Girls' Angle Bulletin

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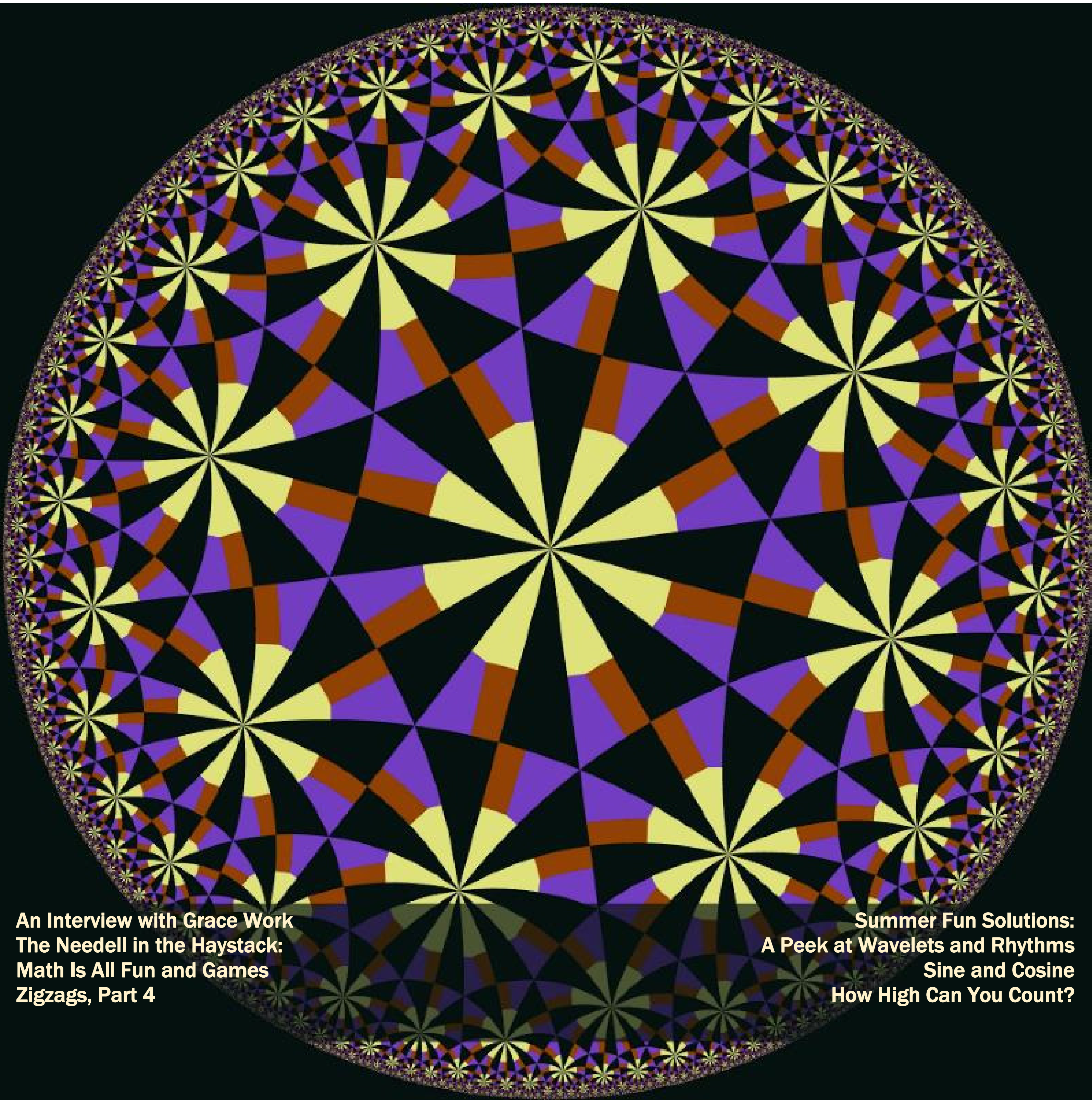
To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

Learning mathematics requires concentration, and in today's world, with all its distractions, it can be challenging to focus and concentrate on anything at all. We hope that the Girls' Angle's club can serve you as a place where you can do math in peace. - Ken Fan, President and Founder

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Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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On the cover: A hyperbolic tiling created using *KaleidoTile*, a computer program by mathematician Jeff Weeks. See: geometrygames.org/KaleidoTile/index.html

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An Interview with Grace Work

Grace Work is Girls' Angle's new Head Mentor. She earned her doctoral degree in mathematics from the University of Illinois Urbana Champaign and was an Assistant Professor of Mathematics at Vanderbilt prior to assuming the role of Head Mentor. Grace is also a research affiliate of MIT's Department of Mathematics.

Whether you love math or hate math, whether you think you're good at math or bad at math, there will be opportunities for everyone to explore, ask questions, make mistakes, learn, and hopefully find something that sparks your interest and keeps you coming back for more.

Ken: You've got a very interesting path into mathematics. I understand you began college as an English major. Is that correct?

Grace: Yes, from a very young age I read everything I could get my hands on and wrote stories in my free time. English remained one of my favorite subjects throughout high school and it seemed to be the clear choice of major in college.

Ken: How did you go from there to finding a love for mathematics?

Grace: I had always liked math, but had never excelled and hadn't ever considered it as a career path. When I started taking college courses I realized that what I liked about English was still just reading and writing, and that could be done on my own, but what I liked about math was learning new things, and that would be difficult on my own. Thus, in my second semester, I decided I could live with never taking another English class and switched my major to Math.

Ken: After you graduated from college, you got a Master of Science degree in Mathematics Education, but then went on to obtain a doctoral degree in mathematics from the University of Illinois. Could you retrace that history for us?

Grace: Straight out of college I joined the New York City Teaching Fellows program (NYCTF), a program that prepares recent college grads and career changers to teach in high-need schools in NYC. While teaching you also attend night and summer classes to receive your Masters. To receive the Master of Science in Mathematics Education you had to take a couple of higher level math classes. I chose elementary number theory and it was through this course that I really discovered a love for mathematics.

Elementary number theory concerns properties of the integers, and the questions were easy to state and sounded simple, but the proofs were deep and interesting. I would find the problems keeping me up at night and occupying the back of my brain during the day. It was here I first learned about Goldbach's Conjecture: Every even integer greater than 2 can be expressed as the sum of two primes. Simply stated and understood but unproven. It was after taking this course that I started thinking about math graduate school. I had also discovered a love for teaching and thought graduate school would be a way to learn more mathematics and enable me to teach a wider range of students.

Ken: Was graduate school in mathematics what you were expecting? Most mathematicians I've spoken to found graduate school to be quite a challenging

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and salient period of their lives. Was it this way for you?

Grace: I had no idea what to expect when it came to graduate school. I did expect it to be challenging, especially since it had been 3 years since I had graduated undergrad. It was more challenging than I could ever have anticipated, and there were many times throughout the first two years that I felt a huge mistake had been made in accepting me. But there was so much math I had no idea existed, I had never heard of topology or combinatorics; I loved attending seminars and conferences to learn about all these different areas.

I also had no idea what research mathematics would look like. All the problems I had worked on before were already solved with solutions that were known and checked by a professor. Research posed new and different challenges than the coursework, but these were challenges I enjoyed overcoming and I started to feel more like I belonged.

Ken: What advice would you give to an undergraduate math major who is trying to decide about math grad school? What should they consider?

Grace: The biggest advice is to think about why you're thinking about graduate school and what is it that you love about math. Do you love the process of finding the solution to problems? Do you love making and learning from your mistakes? Math grad school is not just about doing well on the subject GRE or getting good grades, math grad school eventually requires research and research requires tenacity.

Ken: One of your research interests is hyperbolic geometry. What is hyperbolic geometry?

Grace: Hyperbolic geometry is one of the things that happens when you try to break the rules. Think about Euclidean (planar) geometry that we all know and love. If we have a line in the plane and a point not on the line then there is exactly one line through that point parallel to the given point. Hyperbolic geometry replaces that statement with given a line and a point not on the line then there are at least two distinct lines through the point that do not intersect the given line.

Ken: What fascinates you about hyperbolic geometry?

Grace: What fascinates me the most about hyperbolic geometry is looking at it in contrast to Euclidean geometry. Besides the earlier example, we also have the fact that in hyperbolic geometry triangles have angles which add up to less than 180°. Polygons also behave in unexpected ways, for example it is possible to build an infinite regular polygon of arbitrary side lengths – in Euclidean geometry the only way to get an infinite regular polygon is to have side lengths tending towards 0. One can visualize the hyperbolic plane in several ways, the one I use the most is the upper half-plane model, which can also be viewed as the part of the complex plane where the imaginary part is positive. In this model the shortest path, called a **geodesic**, between two points lies either on half-circles, with center on the real axis, or straight vertical rays orthogonal to the real axis.

Ken: Are there any books you'd recommend to learn about hyperbolic geometry?

Grace: *Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots* by Francis Bonahon, *Geometry of Surfaces* by John Stillwell, *The Shape of Space* by Jeffrey Weeks, and *Three-Dimensional Geometry and Topology* by William Thurston.

Ken: The word "horocycle" appears in one of the titles of one of your papers. What is a horocycle?

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Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Dr. Grace Work and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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The Needell in the Haystack[1]

Math Is All Fun and Games

by Deanna Needell | edited by Jennifer Silva

Last time, we talked about some graph theory and how we can use those tools to efficiently color maps and find perfect matchings among groups of people. As promised, this time we will explore related ideas that take us into the realm of **game theory**. As the name suggests, the field is full of fun and games – but it is also a rich and challenging subject filled with deep mathematics.

The author was first introduced to game theory as an undergraduate at the University of Nevada through her professor, Dr. Quint. On the first day of class, Dr. Quint offered to play a game, telling the class that the winner would win a prize of $20 (this was quite good motivation for the majority of us students!). The game was played as follows: Each student would write down any number greater than 0 and less than 100, along with their name. Dr. Quint would then collect all of the numbers and compute the average. The student whose number was closest to *half* the average would win.

Pause for a moment and ask yourself what number you would write and why. What assumptions are you making? If each student picks a number from 0 to 100 uniformly at random, the average would be expected to be 50. So might you wish to write down half of that, namely 25?

A fair assumption we might make is that every student is playing optimally, and wants to win. Thus every other student will also make this deduction, and if everyone writes down 25, then half the average will actually be 12.5. So should you write down 12.5 instead? But again, everyone else will be making this same rationalization and if everyone writes down 12.5, the average would be 6.25. Alas! Won't this reasoning go on forever? So what do you end up writing down? After such pondering, we see that, in fact, writing down *any number* greater than zero, under the assumption that everyone is playing optimally, is not an optimal strategy. Ack! This is one fun example[2] that gives a peek into the exciting problems in game theory.[3]

Stable Marriage Problem

In the last article, we considered the perfect matching problem. Given a group of people, each of whom may or may not "like" the other, the problem searches for a pairing such that the members within each pair like each other. We will now consider a related problem, known as the **marriage game** or **stable marriage problem**. Although the field of game theory has an abundance of deep mathematics, we provide a glimpse through a relatively simple example, which will hopefully lead to some fun reading.

The classic variety of the marriage problem, and the reason for the nomenclature, seeks to pair men and women together in an optimal way. We will formulate and motivate the problem with a slightly more modern take. To that end, consider a set of n people (applicants) and a set of n volunteer positions (or companies that have a volunteer posting).

Each person has ranked the companies in order of her preference, and each company has ranked the applicants in order of its preference. The problem is to pair the applicants and jobs together in an optimal way. But what does optimal mean in this context? Some things may seem obvious. For example, if George ranks the airport information job as her first choice, and

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[1] This content supported in part by a grant from MathWorks.

[2] This game is a minor variant of the game "guess 2/3 of the average," which was created by Alain Ledoux.

[3] The author wrote down 3.5 and the average was 17. She did not win. Was everyone playing optimally?

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the airport has ranked George as its first choice, there is no reason that George and the airport shouldn't be matched. There are other examples we can consider as well. Suppose George ranks the airport above the library, Signe ranks the library above the airport, the airport ranks George above Signe, and the library ranks Signe above George. Then if there were an assignment in which George was matched with the library and Signe was matched with the airport, this would not be optimal; indeed, all four parties involved would be happier if the appropriate swap were made. One can generalize this example to more than two pairings as well.

These observations lead to the notion of a **stable matching**. A matching between applicants and positions is said to be **unstable** if (i) there is an applicant A who prefers some given position P over the position to which applicant A is already matched, and (ii) position P also prefers applicant A over the applicant to which position P is already matched. A matching is **stable** if it is not unstable. The stable marriage problem is described by the wish to obtain a stable matching given two sets each consisting of n elements, along with a ranking of the opposite set from each member. Although still known as the "marriage problem," this problem has many other important applications, including the pairing of graduating medical students to residency positions and the assignment of users to internet servers.

Nash Equilibrium

Before diving into some methods for solving the marriage game, we allow ourselves to go on a small walk down a tangential road. If you've heard about the **Nash equilibrium**, perhaps you were reminded of it by the definition of a stable matching. The Nash equilibrium was made famous in pop culture by the movie *A Beautiful Mind*, starring Russell Crowe in the role of the real-life economist John Forbes Nash Jr. The Nash equilibrium applies to a competitive (non-cooperative) game among two or more players. A set of player strategies is said to be such an equilibrium if each player, knowing the equilibrium strategies of the others, would not gain anything by changing her own strategy.

Refer back to the problem from the author's undergraduate game theory class described at the beginning of this article. Does it have a Nash equilibrium strategy?

Sadly, as alluded to in the popular film, such equilibria do not necessarily result in "good" outcomes for all players. For example, one of the author's favorite game shows from the early 2000s, called *Friend or Foe*, utilized the so-called **Prisoner's dilemma**. In the show, two contestants (who were strangers) played as a pair in answering trivia questions and earning money in their pot. At the end of the show, after they had made a substantial amount of earnings, each of the two contestants had to make a decision. Without seeing the other's decision, they each had to decide to be either a "friend" or a "foe." If both selected "friend," they would split the winnings and each take home half of the pot. If one selected "friend" and the other "foe," the one who selected foe would win the entire pot, leaving the one who selected friend with nothing. If both selected "foe," neither would win any money. Morality aside, what would the optimal strategy be?

As a player, if your partner selects foe, you are not going to win any money regardless of what you select. If your partner selects friend, then you win the most money if you select foe. Thus, one would argue that the optimal strategy is to select foe, and this would hold from both player's points of view. Indeed, this strategy is the Nash equilibrium – even though it leaves both players with the worst outcome! A quick Google search reveals some data from the show: from 227 games roughly 45% of them resulted in a friend-friend outcome. This perhaps reinstates some faith in humanity, that there is more at play than simply optimizing a payout.

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Gale Shapley Algorithm

Let us now return to the marriage game. We present a simple method that is guaranteed to produce a stable matching, put forth by Gale and Shapley in 1962. Their algorithm proceeds iteratively as follows: Typically, one set (e.g., the applicants or the companies) is deemed to be the “proposers”; here we allow the companies to be the proposers, but this is arbitrary (a different choice may result in a different matching, but will still be a stable matching). In the first round, each of the companies proposes to its first choice applicant. Each applicant who receives a proposal says “no thanks” to each company except their highest ranked company that proposed, to whom they say “maybe.” When a “maybe” response is given, we’ll say that this applicant and company are “provisionally matched.” And let’s call the applicants and companies who are not provisionally matched “unmatched.” From there, each iteration proceeds similarly. In each following iteration, each unmatched company proposes to its highest-ranked applicant to whom it has not yet proposed (whether or not the applicant is already provisionally matched). Then, each applicant replies “maybe” if they are currently unmatched or if they ranked this company over their current provisional match (in this case, the applicant now rejects their current provisional matched company, which then becomes unmatched). This process is repeated until every applicant and company is provisionally matched. When all participants are provisionally matched, these are taken to be the final matching.

There are a few things worth noting before we run through an example. First, by construction, each applicant and company will be in a provisional match upon termination. In the worst case, a company will end up proposing to every applicant; if there were an unmatched applicant, she would have had to say “maybe” to that company at one point. Second, the resulting matching must be stable. Indeed, suppose applicant A and company C are both matched, but not to each other. At the end of the algorithm, it is not possible for both A and C to prefer each other over their current matches. After all, if C prefers A to its current match, C must have proposed to A before proposing to its current match. If A accepted C’s proposal yet is not matched to C in the end, A must have rejected C for a company she prefers more, thus doesn’t prefer C over her current match. If A said no to C’s proposal, then she was already with a company she preferred over C.

An Example

As a concrete example, consider the case with $n = 4$ applicants and companies. Let us suppose the rankings by each applicant are as follows (“BTR” is “Boston Terrier Rescue”):

Applicant	**1st Choice**	**2nd Choice**	**3rd Choice**	**4th Choice**
George	Airport	Library	BTR	MIT
Signe	MIT	BTR	Airport	Library
Blake	MIT	Library	BTR	Airport
Mo	Library	MIT	Airport	BTR

Let’s suppose the rankings by each company are as follows:

Company	**1st Choice**	**2nd Choice**	**3rd Choice**	**4th Choice**
Library	Signe	George	Mo	Blake
Airport	Blake	Signe	George	Mo
BTR	George	Blake	Mo	Signe
MIT	George	Blake	Mo	Signe

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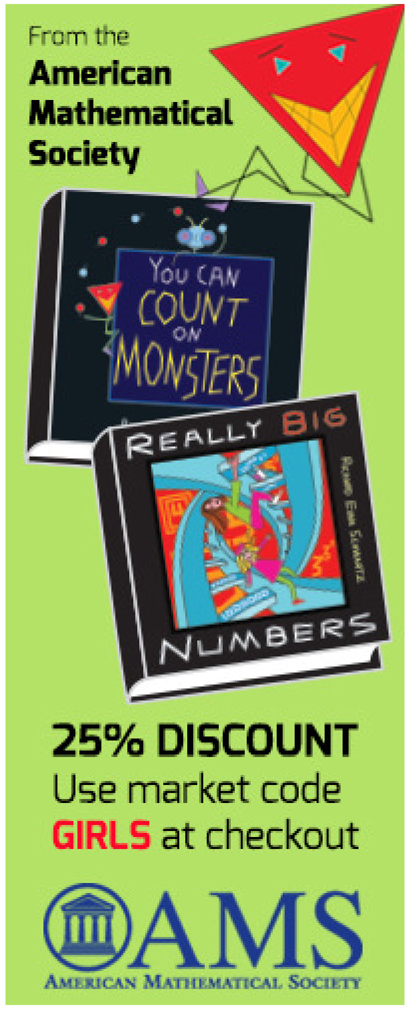

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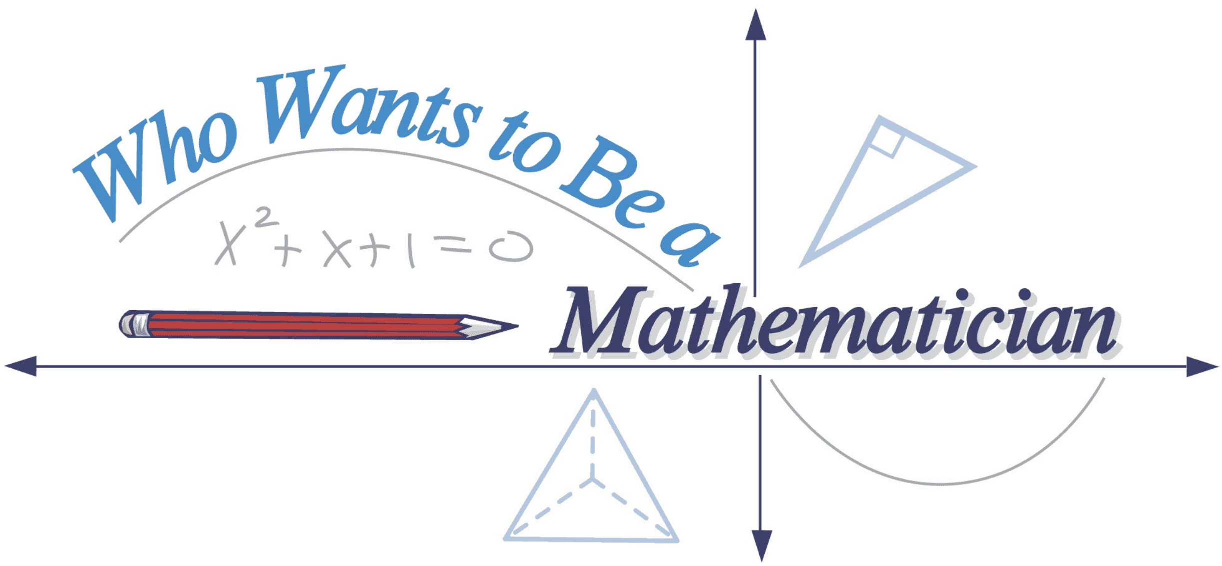

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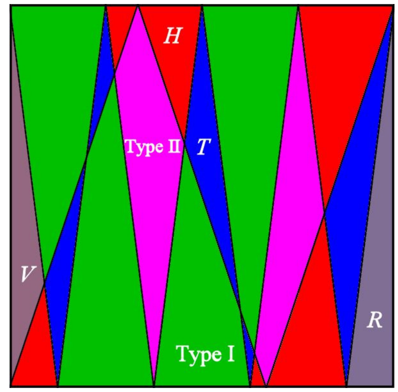
Zigzags, Part 4

by Ken Fan | edited by Jennifer Silva



For your convenience, here are the different types of triangles and quadrilaterals that Emily and Jasmine defined last time. Also, recall that they are assuming that n and m are relatively prime and $n > m$. They found that there are $2m$ triangles of type H and $n - m$ triangles of type T.

Emily: This *is* starting to get complicated!

Jasmine: In fact, I think I already made a mistake when I computed the horizontal side lengths of the two triangles of type H at the ends.

Emily: You did?

Jasmine: Yes. When the two zigzags begin in the same corner, I said that we get a triangle of type H with a horizontal side length *half* the width of the base of one of the isosceles triangles formed by the n-zigzag alone. In actuality, the horizontal side length will be *equal* to the base of one of those isosceles triangles.

Emily: Oh, you're right. The m-zigzag will start in the corner of such an isosceles triangle and exit it on the opposite side, splitting it into two triangles – one of type H and one of type T.

Jasmine: Right. And the base length of the isosceles triangle is actually $2/n$, not $1/n$.

Emily: Good catch!

Jasmine: And then when the zigzags don't come out of the same corner, *that's* when we get a triangle of type H whose horizontal side length is equal to half that of one of the isosceles triangles formed by the n-zigzag. So in this case, the horizontal side length of the resulting triangle of type H will actually be $1/n$… which is what I got before, but through faulty reasoning!

Emily: Good thing you double-checked! So far, we have triangles of type H with horizontal side lengths of $1/n$ or $2/n$, depending on how the two zigzags come out of the corners. Now, what about the other triangles of type H?

Jasmine: For the sake of definiteness, let's focus on the specific case where both zigzags start out from the lower left corner, and on the triangles of type H whose horizontal side is on the bottom edge of the square.

Emily: Okay. To be even more definite, let's place our square in a coordinate grid so that its vertices are (0, 0), (1, 0), (1, 1), and (0, 1). Then the m-zigzag bounces on the interior of the bottom side of the square at the points $(2/m, 0)$, $(4/m, 0)$, $(6/m, 0)$, and so on, and the n-zigzag bounces within the bottom side at the points $(2/n, 0)$, $(4/n, 0)$, $(6/n, 0)$, and so on.

Jasmine: For each point of the form $(2k/m, 0)$, where k is a positive integer and $2k < m$, we must figure out how far it is from the two nearest points of the form $(2j/n, 0)$, where j is an integer.

Emily: This problem really lives within a number line.

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Summer Fun!

In the previous issue, we presented the 2018 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Summer Fun!

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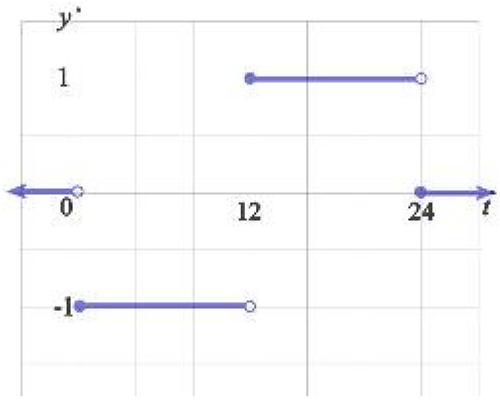
A Peek at Wavelets and Rhythms

by Girls' Angle Staff | edited by Amanda Galtman

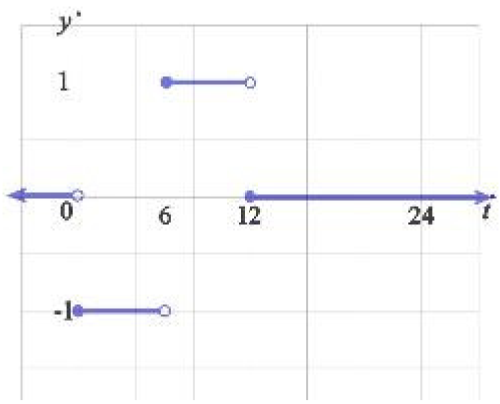
Checkpoint: Plot the function $\psi(t) = \begin{cases} -1 \text{ if } 0 \le t < 12, \\ 1 \text{ if } 12 \le t < 24, \\ 0 \text{ otherwise.} \end{cases}$

Solution: Here is a graph of $y = \psi(t)$:



Checkpoint: To see how this scaling works, plug values into $\psi(2t)$ like $t = 0, 3, 6$, etc., and then plot $\psi(2t)$.

Solution: Here is a graph of $y = \psi(2t)$:



Checkpoint: What scaled function would give the wavelet pattern covering a 6-hour interval? A 3-hour interval?

Solution: For a 6-hour interval, one could use the function $\psi(4t)$. For a 3-hour interval, one could use the function $\psi(8t)$.

Summer Fun!

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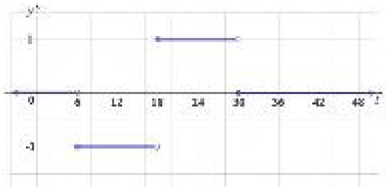
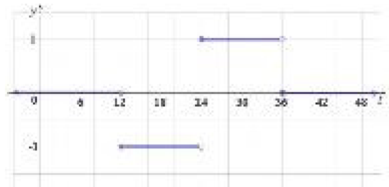
Checkpoint: Plot the functions $\psi(t - 6)$, $\psi(t - 12)$, and $\psi(t - 24)$.

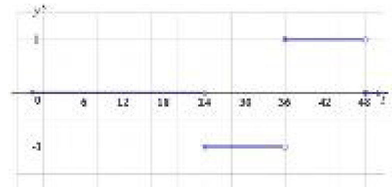
Solutions:



$y = \psi(t - 6)$



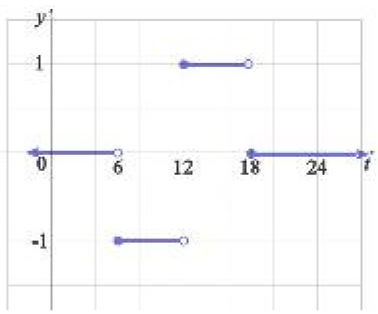
$y = \psi(t - 12)$



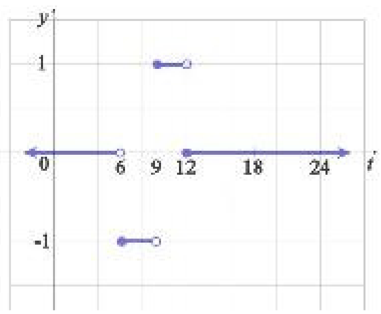
$y = \psi(t - 24)$

Checkpoint: Plot the functions $\psi(2(t - 6))$, $\psi(4(t - 6))$, and $\psi(4(t - 12))$.

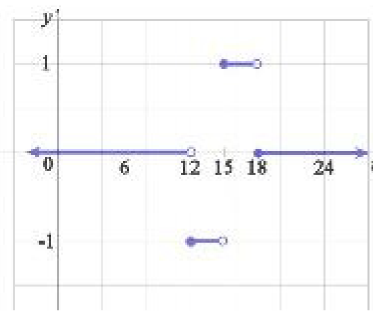
Solutions:



$y = \psi(2(t - 6))$



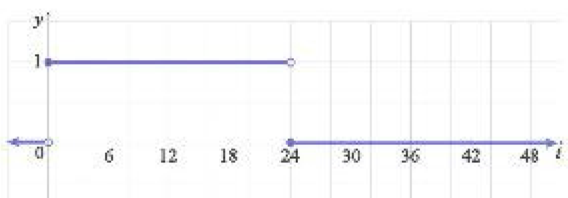
$y = \psi(4(t - 6))$



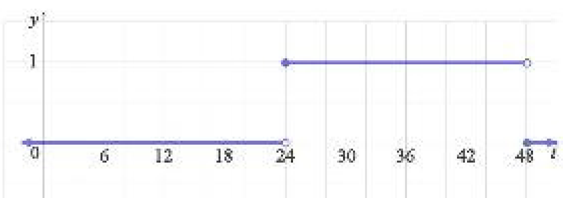
$y = \psi(4(t - 12))$

Checkpoint: Plot $\varphi(t)$ and $\varphi(t - 24)$, where $\varphi(t)$ is the Haar scaling function.

Solutions:



$y = \varphi(t)$



$y = \varphi(t - 24)$

Checkpoint: Use the method described in the Summer Fun Problem Set to sketch the graph of $5\varphi(t) + 3\psi(t) + \psi(2(t - 12))$.

Solution. Here, we provide the associated table and omit the graph.

t	0	3	6	9	12	15	18	21
$5\varphi(t)$	5	5	5	5	5	5	5	5
$3\psi(t)$	-3	-3	-3	-3	3	3	3	3
$\psi(2(t - 12))$	0	0	0	0	-1	-1	1	1
Sum	2	2	2	2	7	7	9	9

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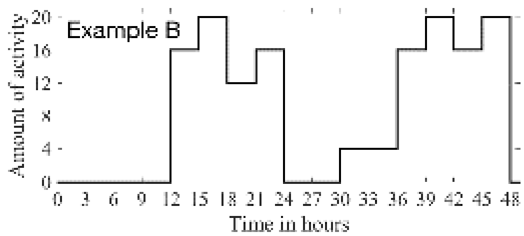
Checkpoint: Why are s_3 and d_3 the same here? How is it connected to zero activity during the daytime?

Solution. If a and b are two consecutive numbers in a pair used to compute a value of s_n and d_n for some n, we would compute $(a + b)/2$ for the relevant value of s_n and $(b - a)/2$ for the relevant value of d_n. These are equal when $(a + b)/2 = (b - a)/2$, or $a + b = b - a$, which simplifies to $a = 0$. Therefore, s_n and d_n are equal when the first numbers of each pair in s_{n-1} are all zero. In this particular example, the values in the table for s_2 and d_2 correspond to 12-hour intervals, so the fact that s_3 and d_3 are the same means that the values of s_2 are zero over the first 12 hours of each 24-hour period.

Checkpoint: Try out the method described in the Summer Fun Problem Set on the graph of Example B below, calculating the arrays s_1, d_1, s_2, d_2, s_3, and d_3, converting into the ψ and φ functions, and then plotting different subsets of terms to explore what features of the data they reveal.



t	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
Ex. B	0	0	0	0	16	20	12	16	0	0	4	4	16	20	16	20

Solution. For s_1 and d_1:

t	0	6	12	18	24	30	36	42
s_1	0	0	18	14	0	4	18	18
d_1	0	0	2	2	0	0	2	2

For s_2, s_3, d_2, and d_3:

t	0	12	24	36
s_2	0	16	2	18
d_2	0	-2	2	0

t	0	24
s_3	8	10
d_3	8	8

From these tables, we can read off that

$$d_1 = 2\psi(4(t-12)) + 2\psi(4(t-18)) + 2\psi(4(t-36)) + 2\psi(4(t-42))$$
$$d_2 = -2\psi(2(t-12)) + 2\psi(2(t-24))$$
$$d_3 = 8\psi(t) + 8\psi(t-24)$$

and

$$s_3 = 8\varphi(t) + 10\varphi(t-24).$$

Therefore, the given graph is the graph of the function $8\varphi(t) + 10\varphi(t-24) + 8\psi(t) + 8\psi(t-24) - 2\psi(2(t-12)) + 2\psi(2(t-24)) + 2\psi(4(t-12)) + 2\psi(4(t-18)) + 2\psi(4(t-36)) + 2\psi(4(t-42))$.

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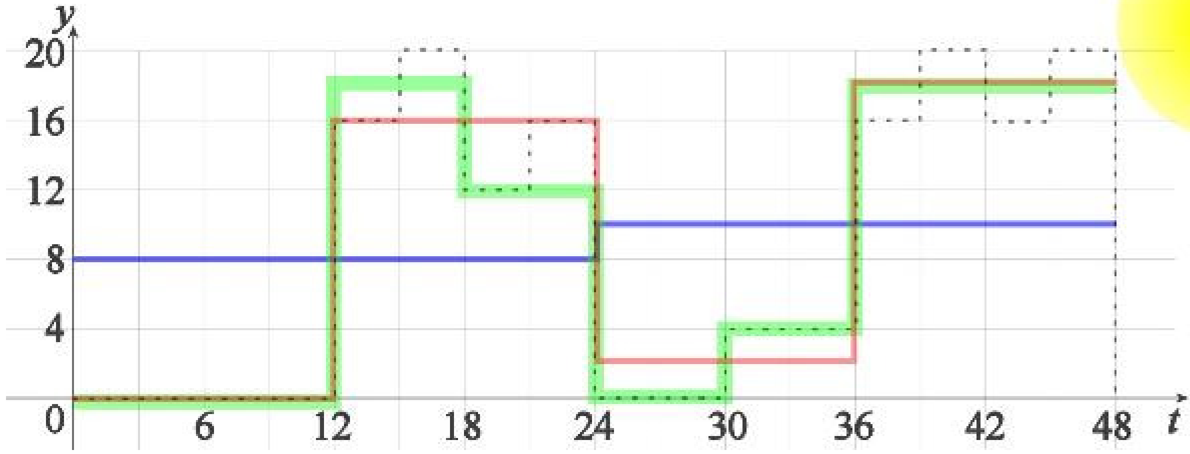

Graphs of different resolutions of the function provided for the problem.

The blue graph, which represents daily averages, is the graph of the sum of the first two terms, $8\varphi(t) + 10\varphi(t - 24)$. The red graph is the graph of the first four terms, $8\varphi(t) + 10\varphi(t - 24) + 8\psi(t) + 8\psi(t - 24)$, and represents the basic day/night variation. The green graph is the graph of the first six terms, $8\varphi(t) + 10\varphi(t - 24) + 8\psi(t) + 8\psi(t - 24) - 2\psi(2(t - 12)) + 2\psi(2(t - 24))$. The dashed black graph is the graph of the original function.

Note: It is a good idea to understand why the method of determining the breakdown of a given function in terms of the Haar wavelet and Haar scaling function works.

The method involves repeatedly taking two numbers, a and b, and computing $(a + b)/2$ and $(b - a)/2$. The key idea is that the values of a and b can be recovered from $(a + b)/2$ and $(b - a)/2$ by adding and subtracting them:

$$a = (a + b)/2 - (b - a)/2 \quad \text{and} \quad b = (a + b)/2 + (b - a)/2.$$

Now suppose that $S(t)$ is constant on h-hour intervals, for some constant h. More precisely, for any integer k, the function $S(t)$ is assumed to be constant for values of t that satisfy $kh \le t < (k + 1)h$. As in the method, we define two functions $s(t)$ and $d(t)$ based on $S(t)$. Both functions are constant for values of t that satisfy $2kh \le t < 2(k + 1)h$. On the interval $2kh \le t < 2(k + 1)h$, let $s(t) = (S(2kh) + S((2k + 1)h))/2$ and let $d(t) = (S((2k + 1)h) - S(2kh))/2$. Using the fact explained in the previous paragraph, we see that

$$S(2kh) = s(2kh) - d(2kh) \quad \text{and} \quad S((2k + 1)h) = s(2kh) + d(2kh). \qquad (*)$$

Another way to express this is to write $S(2kh + t) = \varphi((12/h)t)s(2kh + t) + \psi((12/h)t)d(2kh + t)$ for $0 \le t < 2h$. Notice how the function ψ effects the sign change between the two formulas (*).

In the method, the functions s_{n+1} and d_{n+1} are related to the function s_n exactly as s and d are related to S in the previous paragraph. Thus, one can express s_n in terms of φ, ψ, s_{n+1} and d_{n+1}. One can express s_{n-1} in terms of φ, ψ, s_n, and d_n, or in terms of φ, ψ, s_{n+1}, d_{n+1}, and d_n. The process can continue until one has expressed the original function in terms of a sum of s_{n+1} and the d_k, where $1 \le k \le n + 1$.

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Sine and Cosine

by Whitney Souery and Girls' Angle Staff

2.	**cos *x***	**sin *x***		**cos *x***	**sin *x***		**cos *x***	**sin *x***
A. 0°	1	0	B. 90°	0	1	C. 180°	-1	0
D. 270°	0	-1	E. 360°	1	0	F. -90°	0	-1
G. 45°	$\sqrt{2}/2$	$\sqrt{2}/2$	H. 225°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	I. 60°	1/2	$\sqrt{3}/2$
J. 300°	1/2	$-\sqrt{3}/2$	K. 30°	$\sqrt{3}/2$	1/2	L. 72°	see *	

*We have cos 72° = $(\sqrt{5}-1)/4$ and sin 72° = $\sqrt{5+2\sqrt{5}}$. For one way to deduce this, see *Anna's Math Journal* on pages 14-15 of Volume 8, Number 5 of this Bulletin.

3. The point (cos x, sin x) is on the unit circle centered at the origin, so its distance from the origin is 1. By the Pythagorean theorem, $\cos^2 x + \sin^2 x = 1$.

4. We have sin x = -sin(-x) and cos x = cos(-x).

5. Both have periods of 360° because there are 360° in a full circle.

6. A. Period is 720°. B. Period is 120°. C. This is not periodic. D. Period is 180°.

7. Imagine looking at a vertically hung diagram that illustrates a point on a unit circle centered at the origin. Through the picture, you see your friend, lying on her side staring back at the picture and you, with her body oriented from your right to your left as you go from her head to toe. She sees the distance that you call cos x as the distance sin(90° – x), therefore, cos x = sin(90° – x).

8. We have $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

9. We have $\cos(2x) = 2\cos^2 x - 1 = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ and $\sin(2x) = 2 \sin x \cos x$.

10. It turns out that $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \tan^{-1}(b/a))$.

11. The coordinates of P' are $(x \cos A - y \sin A, x \sin A + y \cos A)$.

12. This can be derived by applying the solutions to Problems 8 and 9:

$$\begin{aligned}
&\sin(2x) + \sin(2y) + \sin(2z) \\
&\quad= \sin(2x) + \sin(2y) + \sin(2(180° - x - y)) \\
&\quad= \sin(2x) + \sin(2y) + \sin(-2(x + y)) \\
&\quad= \sin(2x) + \sin(2y) - \sin(2x)\cos(2y) - \cos(2x)\sin(2y) \\
&\quad= \sin(2x)(1 - \cos(2y)) + \sin(2y)(1 - \cos(2x)) \\
&\quad= 2\sin(2x)\sin^2 y + 2\sin(2y)\sin^2 x \\
&\quad= 4\sin(x)\sin(y)(\cos(x)\sin(y) + \cos(y)\sin(x)) \\
&\quad= 4\sin(x)\sin(y)\sin(x + y) \\
&\quad= 4\sin(x)\sin(y)\sin(180° - z) \\
&\quad= 4\sin(x)\sin(y)\sin(z).
\end{aligned}$$

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How High Can You Count?

by Laura Pierson and Matthew de Courcy-Ireland | edited by Amanda Galtman

1. To find the successor ordinals, we add 1: A. 18 B. $\omega^2 + 1$ C. $\omega^7 \cdot 6 + \omega^4 + 6$

2. A. ω^2 B. ω^ω C. $\omega^2 + \omega \cdot 2$ D. ω^α E. α

3. Consider a successor ordinal, $\alpha + 1$. If we have an increasing sequence of ordinals all less than $\alpha + 1$, they would actually all have to be less than α. The reason is that if one of them were greater than or equal to α, all ordinals after it in the sequence would have to be bigger and would thus be at least $\alpha + 1$. Thus, the limit of the sequence would have to be at most α and not $\alpha + 1$.

4. We can tell whether an ordinal is a successor or a limit based on whether it ends with "+ n" for some positive integer n. For these ordinals we get:

A. Successor ordinal to $\omega + 3$

B. Limit of the sequence $\omega^3 + \omega^2 + \omega$, $\omega^3 + \omega^2 + \omega \cdot 2$, $\omega^3 + \omega^2 + \omega \cdot 3$, …

C. Limit of the sequence ω^ω, ω^{ω^2}, ω^{ω^3}, ω^{ω^4}, ….

5. See the descriptions in the Summer Fun Problem Set.

6. The set 4 is $4 = \{0, 1, 2, 3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$.

7. The elements of these sets are:

A. 0, 1, 2, 3, …, ω, $\omega + 1$, $\omega + 2$, $\omega + 3$, …

B. $\omega \cdot m + n$ for any nonnegative integers m and n

C. All polynomials in ω with nonnegative integer coefficients, i.e., any expression of the form $\omega^n \cdot c_n + \omega^{n-1} \cdot c_{n-1} + \ldots + \omega \cdot c_1 + c_0$, where $c_0, c_1, c_2, \ldots, c_n$, and n are nonnegative integers.

8. Writing out the elements of 1 followed by the elements of ω and counting from the left, we assign an element of ω to each thing on our list. Thus, there are ω things on our list, $1 + \omega = \omega$.

0,	0,	1,	2,	3,	4,	5,	6,	…
↓	↓	↓	↓	↓	↓	↓	↓	
0,	1,	2,	3,	4,	5,	6,	7,	…

9. If we use the same strategy as for the solution to Problem 8, we conclude that $1 + \omega^2 = \omega^2$ and $\omega + \omega^2 = \omega^2$, as shown below.

For $1 + \omega^2$:

0,	0,	1,	2, …	ω,	$\omega + 1$, …	$\omega \cdot 2$, …	$\omega \cdot 3$, …
↓	↓	↓	↓	↓	↓	↓	↓
0,	1,	2,	3, …	ω,	$\omega + 1$, …	$\omega \cdot 2$, …	$\omega \cdot 3$, …

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For $\omega + \omega^2$:

0,	1,	2, …	0,	1,	2, …	ω,	$\omega + 1$, …	$\omega \cdot 2$, …
↓	↓	↓	↓	↓	↓	↓	↓	↓
0,	1,	2, …	ω,	$\omega + 1$,	$\omega + 2$, …	$\omega \cdot 2$,	$\omega \cdot 2 + 1$, …	$\omega \cdot 3$, …

In the case of $1 + \omega^2$, every ordinal that is ω or bigger is matched up with itself. In the case of $\omega + \omega^2$, every $\omega \cdot m + n$ is matched up with $\omega \cdot (m + 1) + n$. Thus, in both cases, every element of the list is matched up with an element of ω^2, with no elements of ω^2 left over. We conclude that there are exactly ω^2 things on the list. In general, for $\alpha < \beta$, we claim that $\omega^\alpha + \omega^\beta = \omega^\beta$. To see this, note that ω^α, $\omega^\alpha \cdot 2$, $\omega^\alpha \cdot 3$, $\omega^\alpha \cdot 4$, … are all less than ω^β, since $\beta \geq \alpha + 1$. If we write out a list as before, our list starts something like this:

0,	1,	2, …	0,	1,	2, …	ω^α, …	$\omega^\alpha \cdot 2$, …	$\omega^\alpha \cdot 3$, …
↓	↓	↓	↓	↓	↓	↓	↓	↓
0,	1,	2, …	ω^α,	$\omega^\alpha + 1$,	$\omega^\alpha + 2$, …	$\omega^\alpha \cdot 2$,	$\omega^\alpha \cdot 3$, …	$\omega^\alpha \cdot 4$, …

We can see that for every nonnegative integer n and ordinal $\gamma < \omega^\alpha$, the ordinal $\omega^\alpha \cdot n + \gamma$ is paired up with $\omega^\alpha \cdot (n + 1) + \gamma$. If we continue writing the list, we see that all ordinals bigger than this (i.e., at least $\omega^{\alpha + 1}$) are paired with themselves, so all elements of our list end up being labeled with an element of ω^β, with none left over.

10. Yes, ordinal addition is associative. The sum $\alpha + \beta + \gamma = (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ means writing out α things, followed by β things, followed by γ things, and then counting how many things there are in total. The total doesn't depend on where we put the parentheses.

11. The expression we want is $\omega^{\alpha_n} \cdot c_n + \ldots + \omega^{\alpha_1} \cdot c_1 + c_0$ where $\alpha_n > \ldots > \alpha_1$.[1] By induction, every nonzero ordinal can be written as a sum of (ordinal) powers of ω, for suppose $\alpha > 1$ is an ordinal and all smaller ordinals can be so expressed. (The base case is $1 = \omega^0$.) Let β be the union of all ordinals γ such that $\omega^\gamma \leq \alpha$. Observe that $\omega^\beta \leq \alpha$ and is the largest ordinal with this property. Therefore, $\alpha = \omega^\beta + \delta$, where $\delta < \alpha$. Either δ is 0 or, by induction, can be written as a sum of (ordinal) powers of ω. By associativity together with Problem 9, whenever we have a smaller power of ω right before a bigger power in our sum, the smaller power gets "absorbed" into the bigger power and goes away. By absorbing then grouping like terms, we arrive at Cantor normal form. When we add two ordinals in Cantor normal form, we get the terms of the first that are at least as big as the biggest term of the second, followed by all the terms of the second.

12. From our characterization of ordinal addition from Problem 11, two ordinals commute if and only if their Cantor normal forms are identical except for possibly the coefficients of their first terms. In such cases, their sum has almost the same Cantor normal form as each summand, except that the leading coefficient is the sum of the leading coefficients of the summands

13. No, subtraction of ordinals doesn't make sense. For instance, if we could define $\omega - 1$, then adding 1 to it would have to give ω. Then ω would be both a limit and a successor ordinal, which we know is impossible.

[1]An error in the problem statement formula in the original version has been corrected in the online version.

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14. From our definition of multiplication, $2 \cdot \omega$ means we write out the elements of 2 (which are 0 and 1) a total of ω times and then count how many things are on our list. This gives us

$$\begin{array}{ccccccccc} 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots \end{array}$$

There are a total of ω things on this list, so $2 \cdot \omega = \omega$, not $\omega \cdot 2$.

15. We claim that $\omega^\alpha \cdot \omega^\beta = \omega^{\alpha+\beta}$. We prove this by induction on β. For the base case, this is true for $\beta = 0$. Now assume it holds for all $\gamma < \beta$. The product $\omega^\alpha \cdot \omega^\beta$ means writing out the elements of ω^α a total of ω^β times (yielding ω^β sublists of length ω^α) and then counting how many things are in the big list. If β is a successor, we can say $\beta = \gamma + 1$ for some γ. Thus, ω^β is a limit of the sequence ω^γ, $\omega^\gamma \cdot 2$, $\omega^\gamma \cdot 3$, …. The first element of the $(\omega^\gamma \cdot n)$-th sublist will be labeled by $\omega^\alpha \cdot (\omega^\gamma \cdot n)$, which the inductive hypothesis implies is $\omega^{\alpha+\gamma} \cdot n$. The limit of this sequence is then $\omega^{\alpha+\beta}$. If β is a limit ordinal, then it is the limit of some increasing sequence β_1, β_2, β_3, … of ordinals. Then, the first element of our ω^{β_n}-th sublist is the $\omega^\alpha \cdot \omega^{\beta_n} = \omega^{\alpha+\beta_n}$-th element of the big list, by the inductive hypothesis. The limit of these ordinals is $\omega^{\alpha+\beta}$, so this must be the total number of things on our list.

16. Yes, ordinal multiplication is associative. The product $\alpha \cdot (\beta \cdot \gamma)$ means writing out the elements of α a total of $\beta \cdot \gamma$ times, which is the same list we get from writing out α a total of β times to give $\alpha \cdot \beta$, and then copying this list γ times.

17. Ordinal multiplication distributes on the left but not on the right. The product $\alpha \cdot (\beta + \gamma)$ means writing out α a total of $\beta + \gamma$ times. This is the same as writing it out β times and then writing it out another γ times, which is what $\alpha \cdot \beta + \alpha \cdot \gamma$ is. Thus, $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. However, we have already seen that $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega \cdot 1 + \omega \cdot 1$. Thus, it is not true in general that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$.

18. When we multiply two ordinals in Cantor normal form, we are adding the left ordinal to itself a bunch of times. When we do this, each copy of the smaller terms gets absorbed by the next copy of the leading term, except possibly one "leftover" copy of them at the end if we are multiplying on the right by a successor ordinal. That is, if we multiply

$$(\omega^{\alpha_n} \cdot c_n + \ldots + \omega^{\alpha_1} \cdot c_1 + c_0) \cdot (\omega^{\beta_m} \cdot d_m + \ldots + \omega^{\beta_1} \cdot d_1 + d_0),$$

if $d_0 = 0$ (i.e., the right ordinal is a limit ordinal), we get

$$\omega^{\alpha_n} \cdot c_n \cdot (\omega^{\beta_m} \cdot d_m + \ldots + \omega^{\beta_1} \cdot d_1) = \omega^{\alpha_n+\beta_m} \cdot c_n d_m + \ldots + \omega^{\alpha_n+\beta_1} \cdot c_n d_1,$$

whereas if $d_0 > 0$ (i.e., the right ordinal is a successor), we get

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$$\omega^{\alpha_n} \cdot c_n \cdot (\omega^{\beta_m} \cdot d_m + \ldots + \omega^{\beta_1} \cdot d_1 + d_0) + \omega^{\alpha_{n-1}} \cdot c_{n-1} + \ldots + \omega^{\alpha_1} \cdot c_1 + c_0$$
$$= \omega^{\alpha_n + \beta_m} \cdot c_n d_m + \ldots + \omega^{\alpha_n + \beta_1} \cdot c_n d_1 + \omega^{\alpha_n} \cdot c_n d_0 + \omega^{\alpha_{n-1}} \cdot c_{n-1} + \ldots + \omega^{\alpha_1} \cdot c_1 + c_0.$$

19. To show that $\omega + 1$ is countable, we pair ω with 0, 0 with 1, 1 with 2, 2 with 3, and so on. This process describes a one-to-one correspondence between the elements of $\omega + 1$ and the elements of ω. For $\omega \cdot 2$, we can pair the elements 0, 1, 2, 3, … with the even numbers and pair the elements ω, $\omega + 1$, $\omega + 2$, $\omega + 3$, … with the odd numbers.

20. The elements of ω^2 are all ordinals of the form $\omega \cdot m + n$, where m and n are nonnegative integers. We can list the ones where $m + n = 0$, followed by the ones where $m + n = 1$, $m + n = 2$, $m + n = 3$, and so on. Every ordinal in ω^2 occupies some finite position on this list.

21. Order each set, and then order the set of sets. Then each element of any of the sets corresponds to an ordered pair (m, n) telling which set it is in and its position in that set. Now, as in the last problem, we can list all the elements with $m + n = 0$, then with $m + n = 1$, $m + n = 2$, $m + n = 3$, and so on.

22. Suppose we have a list of all the real numbers from 0 to 1. Express these numbers in decimal form. If a number has more than one decimal expression, always use the one that terminates (i.e., instead of 0.4999…, use 0.5). Construct a real number by making the decimal in its 10^{-n} place be a 1, unless the nth real number in our list has a 1 in the 10^{-n} place, in which case make it a 2. This new number is different from every number on our list, contradicting the supposition that the list contains all the real numbers from 0 to 1.

23. Suppose our ordinal is written in Cantor normal form, so it is a finite sum of powers of ω. Use Cantor normal form to write every exponent that appears, every exponent within an exponent, and so on. Then, the total sum of the coefficients that appear (including coefficients of 1) at any level of exponent in our ordinal must be finite. There are only a finite number of ordinals where this sum is 1, and then finitely many more where the sum is 2, 3, 4, and so on. Thus, we can list them all in increasing order of coefficient sum, and every ordinal appears somewhere on this list.

Bonus Problems

B1. We omit a picture, as a picture can be idiosyncratic, so will likely differ from valid solutions.

B2. One way to solve this problem is to apply the solution to Problem 18 twice.

B3. The pattern we guess based on the first two cases is that the same integers in reverse order give the coefficients of the powers of ω:

$$(\omega + a_1) \cdots (\omega + a_n) = \omega^n + \omega^{n-1} \cdot a_n + \cdots + a_1.$$

We prove this by induction on the number of terms, n.

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By the induction hypothesis,

$$(\omega + a_1) \cdots (\omega + a_n)(\omega + a_{n+1}) = (\omega^n + \omega^{n-1} \cdot a_n + \cdots + a_1)(\omega + a_{n+1}).$$

We can apply the (one-sided!) distributivee law to expand the product on the right as

$$(\omega^n + \omega^{n-1} \cdot a_n + \cdots + a_1) \cdot \omega + (\omega^n + \omega^{n-1} \cdot a_n + \cdots + a_1) \cdot a_{n+1}.$$

We have $(\omega^n + \omega^{n-1} \cdot a_n + \cdots + a_1) \cdot \omega = \omega^{n+1}$ because the lower-order terms can be absorbed. Also, $(\omega^n + \omega^{n-1} \cdot a_n + \cdots + a_1) \cdot a_{n+1} = \omega^n \cdot a_{n+1} + \omega^{n-1} \cdot a_n + \cdots + a_1$. It follows that $(\omega + a_1) \cdots (\omega + a_n)(\omega + a_{n+1}) = \omega^{n+1} + \omega^n \cdot a_{n+1} + \omega^{n-1} \cdot a_n + \cdots + a_1$, which completes the induction.

B4. Consider the formula from Problem B3 in the case where all the numbers $a_1, a_2, \ldots, a_n$ are distinct. For instance, all permutations of $\omega + 1, \omega + 2, \ldots, \omega + n$ have different products.

B5. Suppose otherwise. Then there is a smallest ordinal that is not equal to a sum of indecomposable ordinals (the minimal criminal). This ordinal is not indecomposable, or else it would be the sum of a single indecomposable ordinal. So it can be written as a sum of smaller ordinals. By definition, all smaller ordinals are sums of indecomposable ordinals, so the minimal criminal is, too. This is a contradiction.

Another way to phrase the argument is as a proof by induction. As base cases, interpret 0 as the empty sum, while 1 is the sum of a single indecomposable ordinal (itself). If α is a sum of indecomposable ordinals, then so is $\alpha + 1$, simply with an extra summand 1. If α is a limit ordinal, then it is either indecomposable or a sum of smaller ordinals. In the latter case, we can assume by induction that the smaller ordinals are themselves sums of indecomposable ordinals.

B6. Find the smallest value of $\varphi(\alpha)$ among the three ordinals. Let's relabel the three ordinals, if necessary, so that $\alpha_1, \ldots, \alpha_k$ are the ones that attain this smallest value of $\varphi(\alpha)$, where k can be 1, 2, or 3. (Recall that $\varphi(\alpha)$ is the largest indecomposable summand of α.) The ordinals α_i for $0 < i \leq k$ can be absorbed on the left in addition. For instance, if $k = 1$, then $\alpha_1 + \alpha_2 + \alpha_3 = \alpha_2 + \alpha_3$. Let γ be the largest indecomposable summand occurring in α_1 and write $\alpha_j = \gamma \cdot c_j + \delta_j$ for $j = 1, \ldots, k$, where c_j is a positive integer and $\delta_j < \gamma$ is an ordinal. Then δ_j can be absorbed into γ wherever δ_j is on the left of γ in the sum. In the case $k = 3$, there are at most 3 sums: $\gamma \cdot (c_1 + c_2 + c_3) + \delta_1$, $\gamma \cdot (c_1 + c_2 + c_3) + \delta_2$, and $\gamma \cdot (c_1 + c_2 + c_3) + \delta_3$.

If $k = 2$, then α_3 can absorb both α_1 and α_2. It follows that

$$\alpha_1 + \alpha_2 + \alpha_3 = \alpha_2 + \alpha_1 + \alpha_3 = \alpha_3,$$

so there are at most 5 sums. If $k = 1$, then α_1 can be absorbed into both α_2 and α_3 wherever α_1 is to the left of either one. This absorption leads to at most 4 different sums $\alpha_2 + \alpha_3$, $\alpha_3 + \alpha_2$, $\alpha_2 + \alpha_3 + \alpha_1$, and $\alpha_3 + \alpha_2 + \alpha_1$.

In all cases, there are at most 5 sums.

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B7. Using the notation from our solution to Problem B6, we need to take $k = 2$. With a little experimentation, we find that the 3 ordinals $\omega + 1$, $\omega + 2$, and ω^2 give us 5 different sums. (The sums are ω^2, $\omega^2 + \omega + 2$, $\omega^2 + \omega + 1$, $\omega^2 + \omega \cdot 2 + 2$, and $\omega^2 + \omega \cdot 2 + 1$.)

B8. We use the same notation as in our solution to Problem B6, except this time we have n ordinals. Among these n ordinals, let γ be the minimal value of $\varphi(\alpha)$ that occurs. We again label the ordinals $\alpha_1, \ldots, \alpha_n$ so that $\varphi(\alpha_i) = \gamma$ exactly for $1 \le i \le k$. For such i, let $\alpha_i = \gamma \cdot c_i + \delta_i$, where c_i is a positive integer and $\delta_i < \gamma$ is an ordinal. These ordinals can be absorbed whenever they occur to the left of an α_j, with $j > k$. We call these the "absorbable" ordinals.

Write A for an absorbable ordinal and N for a non-absorbable one. After absorbing as much as possible, a sum such as AAANANAAA becomes NNAAA. (Here, we omit the plus signs, so "ANA" means an absorbable ordinal plus a non-absorbable ordinal plus an absorbable ordinal, in that order.) There remain some number of "unabsorbed" absorbable ordinals on the right, together with $n - k$ non-absorbable ordinals on the left. The number of possible sums of the N's is at most $f(n - k)$. Let r be the number of non-absorbed A's remaining on the right. Note that more absorption can occur in the sum of the A's. Namely, the remainders δ_i can be absorbed into γ for all but the last absorbable term. The sum of a subset of the absorbable ordinals is equal to $\gamma \cdot c + \delta$, where c is the sum of all the coefficients of γ in the sum and δ is the remainder of the rightmost ordinal in the sum. There are ${}_kC_r$ ("k choose r") ways to choose which of the k absorbable ordinals remain unabsorbed on the right. If $r > 0$, then for each choice, we obtain up to r different sums by permuting the ordinals to have a different δ_i occur last. If $r = 0$, then there is nothing remaining on the right, which gives only 1 possibility.

Thus, the number of "A sums" is at most $r \cdot {}_kC_r$, or 1 in case $r = 0$. Some of these sums might commute with the "N sums" but, if not, there might be as many as $r \cdot {}_kC_r\, f(n - k)$ sums.

Summing over all the possible values of r gives a maximum number of $(1+\sum_{r=1}^{k} r\binom{k}{r}) f(n-k)$ different sums for a given value of k. Therefore, $f(n)$ is bounded by the largest of these values as k varies from 1 to n:

$$f(n) \le \max_k \ (1+\sum_{r=1}^{k} r\binom{k}{r}) f(n-k). \qquad (*)$$

We now give an example to show that this bound is achieved. Let $\alpha_j = \gamma \cdot 2^j + \delta_j$ for $j = 1, \ldots, k$, where the remainders δ_j are distinct ordinals less than γ. Then, all r ways to permute the remaining A's give distinct sums, and none of them commutes with the sum of the N's. This is because the sum of the N's is large enough to absorb them all (it is at least $\omega \cdot \gamma$). It follows that we have equality in the inequality (*).

In principle, this expression can be used to compute $f(n)$ from previously known values. More conveniently, the binomial theorem can help us simplify the sum further:

$$1+\sum_{r=1}^{k} r\binom{k}{r} = k2^{k-1}+1.$$

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(Proof within a proof: Starting from the binomial formula $(x + y)^k = \sum_{r=0}^{k} \binom{k}{r} x^r y^{k-r}$, differentiate both sides with respect to x. Then, set $x = y = 1$ to obtain the result.)
Thus, we may write the sum-counting function as $f(n) = \max_k \ (k2^{k-1}+1) f(n-k)$.

B9. By computing the maximum explicitly, we will show that $f(n) = 81f(n-5)$ for sufficiently large values of n, namely, $n \geq 21$. Indeed, if $n > 5$, we have

$$f(n) = \max_k \ (k2^{k-1}+1) f(n-k) \geq 81 f(n-5),$$

since $k = 5$ is one of the terms appearing in the maximum. Next, we show that $f(n)$ is no larger than $81f(n-5)$.

Suppose k_1 is where the maximum is achieved, so that $f(n) = (k_1 2^{k_1 - 1} + 1) f(n - k_1)$. The value $f(n-k_1)$ is itself given by a maximum, which is assumed at some index k_2. As a result,

$$f(n) = (k_1 2^{k_1-1}+1)(k_2 2^{k_2-1}+1) f(n-k_1-k_2).$$

Continuing in this way for m steps, we find

$$f(n) = f(n-k_1-\ldots-k_m) \prod_{j=1}^{m} (k_j 2^{k_j-1}+1).$$

Observe that $(k2^{k-1}+1)^{1/k}$ is increasing for $k < 5$ and decreasing for $k > 5$, achieving its maximum value of $81^{1/5}$ at $k = 5$. It follows that

$$\prod_{j=1}^{m} (k_j 2^{k_j-1}+1) = \prod_{j=1}^{m} \left((k_j 2^{k_j-1}+1)^{1/k_j} \right)^{k_j} \leq \prod_{j=1}^{m} 81^{k_j/5}.$$

In particular, $f(n) \leq f(n-k_1-\ldots-k_m) \prod_{j=1}^{m} 81^{k_j/5}$, with equality only if all the indices $k_1, \ldots, k_m$ are equal to 5. Since $f(n)$ must be maximal, $f(n) = f(n-5m) \cdot 81^m$. It follows that $f(n) < C\, 81^{n/5}$ for some constant C, so that $f(n)$ grows exponentially. This is much smaller than $n!$, the total number of permutations of n ordinals. The formula also implies that the numbers $f(n)$ have specific factorizations, given by the factors of the first few values of $f(n)$ together with extra factors of 81. The exact pattern depends on n modulo 5:

$$\begin{aligned} f(5m+1) &= 81^m \\ f(5m+2) &= 193 \cdot 81^{m-1} \\ f(5m+3) &= 193^2 \cdot 81^{m-2} \\ f(5m+4) &= 193^3 \cdot 81^{m-3} \\ f(5m+5) &= 33 \cdot 81^m \end{aligned}$$

These patterns in the numbers $f(n)$ were found by Paul Erdős. For more, see "Some Remarks on Set Theory" by P. Erdős in Proceedings of the American Mathematical Society, Vol. 1, No. 2 (Apr., 1950), pp.127-141.

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Calendar

Session 25: (all dates in 2019)

September	12	Start of the twenty-fifth session!
	19	
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

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Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

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Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdococtoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

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Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

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Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

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