Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

An Interview with Christina Eubanks-Turner
An Architect in Wonderland
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Another Diophantine Equation
Notes from the Club

From the Founder

Our 25th session welcomes wonderful new and returning members. A huge Thank You to Anya Bear and Rohan Kundargi of MIT and Cammie Smith Barnes and her colleagues at Google Cambridge for enabling our club to stay abuzz with math without a skip. - Ken Fan, President and Founder

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Girls' Angle welcomes submissions that pertain to mathematics.

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Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Möbius Pumpkin* by C. Kenneth Fan. For more on Möbius transformations, see this issue's *Learn by Doing*. Happy Halloween!

An Interview with Christina Eubanks-Turner

Christina Eubanks-Turner is an associate professor of mathematics at Loyola Marymount University. She earned both her Master of Science and Doctoral degrees in mathematics from the University of Nebraska-Lincoln.

Melissa[1]: When was the moment you knew you wanted to be a mathematician?

Christina: I knew I wanted to be a mathematician later when I was in college.[2] I originally majored in computer science, but I really disliked computer programming. I loved the logical thinking involved in creating computer algorithms, which I realized was just mathematical thinking, but I found the actual coding and learning of the syntax of languages like C++ and Java to be mundane, tedious work. As far as enjoying math when I was growing up, I really did not know I liked math, I just like to solve puzzles and play logic games.

Melissa: What do you mean by "mathematical thinking"?

Christina: There's a thinking that lends well to doing mathematics. Mathematicians have to be precise in their thinking and they have to be good at analyzing definitions and proofs. In fact, even when mathematicians have found a solution to a problem, they often go even further and seek ways to make the solution more efficient or clearer.

...you don't need a special brain to do math.

Melissa: Elsewhere, you spoke about how one of your grade school teachers would give you logic puzzles outside the regular curriculum. Did you do math outside the curriculum throughout your K12 math education?

Christina: Although I did not realize it, I did many things to build my mathematical abilities. At home my mom and I would put together puzzles with thousands of pieces over several days. I would also complete 3-D puzzles and build various replica structures of monuments like the Eiffel Tower. My mom also bought me books that included logical puzzles that I would complete throughout the summers. And in high school I went to two high schools each day: one that taught all subjects and one that only taught science and math. At the high school that only taught science and math, I was taught in classes by people with doctoral degrees.

Melissa: Was your mom a mathematician?

Christina: My mom was a nurse for over 30 years, and she was big on science and math, always regarding both topics as of great importance. In fact, all my aunts are nurses. My mom suggested that I study computer science, but, as I mentioned, I found that I really took to the more mathematical thinking aspects of computer science, so I became a mathematician. I'm the first mathematician that I know of in the family.

Melissa: What kinds of things did you do in your K12, college, or graduate education that enabled you to become a successful mathematician today?

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[1] The interviewer, Melissa Carleton, is an undergraduate at Wellesley College.

[2] Prof. Eubanks-Turner attended Xavier University of Louisiana.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Dr. Christina Eubanks-Turner and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
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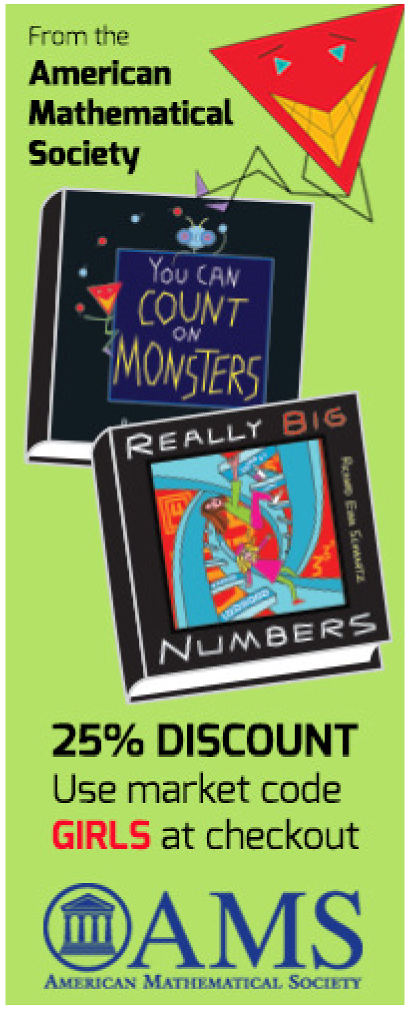

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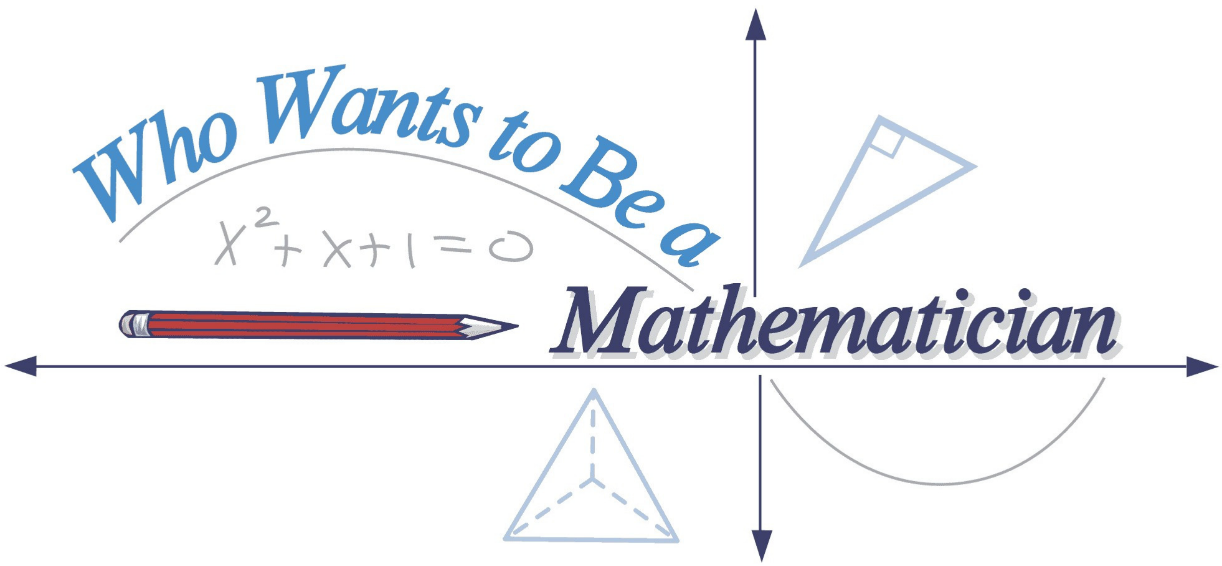
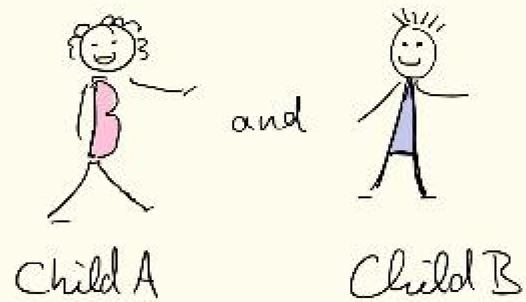


America's Greatest Math Game: Who Wants to Be a Mathematician.

by Konstanze Rietsch

An Architect in Wonderland

As is well known, the Queen of Hearts in Wonderland had a <u>terrible</u> temper. Less well known is that she was also terribly lazy, and that she had two terrible (stickfigure) children, who she named Child A and Child B, quite arbitrarily...



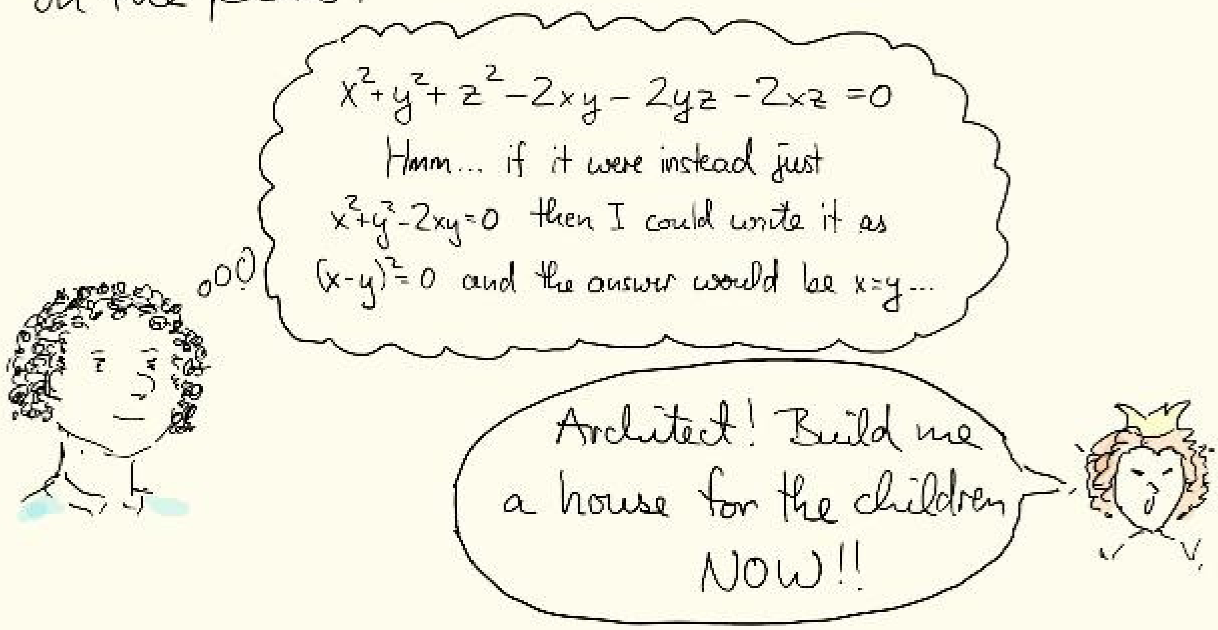
She also had an architect who like everyone else in the kingdom was subject to the queen's every whim, but who nevertheless had a cheery disposition :) and also a love of mathematics!

The Queen of Hearts was particularly lazy in her parenting and Child A and Child B were totally spoiled and generally running wild. And they were also making enormous amounts of noise inside the castle.

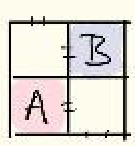
One deafening day the Queen of Hearts decided that she had had enough: those children were wreaking havok in the castle – they needed to move out!

So she called for the architect and told her to build a house on the land behind the castle, just for Child A and Child B.

The architect put aside the diophatine equation* she had just started trying to solve and got to work on the plans.

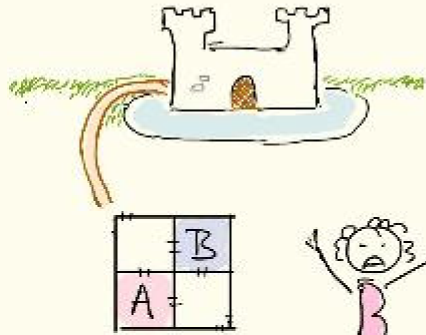


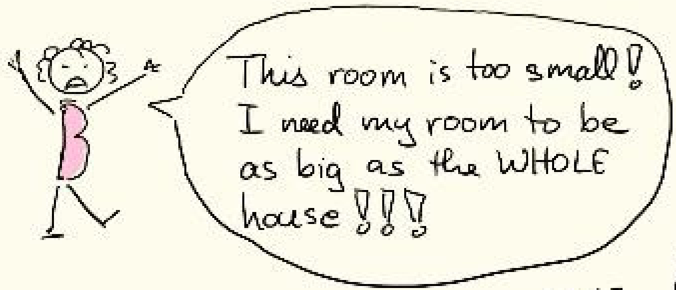
She decided to keep it simple so that she could get back to her diophantine equation quickly, and designed a house with two square rooms of equal size, one for Child A, one for Child B, and two entrance halls.



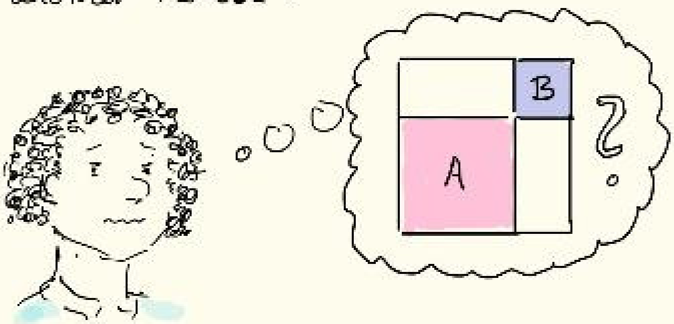
*A diophatine equation is an equation to be solved in integers.

Child A was the first to see the house and bedrooms and immediately complained.



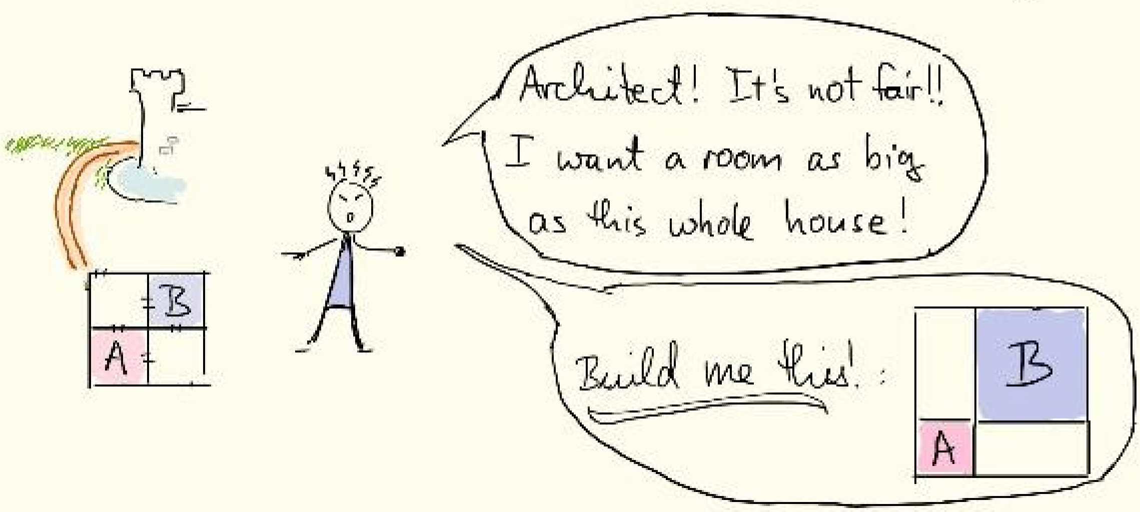


The architect sighed, but went ahead and planned another house:



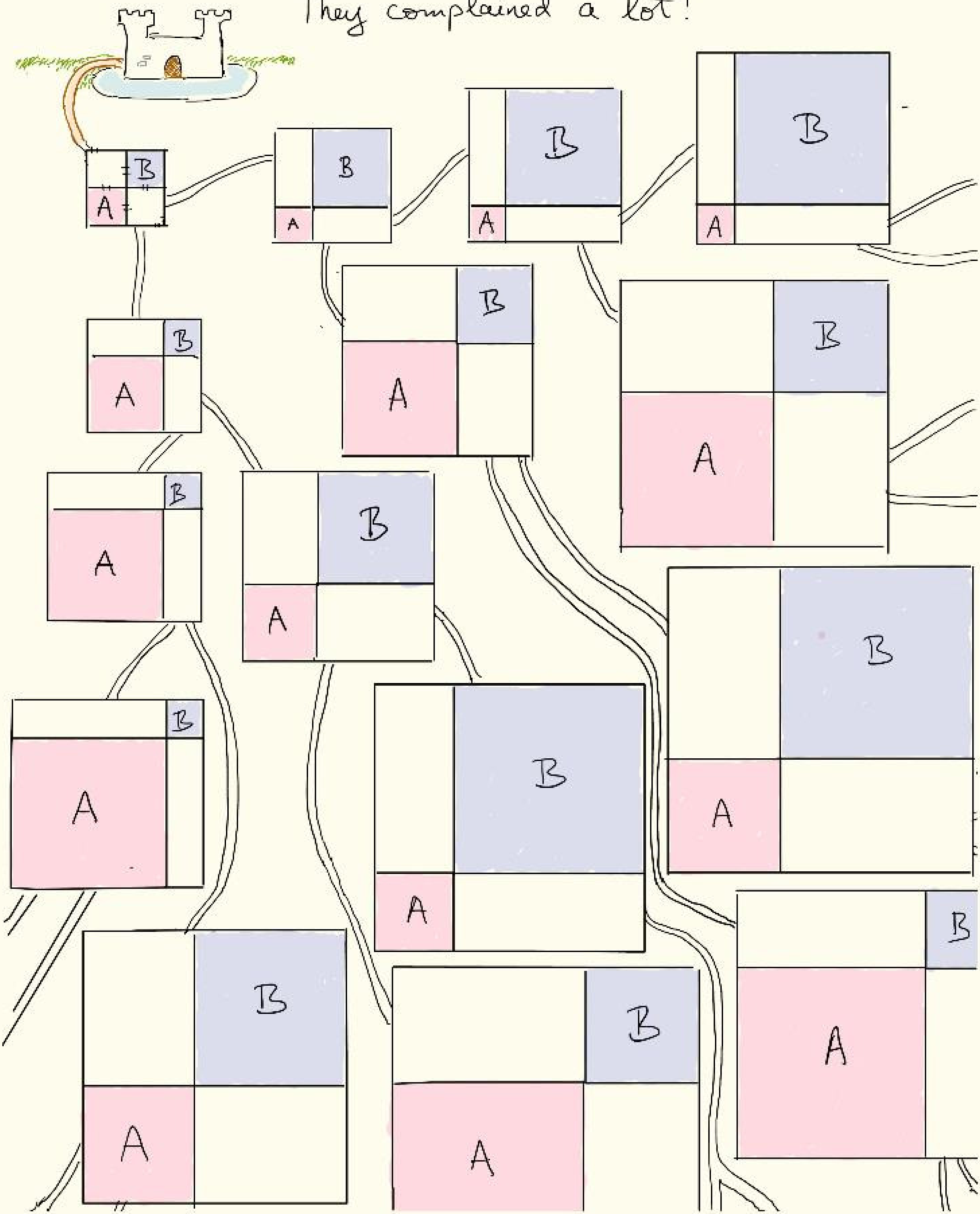
.. and then started construction.

Meanwhile Child B arrived and complained loudly:



Each time a child complained, the architect built another house...

They complained a lot!



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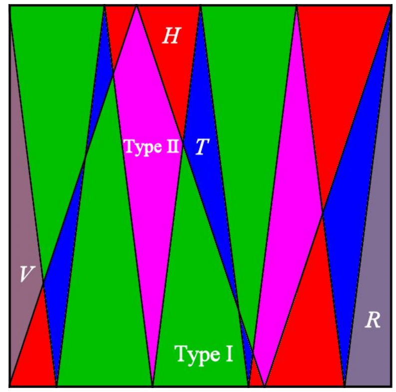
Zigzags, Part 5

by Ken Fan | edited by Jennifer Silva



For your convenience, here are the different types of triangles and quadrilaterals that Emily and Jasmine defined in “Zigzags, Part 3.” Also, recall that they are assuming that n and m are relatively prime and $n > m$. They found that there are $2m$ triangles of type H and $n - m$ triangles of type T.

Emily: Let’s record these results in a table to help us stay organized.

Emily and Jasmine create the table below and fill it in with what they have figured out so far.

Jasmine: Isn’t your piano lesson today?

Emily glances at her watch.

Emily: Oh, gosh! Yes, it is. I need to run, but I’m so curious to know if your dream is actually true.

Jasmine: I can wait for you.

Emily: I wish I could skip my lesson and do math. All we’re going to do is select the pieces for my upcoming recital, but I already know what I want to perform.

Jasmine: Well, you shouldn’t miss your piano lesson. I’ll tell you what: let’s ride the bus there together. I’ll go to Cake Country, and you can join me there after your lesson. I’m in the mood for one of Mr. ChemCake’s signature brownie bonanzas. I’ll save one for you.

Emily: It’s a deal! We’d better get going.

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Learn by Doing

Plane Geometry and Complex Numbers

by Addie Summer | edited by Amanda Galtman

Complex numbers are a wonderful tool for studying plane geometry. In this Learn by Doing, we illustrate with a variety of examples that we hope will inspire you to explore further.

But first, let's review complex numbers and the complex plane. If you're already comfortable with these concepts, skip ahead to Problem 12.

Complex numbers are what you get when you try to create a square root of -1.

Our main goal in this Learn by Doing is to apply complex numbers to problems in plane geometry. To achieve that goal more efficiently, we introduce complex numbers in a formal rather than exploratory way. This has the advantage of being quick and efficient, and it immediately connects complex numbers with points in a plane. The disadvantage is that the motivations are obscured because the method is retrospective.

In a coordinate plane, points are specified by ordered pairs (a, b), where a and b are real numbers. We now define an addition and a multiplication for points of the coordinate plane:

Addition: $(a, b) + (c, d) \equiv (a + c, b + d)$.

Multiplication: $(a, b)(c, d) \equiv (ac - bd, ad + bc)$.

(As with multiplication of real numbers, we'll denote multiplication by juxtaposition or the symbol "·".) The first item of business is to establish that these definitions satisfy the basic laws of arithmetic.

Throughout this Learn by Doing, a, b, c, d, e, and f always denote real numbers.

If this is your first exposure to complex numbers, the formula that we gave for multiplication must look bizarre. Here's how it comes about:

Let's assume that there is a number whose square is equal to -1 and that it interacts with real numbers via addition and multiplication while obeying all the basic laws of arithmetic (i.e., the commutate, associative, and distributive laws of addition and multiplication). Call this number i, so that $i^2 = -1$. We can then multiply i by the real number b to get bi, and then add this to the real number a to get $a + bi$. If we multiply $a + bi$ by another such number $c + di$, we find:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di)\\ &= ac + adi + bci + bdi^2\\ &= ac + (ad + bc)i - bd\\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

If we associate the number $a + bi$ with the point (a, b) in the coordinate plane, the result of the above computation translates to

$$(a, b)(c, d) = (ac - bd, ad + bc),$$

which is what we gave for the definition of multiplication.

Perhaps you are concerned about whether introduction of i is logically incompatible with the basic laws of arithmetic, or whether there are other numbers not of the form $a + bi$ (in which case, our multiplication law would be incomplete). The formal approach to complex numbers that we are using in this Learn by Doing addresses the second concern by its very construction and systematically establishes the laws of arithmetic in Problems 1-11.

It is a retrospective, "we already know the answer," approach.

1. Show that addition of points is commutative and associative. That is, show that

$$(a, b) + (c, d) = (c, d) + (a, b)$$

$$\text{and } ((a, b) + (c, d)) + (e, f) = (a, b) + ((c, d) + (e, f)).$$

2. Show that the origin, $(0, 0)$, is the additive identity. That is, show that $(a, b) + (0, 0) = (a, b)$.

3. Show that every point (a, b) has an additive inverse. That is, find x and y such that

$$(a, b) + (x, y) = (0, 0).$$

4. Show that multiplication of points is commutative and associative. That is, show that

$$(a, b)(c, d) = (c, d)(a, b)$$

and $((a, b)(c, d))(e, f) = (a, b)((c, d)(e, f))$.

5. Show that the point $(1, 0)$ is the multiplicative identity. That is, show that $(a, b)(1, 0) = (a, b)$.

6. Show that every point (a, b), other than $(0, 0)$, has a multiplicative inverse. That is, find x and y such that $(a, b)(x, y) = (1, 0)$.

If you are having trouble with Problems 1-7, here's a solution to the first part of Problem 4.

We must show that $(a, b)(c, d) = (c, d)(a, b)$.

Using the definition for multiplication, we find

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

and $(c, d)(a, b) = (ca - db, cb + da)$.

These represent the same point if they have the same horizontal and vertical coordinates, that is, if $ac - bd = ca - db$ and $ad + bc = cb + da$. But these equalities follow because addition and multiplication of real numbers are commutative.

7. Show that multiplication distributes over addition. That is, show that

$$(a, b)((c, d) + (e, f)) = (a, b)(c, d) + (a, b)(e, f).$$

Any set on which we can define an addition and multiplication that have the properties described in Problems 1-7 is known as a **field**. What you have done so far is to show that with the addition and multiplication defined above, the plane becomes a field of numbers. These numbers are known as **complex numbers**.

Let z and w be complex numbers. By "$z - w$", we mean "z plus the additive inverse of w," and by "z/w" (i.e., "z divided by w"), we mean "z times the multiplicative inverse of w." Note that division by $(0, 0)$ is undefined.

Next, let's take a look at some of the basic properties of complex numbers.

8. Show that the horizontal axis, i.e., the set of points $\{(a, 0) \mid a \text{ is a real number}\}$, is closed under addition and multiplication. In other words, if you add or multiply two such points, the result is also on the horizontal axis.

9. Convince yourself that the horizontal axis is a real number line. In other words, the field of complex numbers contains the field of real numbers as a "subfield." (In addition to solving Problem 8, you have to check that the horizontal axis contains the additive identity $(0, 0)$ and the multiplicative identity $(1, 0)$, and that the horizontal axis is closed under both additive and multiplicative inverses. However, you don't have to check commutativity, associativity, or the distributive law because they follow from Problems 1, 4, and 7.)

Because of Problem 9, we associate the real number a with the complex number $(a, 0)$.

10. Find the two square roots of -1 (which we are now thinking of as $(-1, 0)$).

Spoiler Alert! Let's define i to be the point (0, 1), so that $i^2 = -1$ (and $(-i)^2 = -1$).

11. Show that $(a, b) = a + bi$. Remember, we are associating a real number x with the point $(x, 0)$ and we have just defined i to be the point (0, 1), so $a + bi$ is shorthand for $(a, 0) + (b, 0)(0, 1)$.

Geometric interpretations of addition and multiplication

12. Show that the points (0, 0), (a, b), (c, d), and $(a, b) + (c, d)$ form the vertices of a parallelogram. (If you skipped to here from the beginning, we're thinking of the point (a, b) in the coordinate plane as the complex number $a + bi$.)

For this reason, complex number addition is said to obey the "parallelogram law." If we fix a complex number w, then the function that sends the complex number z to $z + w$ corresponds to a translation of the plane. Each point is shifted in the same direction and by the same distance.

The next problem asks you to interpret complex multiplication geometrically. If you have trouble, read on for help with it.

13. Find a geometric interpretation for complex multiplication. If w is a fixed complex number, what does the function that sends the complex number z to wz correspond to geometrically?

If you solved Problem 13, feel free to skip ahead to Problem 16. To understand complex multiplication, it's helpful to use polar coordinates. Let (a, b) be a complex number. Let r be the distance of (a, b) from the origin (0, 0). (By the Pythagorean theorem, we know that $r^2 = a^2 + b^2$.) Let θ be the angle from the positive real axis to the ray that points from the origin through (a, b) as measured counterclockwise. Then $a = r \cos \theta$ and $b = r \sin \theta$. In other words, every complex number can be expressed in the form $(r \cos \theta, r \sin \theta)$.

14. Show that multiplication by $(\cos \alpha, \sin \alpha)$ corresponds to a counterclockwise rotation by α. That is, if $(r \cos \theta, r \sin \theta)$ is a complex number, show that

$$(\cos \alpha, \sin \alpha) \cdot (r \cos \theta, r \sin \theta) = (r \cos(\alpha + \theta), r \sin(\alpha + \theta)).$$

15. Let r be a real number. Show that the function $f(z) = rz$, defined on complex numbers z, corresponds geometrically to a dilation by a factor of r with center of dilation the origin.

Notice that $(r \cos \theta, r \sin \theta) = r (\cos \theta, \sin \theta)$. Because multiplication is associative, we can determine the effect of multiplying by $r (\cos \theta, \sin \theta)$ by first multiplying by $(\cos \theta, \sin \theta)$, and then multiplying by r. In this way, we can see that multiplication by $r (\cos \theta, \sin \theta)$ corresponds to a counterclockwise rotation by θ followed by a dilation by a factor of r with center of dilation at the origin. Since r can be any positive real number and θ can be any angle, all rotations and dilations centered at the origin correspond to multiplication by some complex number.

To summarize, in the complex plane, addition by a constant corresponds to translation, and multiplication by a constant corresponds to a rotation and dilation, both centered at the origin.

16. By what complex number should you multiply if you want to effect a 90° rotation counterclockwise about the origin?

17. Let $f(z) = w + (\cos\theta, \sin\theta)(z - w)$, where w is a complex number and θ is a real number. Show that $f(z)$ is the counterclockwise rotation of z about the point w through the angle θ.

Complex Conjugation

Given a complex number (a, b), we define its **complex conjugate** by $\overline{(a,b)} \equiv (a,-b)$. Geometrically, complex conjugation is reflection in the real (horizontal) axis.

18. Let z and w be complex numbers. Show that $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z} \cdot \bar{w}$.

In other words, if you have an algebraic identity involving complex numbers, the identity remains valid if you replace every number with its complex conjugate.

19. Let z be a complex number. Show that $\sqrt{z\bar{z}}$ is the distance of z from the origin.

For real numbers x, the distance of x from the origin is given by $|x|$. We extend the absolute value function to complex numbers by defining $|z| = \sqrt{z\bar{z}}$.

Applications

Solve these problems using complex numbers.

20. A bug starts at the origin and begins a journey. The bug walks due east for 1 unit, then turns left 90° and walks 2 units, then turns left 90° and walks 4 units, etc. After each leg of the journey, the bug turns left 90° and walks straight for double the distance of the previous leg. After 100 legs, where does the bug end up?

21. A bug starts at the origin and begins a journey. The bug walks due east for 1 unit, then turns left 45° and walks 2 units, then turns left 45° and walks 4 units, etc. After each leg of the journey, the bug turns left 45° and walks straight for double the distance of the previous leg. After 100 legs, where does the bug end up?

22. A bug starts at the origin and begins a journey. The bug walks due east for 1 unit, then turns left 90° and walks 1/2 unit, then turns left 90° and walks 1/4 unit, etc. After each leg of the journey, the bug turns left 90° and walks straight for half the distance of the previous leg. What point is the bug ultimately heading for?

23. The law of cosines states that in triangle ABC, $BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos A$. Place the triangle in the complex plane so that its vertices are located at 0, z, and w. Use complex numbers to deduce the law of cosines.

24. This problem is essentially Problem 15 from the 2017 AIME I. A right triangle has legs of length a and b. Find the area of the smallest equilateral triangle that has a vertex on each of the sides of the right triangle.

25. Let α and β be real numbers. Rotate about the origin by α degrees and then by β degrees. The result is a rotation about the origin by $\alpha + \beta$ degrees. Did you know that if you rotate by α degrees and then by β degrees about a *different center* from the first rotation, the result is *still* a

rotation by $\alpha + \beta$ degrees about some center, provided that $\alpha + \beta$ is not a multiple of 360°? Show this. What happens when $\alpha + \beta$ is a multiple of 360°?

26. Use complex multiplication to show that $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

27. This is Problem 2 from the 2009 AIME I. There is a complex number z with imaginary part 164 and a positive integer n such that $z/(z + n) = 4i$. Find n. (The **imaginary part** of a complex number (a, b) is b.)

Spoiler Alert! Instead of solving Problem 27 algebraically, let's think geometrically. The equation can be written $z = 4i(z + n)$, which says that z is $z + n$ rotated 90° counterclockwise about the origin and scaled by a factor of 4 (from the origin). Hence, the points 0, z, and $z + n$ are vertices of a right triangle with one leg 4 times the length of the other. Since n is an integer, z and $z + n$ have the same imaginary part, namely, 164. Geometrically, this means that the hypotenuse of our right triangle is horizontal and has length n, and the altitude of the triangle to the hypotenuse has length 164. By computing the area of this right triangle in two ways, we find that $82n = 2|z + n|^2$. By the Pythagorean theorem, $n^2 = |z + n|^2 + (4|z + n|)^2 = 17|z + n|^2$. We eliminate $|z + n|$ and obtain the equation: $82n = 2(n^2/17)$. Thus, $n = 82(17)/2$.

Collinearity

Through any two points in a plane, there is a unique line. But when do three points lie on the same line?

28. Can you come up with an explicit mathematical function $f(u, v, w)$ whose inputs are three distinct complex numbers u, v, and w and which returns a real number if and only if u, v, and w are collinear?

Spoiler Alert! This problem doesn't have a unique answer, so if you came up with something different from the following, please don't assume that yours isn't a solution. One possibility is to define $f(u, v, w) = (u - w)/(v - w)$.

29. Let u, v, and w be distinct complex numbers. Prove that $(u - w)/(v - w)$ is real if and only if u, v, and w are collinear.

The Cross Ratio

Drawing inspiration from geometry, we will devise a nifty way to determine if four distinct complex numbers u, v, w, and z are collinear or cocircular. Suppose, for now, that the four numbers are cocircular, and imagine the angles wuz and wvz. If you've taken a geometry class, recall that angles inscribed in a circle that cut off the same arc have the same measure. If you didn't know this, see if you can prove it, or check out *Math, A Magical Substance* on pages 13-15 of Volume 5, Number 2 of this Bulletin.

In the other direction, fix two points A and B in the plane. Given a constant between 0° and 360°, the locus of points C such that the measure of the angle BCA equals that constant is an arc of a circle. Note: We measure the angle counterclockwise from the ray CA to the ray CB, yielding a measure between 0° and 360°. (If we instead took the measure of the angle to be between 0° and 180° by allowing oneself to measure either clockwise or counterclockwise, then the locus would be a union of two circular arcs that are reflections of each other in the line AB. Please check this.) Caveat: The locus consists of all points of a circle on one side of the line AB.

In other words, if we pick four distinct complex numbers u, v, w, and z, and if the measures of the angles wuz and wvz are equal to the same number between 0° and 180° or 180° and 360° (where we measure as described in the previous paragraph), then the four points u, v, w, and z are cocircular. Now observe that $(w - u)/(z - u)$ is a complex number of the form r (cos α, sin α), where α is the measure of the angle wuz and $r = |w - u|/|z - u|$, which is a real number. Similarly, $(w - v)/(z - v)$ is a complex number of the form s (cos β, sin β), where β is the measure of the angle wvz and s is a real number. This suggests the possibility that if u, v, w, and z are cocircular, then the ratio of the ratios $(w - u)/(z - u)$ and $(w - v)/(z - v)$ will be a real number. The reason we write "possibility" is because of the caveat in the previous paragraph. Still, we forge ahead and define the **cross ratio** $(u, v; w, z)$ to be this ratio of ratios:

$$(u, v; w, z) \equiv \frac{w-u}{z-u} / \frac{w-v}{z-v}.$$

30. Let u, v, w, and z be distinct complex numbers. Prove that $(u, v; w, z)$ is a real number if and only if u, v, w, and z lie on the same line or the same circle.

31. There are 24 permutations of u, v, w, and z. Which ones preserve $(u, v; w, z)$?

32. Show that the cross ratio is invariant under translations, rotations, and dilations.

33. Define the function $f(z) = 1/z$ on the complex plane minus the origin. Show that the image of a line or a circle in the complex plane under the function f is, again, a line or a circle. Which lines are mapped to circles? Which circles are mapped to lines?

A Möbius transformation is a transformation of the complex plane that sends the complex number z to $(Az + B)/(Cz + D)$, where A, B, C, and D are complex numbers that satisfy $AD - BC \neq 0$. (We can say that the Möbius transformation is undefined for $z = -D/C$. Alternatively, we can extend the complex plane by adding a "point at infinity" and say that $-D/C$ is mapped to this point at infinity and the point at infinity is mapped to the point A/C.)

34. Show that Möbius transformations send lines and circles to lines and circles. (Hint: Show that every Möbius transformation can be composed of translations, rotations, dilations, and the function $f(z) = 1/z$. Alternatively, show that the cross ratio is invariant under Möbius transformations.)

35. If you're bothered by having to consider both lines and circles in Problems 30-34, consider stereographic projection, i.e., center a unit sphere at the origin of the complex plane and send the complex number z to the point (other than the north pole) where the sphere intersects the line through z and the north pole. This projection sends the plane onto the sphere minus the north pole. You could even think of the north pole as the "point at infinity." Show that lines and circles in the plane all become circles on the sphere under stereographic projection.

Problem 35 suggests that perhaps we should be talking about the "complex sphere" instead of the "complex plane"! Indeed, the complex sphere is known as the Riemann sphere.

Another Diophantine Equation

by Lightning Factorial | edited by Jennifer Silva

If you enjoyed Konstanze Rietsch's article "An Architect in Wonderland" on page 7, here's another family of Diophantine equations for you to explore. (A Diophantine equation is one that is to be solved in integers.)

As a starting point, let's take the Pythagorean relationship $a^2 + b^2 = c^2$. What solutions are there to this equation in integers a, b, and c? In geometric terms, which lattice points in the plane are a whole number distance from the origin?

A well-known way to find the solutions is to first observe that if $c \neq 0$, then we can divide by c^2 on both sides to obtain $(a/c)^2 + (b/c)^2 = 1$. If a, b, and c are integers, then a/c and b/c are rational numbers. So instead of looking for solutions in integers to $a^2 + b^2 = c^2$, we seek solutions to $x^2 + y^2 = 1$ in rational numbers x and y. (If $c = 0$, what integer values of a and b satisfy $a^2 + b^2 = c^2$?)

The equation $x^2 + y^2 = 1$ describes a circle of radius 1 centered at the origin. This circle passes through the points (1, 0), (0, 1), (-1, 0), and (0, -1), so we know there are at least four rational solutions to $x^2 + y^2 = 1$. Suppose (p, q) is another rational solution, i.e., suppose $p^2 + q^2 = 1$. Here's the idea: observe that the line that passes through the points (-1, 0) and (p, q) has a rational slope, and, conversely, any line through (-1, 0) with rational slope must intersect the circle $x^2 + y^2 = 1$ in points with rational coordinates. To see this, note that the slope of the line that passes through (-1, 0) and (p, q) is $q/(p + 1)$; if p and q are rational, so is $q/(p + 1)$. Conversely, the line though the point (-1, 0) with slope m can be described by the equation $y = m(x + 1)$. To find out where this line intersects the circle $x^2 + y^2 = 1$, we must solve the two equations simultaneously. By substituting $m(x + 1)$ for y in the equation for the circle, we get the equation $x^2 + m^2(x + 1)^2 = 1$, which is a quadratic equation in x. If m is a rational number, then the coefficients of this quadratic equation are also rational numbers; therefore, by Vieta's formulas, the sum and product of the roots will be rational. However, we already know that -1 is a root, which means that the other root, call it p, must also be rational. This other root p is the horizontal coordinate of the point of intersection of the line and the circle other than (-1, 0). If we call this point of intersection (p, q), then $q = m(p + 1)$. And since m is rational, so is q.

In this way, we can parameterize all the rational points on the circle $x^2 + y^2 = 1$ by the slope m of the line through (-1, 0). For example, if $m = 1/2$, then $(p, q) = (3/5, 4/5)$. By rationalizing the denominators, we get the solution $3^2 + 4^2 = 5^2$ to the Pythagorean equation.

Please take a moment to work out the idea just described in detail.

Let's look at another approach to finding integer solutions to $a^2 + b^2 = c^2$. Suppose that (a, b, c) is a solution to $a^2 + b^2 = c^2$. Notice that (b, a, c) is also a solution, as are $(\pm a, \pm b, \pm c)$, for any of the eight possible choices of sign. Let's call these various changes of sign and the switching of the first two coordinates "Pythagorean moves." While you can use these moves to get new solutions from known solutions, the new solutions you get aren't that different. However, let's throw in one more transformation to our collection of Pythagorean moves. This transformation sends the solution (a, b, c) to $(a + 2b + 2c, 2a + b + 2c, 2a + 2b + 3c)$. For example, (3, 4, 5) is sent to (21, 20, 29).

Show that this last move also sends solutions of the Pythagorean equation to solutions.

By the way, notice that if (a, b, c) is a solution to the Pythagorean equation, then so is (da, db, dc) for any number d. For this reason, to find all integer solutions to $a^2 + b^2 = c^2$, it suffices to find the solutions (a, b, c) such that a, b, and c have no common factor. Such solutions are called **primitive**. If you know all the primitive solutions to the Pythagorean equation, then you can find all the solutions by multiplying the primitive ones by various whole numbers.

Show that every primitive solution to the Pythagorean equation can be obtained from the solution (1, 1, 0) and applying Pythagorean moves repeatedly in various ways.

Pretty neat, huh?

But… we said that the Pythagorean equation will be our starting point. So for the next equation, consider the Diophantine equation $a^2 + b^2 + c^2 = d^2$. For example, one solution to this Diophantine equation is $1^2 + 2^2 + 2^2 = 3^2$.

Can you adapt the slope parametrization method described earlier to find all solutions to the Diophantine equation $a^2 + b^2 + c^2 = d^2$?

Are you wondering if there is a way to find solutions to $a^2 + b^2 + c^2 = d^2$ using an analogue to the Pythagorean moves? Well, there is! Let (a, b, c, d) be a solution to the Diophantine equation $x^2 + y^2 + z^2 = w^2$. Our collection of moves will consist of the following transformations of (a, b, c, d):

I. Any permutation of the first 3 coordinates.

II. Any sign changes to any of the 4 coordinates.

III. The transformation that sends (a, b, c, d) to

$$(-b - c + d, -a - c + d, -a - b + d, -a - b - c + 2d).$$

Please check that all of these transformations send solutions to solutions.

Show that every primitive solution to $x^2 + y^2 + z^2 = w^2$ can be obtained from the solution (1, 1, 0, 0) and applying these moves repeatedly in various ways.

Isn't that neat?

You've probably predicted what Diophantine equation comes next. What can you say about the Diophantine equation

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = x^2,$$

where n is a fixed positive integer? Let us know what you discover!

The method described here for generating primitive solutions may be found in the book *Infinite Dimensional Lie Algebras* by Victor Kac.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 25 - Meet 1 September 12, 2019	Mentors:	Emily He, Adeline Hillier, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Christine Soh, Rebecca Whitman, Jasmine Zou

Ten new members joined us for our first meet of Session 25. In accordance with Girls' Angle tradition, our mentors conducted interviews with each new member to learn more about her. We were interested in learning what she likes or dislikes about math, what subjects she enjoys, what she likes to do and think about, whether she prefers to work alone or in groups, whether or not she likes challenges, and any other details about her learning and preferences that we can glean. We also wanted to learn about her expectations and goals, both for her time at Girls' Angle and with respect to math overall.

Alongside these interviews, several math activities were going on organized into various stations. Many returning members had projects they were continuing from the summer or spring; others embarked on new projects. All in all, it seemed as if the math picked up as if there were no summer break!

Session 25 - Meet 2 September 19, 2019	Mentors:	Talia Blum, Katie Gravel, Emily He, Adeline Hillier, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Melissa Sherman-Bennett, Christine Soh, Savannah Tynan, Rebecca Whitman

It is a fairly well-known fact that the sum of the angles of a triangle is 180°, but can you prove it? Can you use this fact to find the sum of the interior angles in a polygon with n sides?

Session 25 - Meet 3 September 26, 2019	Mentors:	Adeline Hillier, Jenny Kaufmann, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Savannah Tynan, Rebecca Whitman, Angelina Zhang, Jasmine Zou

How can you draw a realistic depiction of a 3D world? If you try to make a drawing of a complex interior with windows, doors, desks, chairs, tables, bookshelves, and a staircase, you will likely find that "eyeballing" the drawing will only get you so far. Things will start to look off. There's interesting mathematics at work in such drawings, and if you study this math of **perspective drawing** and adhere to the math instead of going with what you intuit, you'll be able to produce depictions of 3D worlds that are spot on.

Session 25 - Meet 4 October 3, 2019	Mentors:	Katie Gravel, Tina Lu, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Melissa Sherman-Bennett, Christine Soh, Savannah Tynan, Rebecca Whitman, Angelina Zhang

How random do you think you can be? Mentor Rebecca Whitman challenged members to produce lists of coin flip outcomes both by faking them and by using actual coin tosses, and see if they could fool her as to which lists were made using actual coin flips. She boldly claimed that she couldn't be so fooled! The members tried, but Rebecca could not be fooled. How do you think she did this?

Session 25 - Meet 5 October 10, 2019	Mentors:	Adeline Hillier, Jenny Kaufmann, Rebecca Nelson, Kate Pearce, Laura Pierson, Melissa Sherman-Bennett, Christine Soh, Rebecca Vessenes, Karissa Wenger, Rebecca Whitman

What is division? Try to imagine yourself living at a time before division was defined. How do you think you would have defined it? What properties would you want your division operation to have?

Session 25 - Meet 6 October 17, 2019	Mentors:	Emily He, Adeline Hillier, Tina Lu, Kate Pearce, Laura Pierson, Nehar Poddar, Melissa Sherman-Bennett, Christine Soh, Rebecca Vessenes, Karissa Wenger, Angelina Zhang

Probably for as long as there has been human civilization, there has been the problem of creating a method to communicate with friends in such a way that your foes cannot figure out what you are saying, even if they are eavesdropping. Mathematicians have been drawn to this problem, and modern solutions involve sophisticated mathematics from number theory. What solutions can you devise for this secret communication problem?

And if you'd like to try your hand at deciphering a secret message, try to figure out this one: "Sd sc swzyccslvo dy lo k wkdrowkdsmskx gsdryed losxq k zyod sx cyev." - Cypik Uyfkvofcukik.

Session 25 - Meet 7 October 24, 2019	Mentors:	Emily He, Adeline Hillier, Jenny Kaufmann, Tina Lu, Aileen Ma, Kate Pearce, Laura Pierson, Savannah Tynan, Rebecca Vessenes, Karissa Wenger, Angelina Zhang
	Visitor:	Sally Seaver, Biotechnology Consultant

Sally Seaver was born in Marblehead, Massachusetts, attended high school in New York, then went to Harvard-Radcliffe for college, earned a doctoral degree in chemical physics from Stanford, then did a postdoc at the Louis Pasteur University Institute at the University of Strasbourg in France. After serving on the faculty at Vanderbilt, she left academia for industry, eventually founding her own company: Seaver Consulting. There, she worked with pharmaceutical corporations by helping them to communicate their goals and products to non-scientists. She also helped to formulate an industry standard for evaluating drug efficacy. Today, she owns two horses that compete in dressage at the Grand Prix level. She delivered to us three main messages:

- Be comfortable with math so you are able to appropriately challenge people's mathematical arguments.
- There is no such thing as failure. Experiences that feel like failure are simply more data for you to learn from, so be open to trying new things.
- The only stupid question is the one you didn't ask.

Calendar

Session 25: (all dates in 2019)

September	12	Start of the twenty-fifth session!
	19	
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	

Session 26: (all dates in 2020)

January	30	Start of the twenty-sixth session!
February	6	
	13	
	20	No meet
	27	
March	5	
	12	
	19	
	26	No meet
April	2	
	9	
	16	
	23	No meet
	30	
May	7	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdococtoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________