summer Fun, not Summer Torture!

Summer Fun!

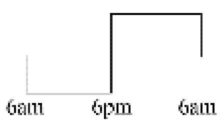
A Peek at Wavelets and Rhythms

by Tanya Leise

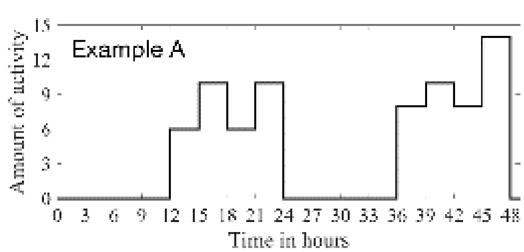
What are wavelets and why use them to study circadian rhythms? Let's start by thinking about a typical 24-hour activity pattern of a mouse. She'll run in her wheel, push her bedding around, and eat during the night, then stay quiet through most of the daytime. The pattern is like a square wave, quiet all day then active all night, repeating every 24 hours:



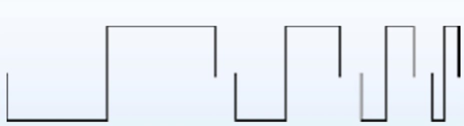
A **wavelet** captures the basic pattern for a single off-on cycle, in this case, a "square wavelet":



This pattern is called a *Haar wavelet*. We can use it as a building block to find the different subpatterns contained in an overall activity pattern of the mouse, which will be more complicated than the simple square wave. Suppose the mouse is sometimes more active, sometimes less active, like this:



Subpatterns happen over a few hours, rather than over a full day, so we need shorter scale versions of our wavelet to work with:



To accomplish this scaling mathematically, let's think about the wavelet as a function $\psi(t)$ that looks like our basic pattern:

$$\psi(t) = \begin{cases} -1 \text{ if } 0 \le t < 12, \\ 1 \text{ if } 12 \le t < 24, \\ 0 \text{ otherwise.} \end{cases}$$

Checkpoint: Plot this function $\psi(t)$.

To shrink the wavelet down to cover 12 hours instead of 24 hours, we scale it by a factor of 2: $\psi(2t)$.

Checkpoint: To see how this scaling works, plug values into $\psi(2t)$ like $t = 0, 3, 6$, etc., and then plot $\psi(2t)$.

Checkpoint: What scaled function would give the wavelet pattern covering a 6-hour interval? A 3-hour interval?

In general, we can shrink the time interval in half k times for the wavelet using $\psi(2^k t)$.

We'll also need to shift the wavelet in time to put them at the right places in the overall pattern. A wavelet that starts an hour later is $\psi(t - 1)$, which equals -1 for $1 \le t < 13$ and 1 for $13 \le t < 25$ (and zero otherwise).

Checkpoint: Plot the functions $\psi(t - 6)$, $\psi(t - 12)$, and $\psi(t - 24)$.

What if we combine the scaling and shifting? For example, $\psi(2(t - 12))$. Look at the inequalities in the definition. This function equals -1

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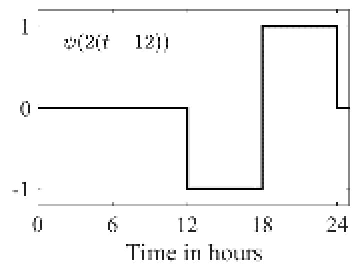
when $0 \le 2(t-12) < 12$. Divide both sides by 2, then add 12 to obtain $12 \le t < 18$.

Similarly rearrange the second inequality and then graph the result:



Checkpoint: Plot the functions $\psi(2(t-6))$, $\psi(4(t-6))$, and $\psi(4(t-12))$.

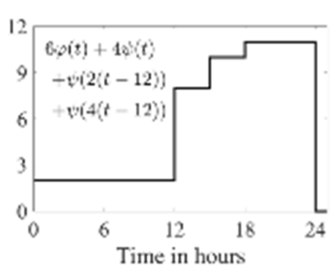
We can shrink and shift our Haar wavelet pattern to create a flexible set of building blocks: $\psi(2^k(t-m))$. These all have a zero average, so to capture nonzero daily averages, we will also need a companion function $\varphi(t)$ called the **Haar scaling function**:

$$\varphi(t) = \begin{cases} 1 & \text{if } 0 \le t < 24, \\ 0 & \text{otherwise.} \end{cases}$$

Checkpoint: Plot $\varphi(t)$ and $\varphi(t-24)$.

We can add together combinations of these functions to make more interesting graphs. For example, here is the graph of

$6\varphi(t) + 4\psi(t) + \psi(2(t-12)) + \psi(4(t-12))$:



A quick way to figure out what the graph looks like is to write down the value on each 3-hour interval for each part, then add together those arrays of numbers. The 1st number is for $0 \le t < 3$, the 2nd is $3 \le t < 6$, and so on.

t	0	3	6	9	12	15	18	21
$6\varphi(t)$	6	6	6	6	6	6	6	6
$4\psi(t)$	-4	-4	-4	-4	4	4	4	4
$\psi(2(t-12))$	0	0	0	0	-1	-1	1	1
$\psi(4(t-12))$	0	0	0	0	-1	1	0	0
Sum	2	2	2	2	8	10	11	11

Checkpoint: Use this method to sketch the graph of $5\varphi(t) + 3\psi(t) + \psi(2(t-12))$.

There's an easy way to work backwards from the values in the graph to deduce how to express it as a combination of ψ and φ functions, which is the essential goal of wavelet analysis. First make an array with the value every 3 hours of the Example A graph (t is the left end of each interval):

t	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
Ex. A	0	0	0	0	6	10	6	10	0	0	0	0	8	10	8	14

Create a new array s_1 by summing each pair of numbers and dividing by two. For example, the 3rd entry will be $(6 + 10)/2 = 8$. Similarly, create a new array d_1 by taking the differences of pairs (2nd number minus 1st number), divided by two. This process will double the length of the time intervals and the new arrays will be half as long:

t	0	6	12	18	24	30	36	42
s_1	0	0	8	8	0	0	9	11
d_1	0	0	2	2	0	0	1	3

Repeat this process on s_1 to obtain arrays s_2 and d_2, then use s_2 to obtain s_3 and d_3:

t	0	12	24	36
s_2	0	8	0	10
d_2	0	0	0	1

t	0	24
s_3	4	5
d_3	4	5

Checkpoint: Why are s_3 and d_3 the same here? How is it connected to zero activity during the daytime?

The s_j arrays tell us the coefficients that go with the φ functions, shifted by the t value. In particular, s_3 is shorthand for the function $4\varphi(t) + 5\varphi(t-24)$, capturing the daily average activity. We only use this last s_3 array, but we'll use all of the d_j arrays.

The d_j arrays tell us the coefficients that go with the ψ functions, shifted by the t value. These capture the details of how activity is changing around the average, with finer details as the interval repeatedly shrinks in half. Thus, d_3 gives $4\psi(t) + 5\psi(t-24)$, which tells us that there is an off-on cycle in activity on both days. And d_2 gives $\psi(2(t-36))$, where we scale by 2 because we have cut the 24-hour intervals in half to yield 12-hour intervals. And d_1 gives

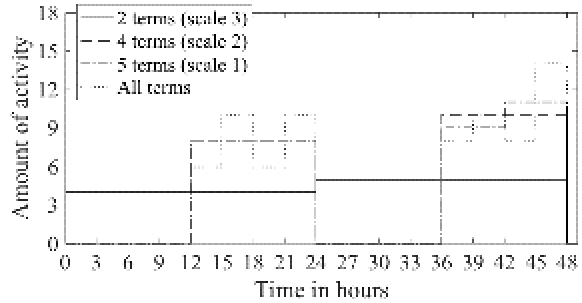
$$2\psi(4(t-12)) + 2\psi(4(t-18)) \\ + \psi(4(t-36)) + 3\,\psi(4(t-42)),$$

where we scale by 4 because we have cut the 24-hour intervals into quarters to yield 6-hour intervals.

Putting these together, Example A can be written as the combination

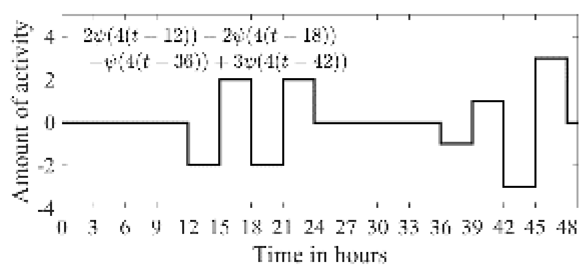
$$\begin{aligned} &4\varphi(t) + 5\varphi(t-24) \\ &+ 4\psi(t) + 5\psi(t-24) \\ &+ 2\psi(4(t-12)) + 2\psi(4(t-18)) \\ &+ \psi(4(t-36)) + 3\,\psi(4(t-42)). \end{aligned}$$

The φ terms set the daily averages as a base. The first two ψ terms set the basic day-night off-on pattern, then the further ψ terms add details at increasingly finer scales. This type of decomposition into functions at different scales is called a *multiresolution analysis*.



Writing the activity as a combination of ψ and φ functions helps us study the mouse's circadian rhythms by letting us focus on different time scales in the data. If we want to see the basic day-night pattern in a clean form, we use only the terms coming from the last scale (s_3 and d_3 in Example A). If we want to zero-center the activity data, we use only the sum of the ψ functions, which removes the daily averages from the data. To remove high frequency jitter, we can discard the ψ terms that come from d_1, which works best if the data's time intervals are short, e.g., a few minutes long.

In Example A, we may be interested in the pattern occurring during each night that is indicated by d_1 (the last 4 ψ terms):



Isolating this rhythm from the overall day-night rhythm lets us see it more clearly. This separation also allows us to assess how this nighttime rhythm varies from night to night. In general, isolating the terms at particular scales can help us see what interesting patterns might be hidden within the data.

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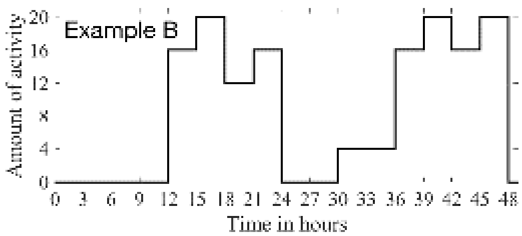
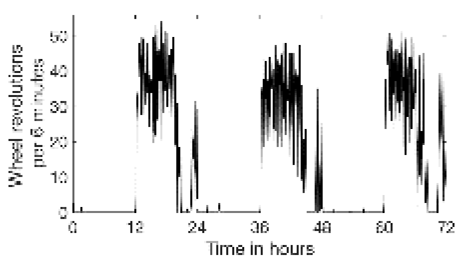
Checkpoint: Try out this method on the graph of Example B below, calculating the arrays s_1, d_1, s_2, d_2, s_3, and d_3, converting into the ψ and φ functions, and then plotting different subsets of terms to explore what features of the data they reveal.

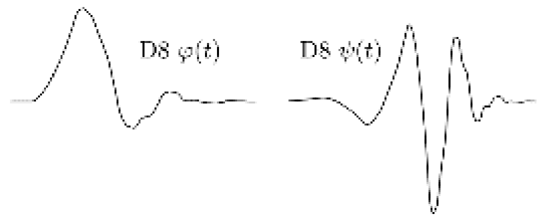


t	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
Ex. B	0	0	0	0	16	20	12	16	0	0	4	4	16	20	16	20

Real data is more complicated than these simple step functions, and so are the wavelet functions that we use to analyze them. However, the underlying idea is the same, and the multiresolution analysis approach provides a powerful framework for many kinds of data: audio, image, seismic, biological, etc.



To analyze data like the mouse wheel-running activity shown above, I use a popular type of discrete wavelet developed by Ingrid Daubechies, a world-renowned mathematician specializing in wavelets.[1] Wavelet and scaling functions need to satisfy certain special properties, so we can't just choose anything as our building block pattern. Fortunately, many wavelet families are available to choose from. I chose the particular Daubechies wavelet D8 after exploring which did the best job of isolating the circadian pattern for all the different types of data I use: activity, body temperature, and gene expression data.



Observe that the Haar ψ and φ satisfy

$$\varphi(t) = \varphi(2t) + \varphi(2t - 24),$$
$$\psi(t) = -\varphi(2t) + \varphi(2t - 24).$$

This pair of equations is called the *two-scale relation*. The D8 ψ and φ satisfy a two-scale relation with 8 terms on each right-hand side, where 8 coefficients h_k play a key role in computing the s_j and d_j arrays:

$$\varphi(t) = \sum_{k=0}^{7} h_k \varphi(2t - 24k),$$
$$\psi(t) = \sum_{k=0}^{7} (-1)^k h_{7-k} \varphi(2t - 24k),$$
$$h \approx [.33\ 1.01\ 0.89\ -0.04\ -0.26\ 0.04\ 0.05\ -0.01].$$

Where the Haar s_j and d_j use computations on pairs of data values (Haar is also called D2), D8 calculates a weighted average of 8 data values using the h_k, then moves over 2 spots and repeats, creating new arrays s_{j+1} and d_{j+1} that are half the length of the previous s_j. The halving nature of this algorithm requires the data to have length equal to a power of 2 – you can add zeros to the end if needed to make it such a length.

I hope you enjoyed this brief overview, which only scratched the surface of the beautiful theory of wavelets and their applications – there's much more to learn if you are interested!

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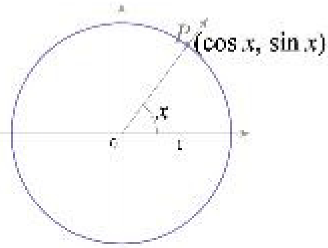
[1]See www.simonsfoundation.org/2019/06/12/making-wavelets-a-profile-of-ingrid-daubechies/

Sine and Cosine

by Whitney Souery

This Summer Fun problem set is intended for people who haven't learned about the sine and cosine function and are up for a challenging way to learn about them through problem solving.

In the coordinate plane, consider a ray emanating from the origin. Suppose that the ray makes an angle x as measured counterclockwise from the positive horizontal axis. Let P be the point where this ray intersects the circle of radius 1 centered at the origin. By definition, the Cartesian coordinates of P are (cos x, sin x).



1. Sketch graphs of the functions cos x and sin x.

2. Determine the exact values of cos x and sin x for the following angles x:

A. 0°	B. 90°	C. 180°	D. 270°	E. 360°	F. -90°
G. 45°	H. 225°	I. 60°	J. 300°	K. 30°	L. 72°

3. Explain why $\cos^2 x + \sin^2 x = 1$.

4. How are sin x and sin($-x$) related? How are cos x and cos($-x$) related?

A function f is said to be **periodic** if there exists $p > 0$ such that $f(x + p) = f(x)$ for all x. The **period** of a periodic function f is the smallest $p > 0$ such that $f(x + p) = f(x)$.

5. What are the periods of cos x and sin x?

6. For each of the following functions, determine if it is periodic, and if so, determine its period.
   A. $\sin(x/2)$ B. $\cos(3x + 1)$ C. $\sin(x^2)$ D. $\cos^2(x)$

7. Exploit the symmetry of a circle to prove that $\cos x = \sin(90° - x)$, where x is measured in degrees.

8. Express $\cos(x + y)$ and $\sin(x + y)$ in terms of cos x, sin x, cos y, and sin y.

9. Use your answer to Problem 8 to give formulas that express $\cos(2x)$ and $\sin(2x)$ in terms of cos x and sin x.

10. Let a and b be constants. Determine values of c and d so that $a \sin x + b \cos x = c \sin(x + d)$.

11. Let $P = (x, y)$. Let P' be the image of P under counterclockwise rotation about the origin by A degrees. What are the coordinates of P' in terms of x, y, and A?

12. Suppose that $x + y + z = 180°$. Prove that $\sin(2x) + \sin(2y) + \sin(2z) = 4 \sin x \sin y \sin z$.

Summer Fun!

How High Can You Count?

by Laura Pierson and Matthew de Courcy-Ireland

What are ordinals?

You probably know that if you start counting 0, 1, 2, 3, 4, …, you could go on forever. But what if *after* forever, you keep going? This give rise to the **ordinals**.

Let's call the next number after all the positive integers ω (**omega,** the last letter of the Greek alphabet). This is the first infinite ordinal. Now, we just keep adding 1 to get $\omega + 1$, $\omega + 2$, and so on. Then, the first ordinal bigger than all of these is $\omega + \omega$, or $\omega \cdot 2$ (but not $2 \cdot \omega$, for reasons we will see later). Similarly, we get $\omega \cdot 2 + 1$, $\omega \cdot 2 + 2$, …, $\omega \cdot 3$, …, $\omega \cdot 4$, … and eventually, we will get ω^2. If we keep going, we can count to ordinals like $\omega^3 \cdot 3 + 7$, ω^ω, ω^{ω^ω}, … (think about how!).

Basically, there are two ways to build new ordinals:

- **Successor ordinals** are defined by adding 1 to the previous ordinal.
- **Limit ordinals** are defined as the first ordinal bigger than an infinite increasing sequence of smaller ordinals.

1. What should the successors of the following ordinals be?
   A. 17   B. ω^2 C. $\omega^7 \cdot 6 + \omega^4 + 5$

2. What should the limits of the following increasing sequences of ordinals be?
   A. $\omega \cdot 2, \omega \cdot 3, \omega \cdot 4, \ldots$ B. $\omega, \omega^2, \omega^3, \omega^4, \ldots$
   C. $\omega^2 + \omega + 2, \omega^2 + \omega + 4, \omega^2 + \omega + 6, \omega^2 + \omega + 8, \ldots$
   D. $\omega^{\alpha_1}, \omega^{\alpha_1}, \omega^{\alpha_1}, \ldots$, where $\alpha_1, \alpha_2, \alpha_3, \ldots$ is an increasing sequence of ordinals with limit α.
   E. $\alpha_1 + 1, \alpha_2 + 1, \alpha_3 + 1, \ldots$, where $\alpha_1, \alpha_2, \alpha_3, \ldots$ is an increasing sequence of ordinals with limit α.

3. Show that no ordinal can be both a limit ordinal and a successor ordinal.

4. Determine whether the following ordinals are successors or limits, by giving either the previous ordinal or an infinite increasing sequence whose limit is that ordinal.
   A. $\omega + 4$ B. $\omega^3 + \omega^2 \cdot 2$ C. ω^{ω^ω}

Ordinal Arithmetic

Now that we have these things called ordinals, it would be nice if we could do things with them in general, like adding and multiplying them.

5. Before reading on, think about how you would define arithmetic of ordinals. How can we define addition of positive integers? How about multiplication? What would this look like if we extend it to infinite ordinals?

Summer Fun!

Here's one way we can think about addition. If we want to add 3 and 5, we line up 3 things, then line up 5 things after them, then count how many things we have total. Now if we want to add ω and 5 we can try to do the same thing. We line up ω things, then line up 5 things, then count how many things we have total. To understand how to do this, let's define what we mean a little more formally.

In set theory, the only things that exist are sets. We can have sets of sets, and sets of sets of sets, and so on, but we can never have things that aren't sets. Thus, we will define each ordinal α to be a *set* containing exactly α elements. Specifically, *its elements are all the previous ordinals***.**

We start with defining 0 as the **empty set**: $0 \equiv \{\} = \emptyset$. Similarly, we can define

$$\begin{aligned} 1 &= \{0\} = \{\emptyset\}, \\ 2 &= \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \\ 3 &= \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \end{aligned}$$

and so on. We can now define ω to be the set containing all the natural numbers.

6. Following this notation, write out the elements of the set 4 just using sets.

7. How can you describe the elements of the following sets? (You can write the elements as ordinals, not as sets, if you wish.) A. $\omega \cdot 2$ B. ω^2 C. ω^ω

Let's get back to addition. When we add 3 and 5, we can say we're listing the elements of the set 3, then listing the elements of the set 5 after them, like this: 0, 1, 2, 0, 1, 2, 3, 4.

Now we count from the left how many elements are in this list by assigning each list element to an ordinal, starting from 0 (at right).

0,	1,	2,	0,	1,	2,	3,	4
↓	↓	↓	↓	↓	↓	↓	↓
0,	1,	2,	3,	4,	5,	6,	7.

We see that we have paired up each element of our list with exactly one element of the set 8, in increasing order from left to right. Thus, we say that our set has 8 elements. (If you want the fancy terminology, we say that this is an **order-preserving bijection** of our list with the set 8, and thus our list has **order type** 8.) Note that 8 is the *set of numbers* we assigned to things on our list, *not* the biggest number we assigned to a thing on our list.

Now if we want to add ω and 5, we do the same thing. We list the elements of the set ω, then list the elements of the set 5, then assign each thing on our list to an ordinal, counting from the left:

0,	1,	2,	3,	…,	0,	1,	2,	3,	4
↓	↓	↓	↓		↓	↓	↓	↓	↓
0,	1,	2,	3,	…,	ω,	$\omega + 1$,	$\omega + 2$,	$\omega + 3$,	$\omega + 4$.

We see that we count up to $\omega + 4$, which is the last element of the set $\omega + 5$. Thus, there are $\omega + 5$ elements in the set.

8. Addition of integers is **commutative**, meaning order doesn't matter, i.e., $a + b = b + a$. show that addition of ordinals is *not* commutative. In particular, show that $1 + \omega = \omega \neq \omega + 1$.

9. What is $1 + \omega^2$? How about $\omega + \omega^2$? What can you say in general about $\omega^\alpha + \omega^\beta$, where α and β are ordinals with $\alpha < \beta$?

10. Addition of integers is also **associative**, meaning we can rearrange parentheses, i.e., $(a + b) + c = a + (b + c)$. Is ordinal addition associative?

11. Show that every ordinal can be uniquely written in the form $\omega^{\alpha_n} \cdot c_n + \ldots + \omega^{\alpha_1} \cdot c_1 + c_0$ for some $n \geq 0$, where $\alpha_n > \ldots > \alpha_1$ are nonzero ordinals, $c_1, \ldots, c_n$ are positive integers, and c_0 is a nonnegative integer. (This is called the **Cantor normal form**). How can you describe the sum of two ordinals in terms of Cantor normal forms?

12. As we've seen, addition of ordinals is not commutative in general. However, some ordinals do commute with each other (under addition). Can you come up with examples of ordinals that do commute? Can you characterize in general which pairs of ordinals commute? (Think about Cantor normal forms.)

13. Does it make sense to define subtraction of ordinals?

Now let's go on to multiplication. Multiplication is repeated addition; for instance,

$$3 \cdot 5 = 3 + 3 + 3 + 3 + 3 = 5 + 5 + 5.$$

However, in the case of ordinals, these are not necessarily the same, so we'll choose the first one and define $\alpha \cdot \beta$ to mean β copies of α added together. That is, we'll list out the elements of the set α a total of β times, and then count how many things are on our list.

14. Under this definition, what is $2 \cdot \omega$? You should find that it is *not* the same as $\omega \cdot 2$! Thus, multiplication of ordinals is *not* commutative.

15. Is it true that $\omega^\alpha \cdot \omega^\beta = \omega^{\alpha + \beta}$? (You might want to proceed inductively, considering separately the cases where β is a limit or a successor.)

16. Is multiplication of ordinals associative?

17. Does ordinal multiplication satisfy the distributive property, $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ and $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$? (You should check these two identities separately since they might behave differently!)

18. How can you describe the product of two ordinals in terms of their Cantor normal forms?

Summer Fun!

How Big are They Really?

Another question we can ask about ordinals is how big they are. Of course, we already have one sense of their size, in that we can list them out in order and ones that come later in the list are bigger, so they're all different sizes. But it seems like adding one thing to an infinite set doesn't make it that much bigger… after all, it's still infinite, right?

Say we have some red balls and some green balls, and we want to know if we have the same number of each color. Well, one way to do this is to try to pair up each green ball with a red ball, and if we can do this without running out of either color, there must be the same number. Using this idea, we say that two sets have the same **cardinality** if there is some way of pairing up the elements of the two sets without running out (called a **one-to-one correspondence** or a **bijection**). This is different from how we "counted" sets before in that we no longer care which order we pair up the elements.

For instance, the set of positive integers and the set of even positive integers have the same cardinality, because we can pair 1 with 2, 2 with 4, 3 with 6, and so on, and we'll never run out. Thus, these two sets have the same "size" even though one of them is a subset of the other!

We call the possible cardinalities of infinite sets the **cardinals**, and write $\aleph_0$, $\aleph_1$, $\aleph_2$, (this is using the Hebrew letter **aleph**). The smallest of these, $\aleph_0$, is the cardinality of ω, and ordinals of this cardinality are called **countable**.

19. Show that $\omega + 1$ and $\omega \cdot 2$ are countable by putting their elements in one-to-one correspondence with the elements of ω.

20. Show that ω^2 is countable.

21. Show that a countable union of countable sets is countable. (This is very similar to the last problem.)

22. Show that the set of real numbers from 0 to 1 is uncountable. Try proof by contradiction: suppose you've written them all on a list, and then find one that's not on your list. (Thus, not all sets are countable!)

23. Show that **all** the ordinals we've written down so far are countable! Thus, as infinite sets go, they're all actually really small!

Bonus Problems

We have seen that addition and multiplication are not commutative. This means that given ordinals $\alpha_1, \ldots, \alpha_n$, one can form several different sums and several different products. What is the largest possible number as a function of n? How many different permutations give different sums? Different products? We'll find the answers to these questions in the following series of bonus problems.

Summer Fun!

B1. Draw a picture showing that $(\omega + a)(\omega + b) = \omega^2 + \omega \cdot b + a$, where a and b are positive integers. Recall that this ordinal is $\omega + b$ copies of $\omega + a$.

B2. Show that $(\omega + a)(\omega + b)\ (\omega + c) = \omega^3 + \omega^2 \cdot c + \omega \cdot b + a$, where a, b, and c are positive integers.

B3. Find a pattern that expresses $(\omega + a_1)\cdots(\omega + a_n)$ in Cantor normal form where $a_1, \ldots, a_n$ are positive integers.

B4. Deduce that it is possible for all permutations of any number of ordinals to yield different products.

It is more subtle to understand how many sums are possible. We have to understand how to decompose an ordinal into a sum of smaller ones, and we need to understand which ordinals can be absorbed into larger ones in a sum.

A non-zero ordinal is called "indecomposable" if it is not equal to the sum of strictly smaller ordinals.

For example, 1 is indecomposable because the only smaller ordinal is 0 and $0 + 0 = 0$. Another indecomposable ordinal is ω because the sum of two smaller ordinals remains finite. Note that $1 = \omega^0$ and $\omega = \omega^1$. In general, the indecomposable ordinals are the powers of ω, including infinite powers using the notion of exponentiation for ordinals.

B5. Show that any ordinal is a sum of indecomposable ordinals.

An indecomposable ordinal "absorbs" addition by any smaller ordinal on the left. For instance, $0 + 1 = 1$, $2019 + \omega = \omega$, and $\omega + \omega^2 = \omega^2$. Let $\varphi(\alpha)$ be the largest indecomposable summand of α. For instance, $\varphi(\omega^2 + \omega) = \omega^2$.

B6. Show that the number of sums obtained by permuting three ordinals $\alpha_1, \alpha_2, \alpha_3$ is at most 5.

B7. Give an example of three ordinals where there really are five different sums.

B8. Let $f(n)$ be the largest number of sums that can be obtained from n ordinals, for instance $f(2) = 2$ and $f(3) = 5$. Give a recursive formula for computing $f(n)$ from the previous values $f(2)$, $f(3), \ldots, f(n - 1)$.

B9. Use your recursive formula to show that the number of sums is much less than the number of permutations. What other patterns do you notice in the numbers $f(n)$?

Theses patterns in the numbers $f(n)$ were found by Paul Erdős. For more, see "Some Remarks on Set Theory" by P. Erdős in Proceedings of the American Mathematical Society, Vol. 1, No. 2 (Apr., 1950), pp.127-141.

Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 24 - Meet 11
May 2, 2019

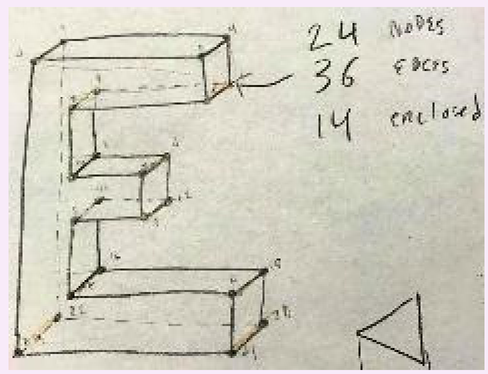
Mentors: Talia Blum, Grace Bryant, Adeline Hillier, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Savannah Tynan, Jane Wang, Rebecca Whitman

Visitor: Eric Lander

Eric Lander, the president and founding director of the Broad Institute, stopped by and gave us an inspirational biographical talk, tracing his days from when he was a student at Stuyvesant to professional mathematician to founding director of the Broad Institute.

Some members counted vertices (nodes), edges, and faces of various polyhedrons, eventually rediscovering Euler's celebrated formula.

Polyhedrons provide many paths of exploration to pursue. While Euler's formula is well-known, there surely remain many beautiful, unknown, facts still to be discovered about them.



Session 24 - Meet 12
May 9, 2019

Mentors: Talia Blum, Kelly Chen, Anna Ellison, Amy Fang, Katie Gravel, Adeline Hillier, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Melissa Sherman-Bennett, Shohini Stout, Jane Wang, Rebecca Whitman, Jasmine Zou

We held our traditional end-of-session Math Collaboration. Try your hand at solving some of the problems.

An isosceles trapezoid's sides are all whole numbers. Its bases have length 200 and x. The other two sides both have length 62. How many possible values are there for x?

Tree stumps are placed at (x, y) where x and y are positive integers. You jump from stump to stump, starting at (1, 1). You can jump to any stump that increases your y-coordinate and doesn't decrease your x-coordinate. How many ways are there to jump to (3, 5)?

How many non-congruent right triangles are there whose sides have integer length and one leg of length 121?

Calendar

Session 23: (all dates in 2018)

September	13	Start of the twenty-third session!
	20	
	27	
October	4	
	11	
	18	
	25	
November	1	
	8	
	15	
	22	Thanksgiving - No meet
	29	
December	6	

Session 24: (all dates in 2019)

January	31	Start of the twenty-fourth session!
February	7	
	14	
	21	No meet
	28	
March	7	
	14	
	21	
	28	No meet
April	4	
	11	
	18	No meet
	25	
May	2	
	9	

Session 25: To be announced…

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

A heartfelt Thank You to Monica Concepcion, Rachel Gesserman, and all employees at the Broad Institute for giving Girls' Angle a marvelous, inspiring home for the past three years!

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton University
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: _______________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ___________________________________