Girls' Angle Bulletin

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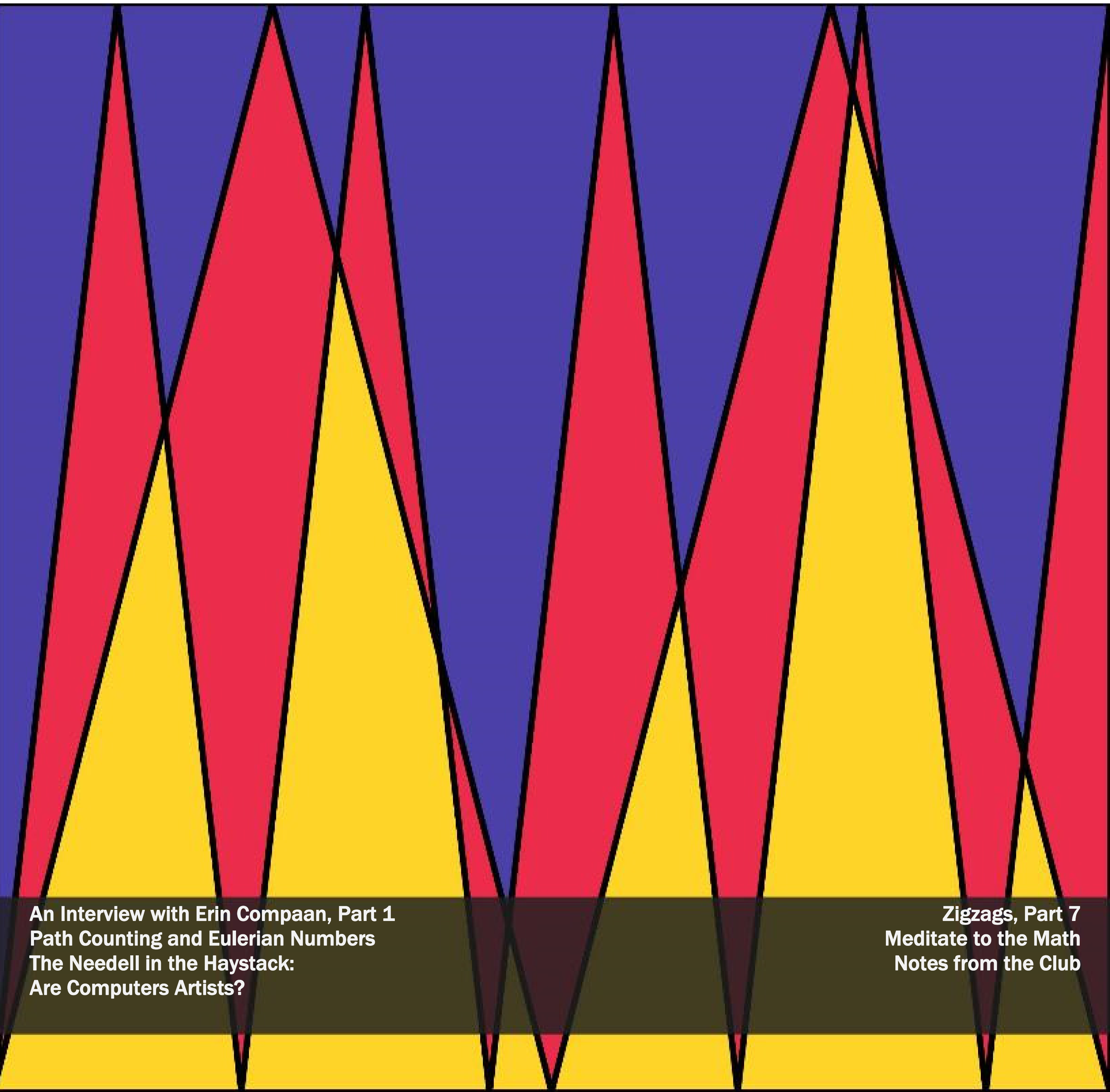
To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

Math projects can be such epic journeys. In this issue, two long adventures come to fruition. Fictional characters Emily and Jasmine discover a nifty theorem and real people Esmé and Molly present their analysis of paths in a street network of their own creation. - Ken Fan, President and Founder

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Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Crisscrossing Zigzags II by C. Kenneth Fan. For an explanation of the color scheme, see *Zigzags, Part 7* on page 16.

An Interview with Erin Compaan, Part 1

Erin Compaan is a mathematician from Draper Labs. Prior to Draper Labs, she was a National Science Foundation Postdoctoral Fellow in Mathematics at the Massachusetts Institute of Technology. She received her doctoral degree in mathematics from the University of Illinois at Urbana-Champaign under the supervision of Nikolaos Tzirakis.

Ken: How did you become interested in mathematics in the first place?

Erin: That is not a very inspiring story. I didn't particularly like math in high school, but I went to college and I looked at all the majors and math looked like it didn't have too many requirements compared to the others. I crossed off a lot of things and math was left so I decided I would try it. I took calculus and discovered that I really liked how you could generalize the sort of not-so-exciting ideas of slope and area from algebra to get some of the continuous analogs.

Ken: That's amusing! Did you actually take calculus in college for the first time?

Erin: Yeah, I did.

Ken: Wow, that's very interesting. Actually quite a few of the people I've interviewed have said that they didn't think about mathematics much until college. Would you characterize yourself in that way?

Erin: Yes, I think so.

Ken: In high school did you take a regular math track?

Erin: Yeah, I took algebra a couple of years, geometry, logic, and that sort of thing.

What I see now is a lot of questions that would not have even occurred to me when I first started studying…

Ken: And that math you did in high school never stood out as something you might want to do for a living?

Erin: No, not really. In fact, it seemed kind of tedious with all the sheets of exercises that you had to do that were all essentially the same.

Ken: Were you interested in science?

Erin: Not particularly. I guess I enjoyed physics. I liked chemistry well enough but my sisters did chemistry and we all went to the same university, so they were still there doing chemistry when I started, and I thought, no, I'm not going to do the same thing as my sisters.

Ken: Are your parents in science or mathematics?

Erin: My dad's an engineer, and my mom stayed home with the kids.

Ken: Even college is still a long way from becoming a professional mathematician, but when, in college, did you realize that you would want to continue studying math after you graduated?

Erin: I think the courses that I took toward the end of the degree were more interesting than I expected them to be. I was not particularly interested in college when I started. I thought I would end up doing some sort of trade as a career. But by the time I finished I was finding math a lot more interesting. Also, I thought it would be easier to go straight from college to math grad school rather than deciding ten years

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Dr. Erin Compaan and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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President and Founder
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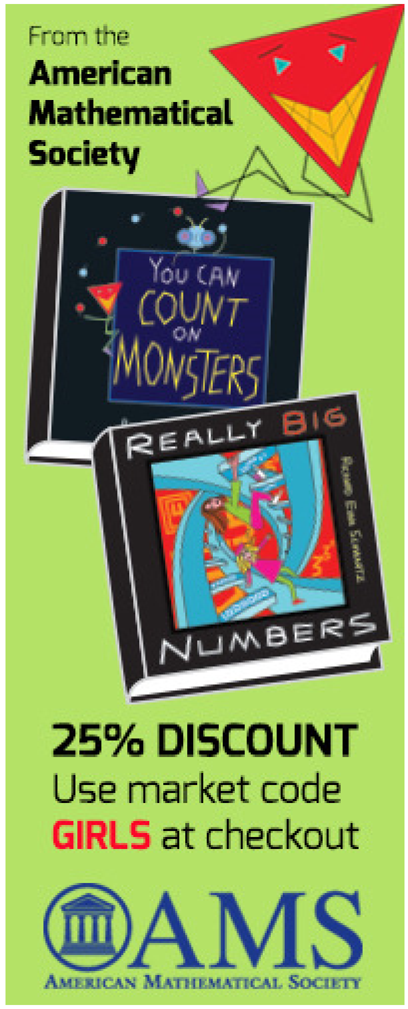

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Girls' Angle

Path Counting and Eulerian Numbers

by Esmé Krom and Molly M. Roughan

Primary guidance and mentorship for this mathematical investigation was provided by Adeline Hillier.

For the last few months, we have been studying path counting problems—problems where you try to count the number of possible paths from a start node to an end node, given different networks of edges connecting pairs of nodes. We first studied networks that had already been created, then started to make our own. We came up with this network:

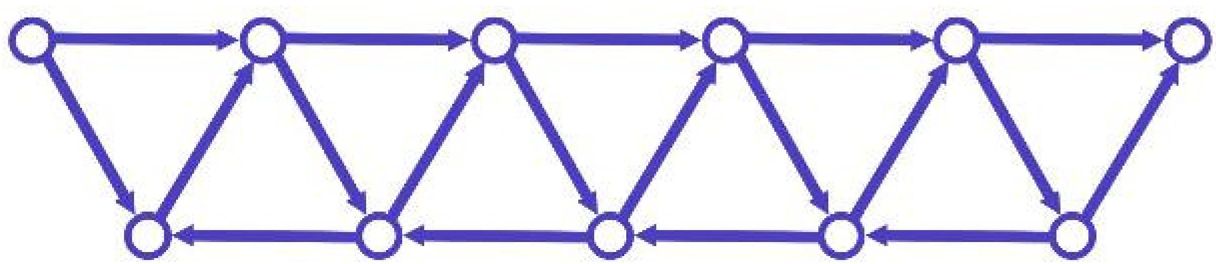

Figure 1. Our path network. The circles are nodes and the arrows running between them are edges. This figure only illustrates the case with 5 nodes in the bottom row, but we considered analogous networks with any number of nodes in the bottom row.

For the purposes of this paper, a path is any route between nodes that traverses the edges, following the directions of the arrows, and such that no node is visited more than once. We have been looking for patterns in the number of paths from the leftmost upper node (the "start" node) to a designated destination node. We began by calculating the number of paths from the start node to each of the upper nodes. The results began to show a pattern. There are two paths from any upper node to the node immediately to its right. If we know the number of paths to an upper node, the number of paths to the node to its right will be double that, because each path to the first node can be extended in two ways to get to the second one. Following that logic, starting at the leftmost upper node, the number of ways to get to each succeeding upper node follows a geometric sequence with a common ratio of 2 and a starting term of 1 (because there is only one way to get from the start node to itself): 1, 2, 4, 8, 16, etc.

When we first created the network, we imagined it to be infinitely long, but we soon realized that if that were so, there would be infinitely many paths to each of the lower nodes. We made the network finite by cutting it off at a certain point, so that the number of paths would be finite, and depend on the length of the network, which we defined to be the number of nodes, y, in the bottom row. We decided to study how the number of paths from the start node to each of the lower nodes related to the length of the network.

A convenient way to calculate the number of paths that begin at the start node, A, and end at a destination node, B, in the bottom row, is to identify an intermediate node, N, in the top row, between A and B, then take the number of paths from A to N, and multiply it by the number of paths from N to B.

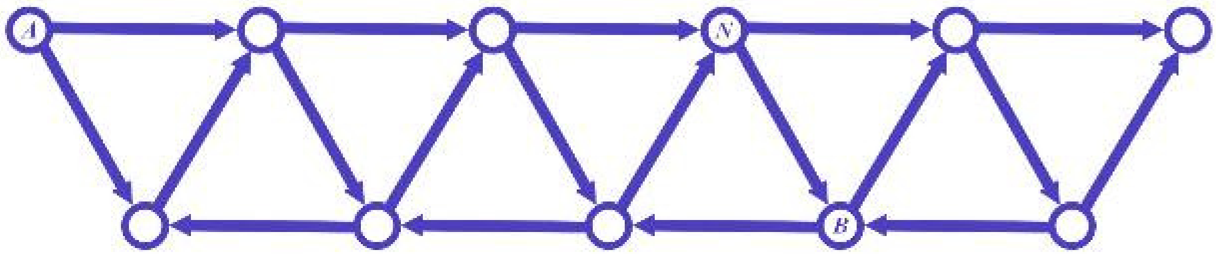

Figure 2. This shows the start node, A, the destination node, B, and an intermediate node between the two, N, used to calculate the number of paths between A and B.

The reason that this computation works—the number of paths from A to B equals the number of paths from A to N times the number of paths from N to B—is because every path from A to B goes through your identified N, and for each path from A to N, you can extend it by each path from N to B to create a path from A to B. (Note that no path from N to B can visit a node passed through on the way from A to N without being forced to revisit N.) Since every path from A to B must pass through N, we capture every possible path from A to B.

This trick allowed us to calculate the number of paths to distant end nodes. To record our calculations, we made a chart, which showed the number of paths to each node on the bottom of the graph and for each network length that we studied.

Position of the destination node, x, along bottom row

Length of the network, y	**1**	**2**	**3**	**4**	**5**
1	1				
2	2	2			
3	3	4	4		
4	4	6	8	8	
5	5	8	12	16	16

Figure 3. The number of paths from the topmost left node in the network to a destination node given by x, for networks of length y.

We decided to make a formula to describe the pattern, so that we wouldn't have to manually count the number of paths each time. To do that, we decided to represent the length of our graph as the number of lower nodes in the network, y, and the different end nodes as x (the leftmost lower node corresponds to $x = 1$).

In order to find a formula for the number of paths from A to B in terms of x and y, we put the number of paths from A to N and N to B in terms of x and y since these relationships were simpler. For these calculations, we established N to be the node that comes directly before B (as in Figure 2).

Length of network, y \ Position of destination node, x, along bottom row	**1**	**2**	**3**	**4**	**5**
1	1				
2	2	2			
3	3	4	4		
4	4	6	8	8	
5	5	8	12	16	16

Figure 4. Summary of the numbers from Figure 3.

We first considered the number of paths from A to N in terms of x and y. Each time you increase y by 1, holding x constant (or each time you increase the length of the network, without changing the destination node), the number of paths from A to N stays the same, so y is irrelevant. We already determined that for a given length of the network, y, the number of paths from A to N follows a geometric sequence with common ratio 2. Specifically, in terms of x, the number of paths from A to N is 2^{x-1}.

We then considered the number of paths from N to B in terms of x and y. We manually counted these values in order to form our graph, and found that the number of paths from N to B fit with the formula $y - x + 1$. We then had to prove that this could apply to any value of x and y. We realized that this formula does make sense. For one thing, each time you increase y by 1 while holding x constant, the number of ways to get from N to B increases by one, and extending the graph by one unit opens up exactly one more path from N to B. And each time you increase x by 1 while holding y constant, the number of ways to get from N to B decreases by 1, because it closes the rightmost loop. Since these are always true, the formula should work for any x and y!

As a result, the number of paths from A to B is: $(y - x + 1)2^{x-1}$.

Notice that our street map can be thought of as consisting of nodes labeled 0 through n for some nonnegative integer n, and then building one-way streets from each integer to the next, from each even integer to the next even integer, and from each odd number to the previous odd number. The original zigzag path gets stretched into a linear path from 0 to 1 to 2 to 3, etc. Although we didn't show pictures of the case where n is odd, our work reveals the numbers of paths from 0 to any node $0 \le k \le n$ and for any n, because the number of such paths is unaffected by adding, if necessary, an even node at the end, since that node would be a sink.

Our path counting problem leads to some interesting integer sequences. The triangle of numbers induced by the description in the previous paragraph is not yet on the Online Encyclopedia of Integer Sequences (OEIS), so we will attempt to add it.

We did find the integer sequence associated to going from the upper left node to one of the bottom row nodes in the OEIS (see OEIS A130128), though it doesn't appear it was found in the way we found it. It was added to OEIS by Gary W. Adamson in 2007. What's more, the sums of the rows in Figure 4 is 1, 4, 11, 26, 57, etc. This sequence is called the **Eulerian numbers** (OEIS A000295), and is apparently well researched. For example, L. Edson Jeffery found that this pattern can be found inside Pascal's triangle. Specifically, it can be found in the row sums of the third sub-triangle. As far as we know, we are the first people to find these sequences using path counting.

We had the time of our lives working on this project, and can't wait to do more!

The Needell in the Haystack[1]

Are Computers Artists?

by Deanna Needell | edited by Jennifer Sidney Silva

My husband Blake is both a mathematician and an artist. His ability to put onto canvas what he has in his mind never ceases to amaze me. On the other hand, my attempt at drawing stick figures often leaves the observer dazed and confused. Art can be a subjective and personal craft, and one might wonder how the ability to produce art is related to mathematics. In our previous installment we explored how neural nets could be used to generate images that looked real, yet were entirely synthetic. In a sense, the computer was "learning" to generate such images. At a high level, I myself view this as a surprising and intriguing blurring of the lines between math and art.

Today, we explore a related problem in the subject of **machine learning**, in which the computer will learn so-called **dictionary atoms** that are used like puzzle pieces to reconstruct images and art. There are diverse applications of math to making art,[2] but today's focus is on one that is closely related to nonnegative matrix factorization, a tool we have already seen in this series.[3] The method we will explore is an online variant, called **online nonnegative matrix factorization** and abbreviated ONMF.

ONMF is motivated by the need to handle large-scale data, such as *streaming* data. In the last few decades, the quantity of data available and the need to effectively exploit this data have been growing exponentially. At the same time, the analysis of modern data presents new challenges that necessitate novel ideas and techniques. Many of these techniques may be classified as **topic modeling** (or **dictionary learning**); these techniques aim to extract important features of a complex dataset so that the dataset can be represented in terms of a reduced number of features known as **dictionary atoms**. Just as the English language is built of words listed in a dictionary, one can also try learning good dictionaries that consist of these dictionary atoms that can be used to build other things, such as images or even art. One of the advantages of topic modeling-based approaches is that the extracted topics are often directly interpretable, as opposed to the arbitrary abstractions of deep neural networks that we saw in the previous article.

Matrix factorization provides a powerful setting for dimensionality reduction and dictionary learning problems. In this setting, we have a data matrix X, which is a d by n matrix of real numbers, i.e., X is in $\mathbf{R}^{d \times n}$, where $\mathbf{R}$ is the set of real numbers. We seek an approximate factorization of X into a product WH, where W, the **dictionary matrix**, is in $\mathbf{R}^{d \times r}$ and H, the **code matrix**, is in $\mathbf{R}^{r \times n}$ (see Figure 1). Because of the way matrix multiplication is defined, each column of the data matrix is approximated by a linear combination of the columns of the dictionary matrix W with coefficients given by the corresponding column of the code matrix H. This matrix factorization problem has been extensively studied under many names, each with different constraints: dictionary learning, factor analysis, topic modeling, and component analysis, to name a few. And it has found countless applications: text analysis, image reconstruction, medical imaging, bioinformatics, etc.

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[1] This content supported in part by a grant from MathWorks.

[2] Incidentally, many of the *Bulletin* covers would be extraordinarily difficult to produce without the use of a high-speed computer.

[3] See "The Needell in the Haystack: Topic Modeling" in Volume 11, Number 3 of this *Bulletin*.

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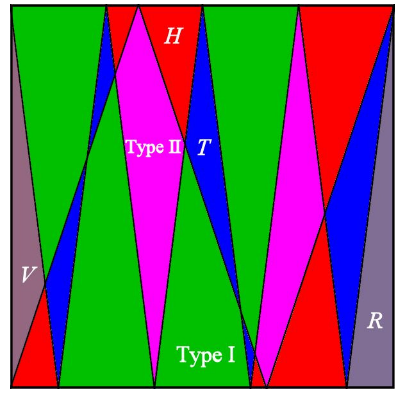
Zigzags, Part 7

by Ken Fan | edited by Jennifer Sidney Silva

Emily: All of the quadrilaterals are formed when the m-zigzag slices through one of the isosceles triangles formed by the n-zigzag. Quadrilaterals of type I are isosceles triangles minus a triangular tip of type T, and quadrilaterals of type II are isosceles triangles minus two triangles of type H.

Jasmine: That means that we have everything we need to compute the areas of the quadrilaterals. From the area of an isosceles triangle formed by the n-zigzag, which is $1/n$, we subtract the total area of any removed triangles.

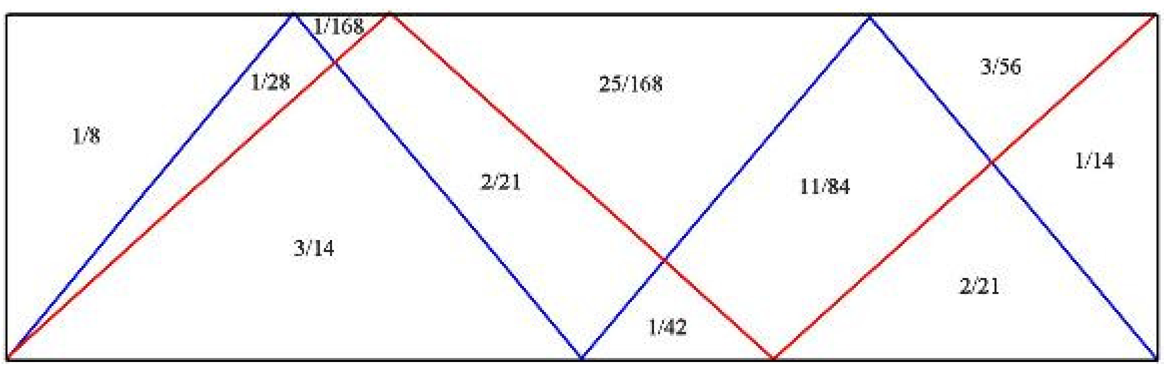
Emily: We're set, then! I'd like to check that the formulas we derived agree with the examples we computed earlier.



For your convenience, here are the different types of triangles and quadrilaterals that Emily and Jasmine defined in "Zigzags, Part 3." Also, recall that they are assuming that n and m are relatively prime and $n > m$. In previous installments, they determined the total number of each type of triangle and their areas.

Emily rummages through her notes and finds those examples.

Emily: Here they are. Let's check the case $(n, m) = (4, 3)$.



(This square has been stretched horizontally to make the numbers easier to read.)

Jasmine: Here, n and m have opposite parity. In that case, we found that there should be one triangle of type R, one triangle of type V, six triangles of type H, and one triangle of type T.

Emily: That all checks out!

Jasmine: The triangle of type R should have area $1/(2n)$, which in this case is $1/(2 \times 4) = 1/8$. Check!

Emily: The triangle of type V should have area $1/(2(m + n))$, which in this case is $1/(2(3 + 4))$, or $1/14$. Check!

Jasmine: According to our table, for the triangles of type H, in units of $1/(nm) = 1/12$, we should find three with horizontal side lengths 2, 4, and 6 on the bottom edge, and three with horizontal side lengths 1, 3, and 5 on the top edge. While I was waiting for you, I found that the areas of these triangles are $\frac{mnh^2}{2(m+n)}$, where h is the horizontal side length. If we set $h = k/(nm)$ and substitute, we find that the area is $\frac{k^2}{2mn(m+n)}$. In this case, the denominator is 168. So we should find triangles on the bottom with areas $2^2/168$, $4^2/168$, and $6^2/168$, which simplify to 1/42, 2/21, and 3/14, respectively. And we should find triangles on the top with areas $1^2/168$, $3^2/168$, and $5^2/168$, which simplify to 1/168, 3/56, and 25/168, respectively.

Emily: Check, check, check, check, check, and check!

Jasmine: Wow, it works! And finally, the lone triangle of type T should have T-length 1, in units of $1/(nm)$, and – as we computed – should have area $\frac{1}{(nm)^2}\frac{m^2n}{n^2-m^2}$, which works out to 1/28.

Emily: That's what we found, so everything checks out perfectly! Writing those areas as simplified fractions caused me to completely miss the regularity in the scale factors.

Jasmine: Sometimes it's helpful to *not* simplify fractions, I guess!

Emily: Although we know the areas of all of the triangles, it's a little harder to determine the exact location of a triangle with a specific area.

Jasmine: That's true. The exact locations of specific triangles depend on solving a congruence problem in modular arithmetic. But we could compute the total area of all triangles of each type. For triangles of type H and type T, the computation would essentially come down to computing a sum of consecutive perfect squares, and it's known that the sum of the first N perfect squares is equal to $N(N + 1)(2N + 1)/6$. For fun, do you want to compute these totals?

Emily: Sure, let's do that! The total area of the triangles of type H along the bottom edge of the square will be the area of the unit triangle of type H (in units of $1/(nm)$), which is $\frac{1}{2mn(m+n)}$, multiplied by either $2^2 + 4^2 + 6^2 + \ldots + (2m)^2$ or $1^2 + 3^2 + 5^2 + \ldots + (2m - 1)^2$. For even squares, we have $2^2 + 4^2 + 6^2 + \ldots + (2m)^2 = 4(1^2 + 2^2 + 3^2 + \ldots + m^2) = 4m(m + 1)(2m + 1)/6$, so when the triangles of type H along the bottom edge have even horizontal lengths (in units of $1/(nm)$), their total area will be $\frac{(m+1)(2m+1)}{3n(m+n)}$. In our example with $(n, m) = (4, 3)$, this works out to 4×7 over … um … 1/3! Gosh, that simplified more than I was expecting, but I don't think we can expect a small denominator like that in general.

Jasmine: No. But in that formula $\frac{(m+1)(2m+1)}{3n(m+n)}$, if you set n equal to $m + 1$, it simplifies to 1/3.

Emily: You're right! So in the case where you have an m-zigzag and an $(m + 1)$-zigzag, the triangles of type H along one of the sides will comprise exactly 1/3 of the area of the square!

Jasmine: For the type H triangles with odd horizontal lengths (in units of $1/(nm)$), the total area will come out to $\frac{1}{2mn(m+n)}(1^2 + 3^2 + 5^2 + \ldots + (2m - 1)^2)$. What's the sum of the first m odd perfect squares?

Emily: We can compute that by subtracting the sum of the first m even squares from the sum of the first $2m$ squares:

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \ldots + (2m-1)^2 &= \frac{2m(2m+1)(4m+1)}{6} - \frac{4m(m+1)(2m+1)}{6} \\ &= \frac{m(2m+1)((4m+1)-2(m+1))}{3} \\ &= \frac{m(2m+1)(2m-1)}{3}. \end{aligned}$$

Jasmine: So the total area in this case comes out to $\frac{(2m+1)(2m-1)}{6n(m+n)}$. In the case $(n, m) = (4, 3)$, this equals 5/24, which does equal 1/168 + 3/56 + 25/168.

Emily: Say, Jasmine, in the case where $n = m + 1$, the triangles of type H along the side where they have even horizontal lengths (in units of $1/(nm)$) …

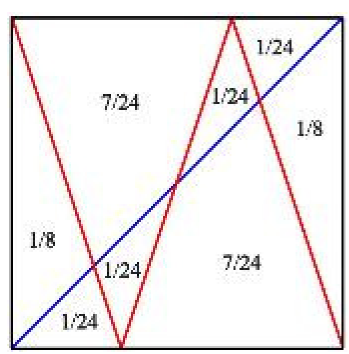
Jasmine: … have total area 1/3 …

Emily: Yes, but they also happen to occupy the entire length of that side. After all, the sum of their horizontal lengths is $2 + 4 + 6 + \ldots + (2m) = m(m + 1)$ in units of $1/(nm)$, and when n equals $m + 1$, that's exactly 1 unit in absolute terms. So when the triangles of type H form a mountain landscape running across the full length of the square, *that*'s when their total area is 1/3.

Jasmine: Amusing, but it only happens in the very special case where n and m are consec…

Jasmine pauses, her eyes darting back and forth over the numbers, as she seems to notice something.

Jasmine: In our example, the odd-horizontal length triangles of type H have total area 5/24, which is shy of 1/3 by 3/24, or 1/8. But 1/8 is exactly the area of the triangle of type R here, and it appears that the triangle of type R and the triangles of type H along the top side fill out the entire side length. Could it be that the total area of *all* the triangles along the top or the bottom side always comes out to 1/3?



Emily: I really doubt that! It's probably just a coincidence. In fact, take a look at this 1-zigzag/3-zagzag case. There's only one triangle along the bottom, and it definitely does not have area 1/3.

Jasmine: Oh yeah, you're right. I just thought it was so uncanny that in the 3-zigzag/4-zigzag case, the area of the type *R* triangle happened to be exactly what was needed to bring the area of the odd-horizontal length triangles of type *H* to 1/3. In this 1-zigzag/3-zigzag case, the upper and lower sides are essentially the same because of 180° rotational symmetry, and each side has only one triangle of area 1/24. Definitely not 1/3.

Emily: Hey, wait a sec!

Jasmine: What?

Emily: 1/3 – 1/24 = 7/24, and look! There's a type *I* quadrilateral of area 7/24!

Jasmine: That is true. Say, do you think it could be that …

Emily: … the total area of *all of the pieces* with a side along the bottom edge …

Jasmine: … is always 1/3?

Emily and Jasmine go through all of the examples they computed to check this.

Emily: I can't believe it! It's true in every single case we've worked out so far!

Jasmine: What we're suggesting here is that the two zigzags split the square into three regions: the part below both zigzags, the part above both zigzags, and the part in between the two zigzags. And these three parts split the area of the square into thirds! It can't be, can it?

Emily: We have all of the formulas. We can check if that's true in general.

Jasmine: Okay, let's do that!

Emily: We might as well proceed systematically through all of the cases. Let me get out the relevant notes.

Emily takes out the papers with the tables showing the numbers of each type of triangle and quadrilateral (see the tables in Volume 12, Number 4 and Volume 13, Number 2).

Emily: Let's start with the case where n and m have opposite parity.

Jasmine: Okay. In that case, we can rotate and flip as necessary so that both zigzags begin in the lower left corner. When we do that, there will be one triangle of type *R*, and its base will be on the upper side. There will be one triangle of type *V*, but that sits between the two zigzags so won't contribute to our area computation. We'll have $2m$ triangles of type *H*, and the even-horizontal length ones (in units of $1/(nm)$) will run along the bottom edge, while the odd-horizontal length ones will run along the top edge. There will be $n - m$ triangles of type *T*, and each will be the tip of an isosceles triangle (formed by the n-zigzag). If these tips are removed from the isosceles triangle, $n - m - 1$ of them will leave a quadrilateral of type I and one will leave a triangle of type *H*, specifically the one with *T*-length $n - m$ (in units of $1/(nm)$). Furthermore, the ones with even *T*-length will have their tips on the bottom edge, while the ones with odd T-length will have their tips on the top edge. Lastly, there will be $m - 1$ quadrilaterals of type II, but these all lie between the two zigzags and do not contribute to our sum.

Emily: That means that along the bottom edge, we will find the even-horizontal length triangles of type H and the quadrilaterals of type I formed by removal of triangles of type T with odd T-length, and that's it. We already computed that the total area of the even-horizontal length triangles of type H is $\dfrac{(m+1)(2m+1)}{3n(m+n)}$.

Jasmine: And for the contribution from the quadrilaterals of type I, we can compute the total area of the isosceles triangles whose tips are chopped off, and subtract the area of their type T triangular tips. We get one for each odd number from 1 to $n - m - 2$.

Emily: If $n - m = 1$, then the whole bottom edge consists of triangles of type H, and that's the case where – as you pointed out – the total area of the even-horizontal length triangles of type H does simplify to 1/3. So let's assume $n - m > 1$. Then there are a total of $(n - m - 1)/2$ quadrilaterals of type I. Using the formula for the sum of the first $(n - m - 1)/2$ odd squares, the total area of these quadrilaterals is

$$\left(\frac{n-m-1}{2}\frac{1}{n}\right)-\frac{m^2n}{n^2-m^2}\left(\frac{1}{(nm)^2}\right)\left(\frac{\frac{n-m-1}{2}(n-m)(n-m-2)}{3}\right)$$

$$\begin{aligned}
&= \frac{n-m-1}{2n}-\frac{1}{n(n^2-m^2)}\left(\frac{(n-m-1)(n-m)(n-m-2)}{6}\right)\\
&= \frac{n-m-1}{2n}-\frac{1}{n(n+m)}\left(\frac{(n-m-1)(n-m-2)}{6}\right)\\
&= \ldots \text{ (skipping lots of algebra) } \ldots\\
&= \frac{(n-m-1)(n+2m+1)}{3n(n+m)}.
\end{aligned}$$

Jasmine: Gosh, that was an algebra workout! I'm not too hopeful that these two expressions are going to add up to 1/3 and not some crazy rational polynomial. Do they?

Emily: Well, let's see.

$$\begin{aligned}
\frac{(m+1)(2m+1)}{3n(m+n)}+\frac{(n-m-1)(n+2m+1)}{3n(n+m)} &= \frac{(m+1)(2m+1)+(n-m-1)(n+2m+1)}{3n(n+m)}\\
&= \frac{2m^2+3m+1+n^2+nm-2m^2-3m-1}{3n(n+m)}\\
&= \frac{n^2+nm}{3n(n+m)} = 1/3!
\end{aligned}$$

Jasmine: Incredible, I can hardly believe it! What about the pieces on the top side?

Emily: On the top side, the triangles of type H have total area $\frac{(2m+1)(2m-1)}{6n(m+n)}$. The quadrilaterals of type I on the top side have total area

$$\left(\frac{n-m-1}{2}\frac{1}{n}\right)-\frac{m^2n}{n^2-m^2}\left(\frac{1}{(nm)^2}\right)\left(\frac{2\left(\frac{n-m-1}{2}\right)\left(\frac{n-m+1}{2}\right)(n-m)}{3}\right)$$

$$= \ldots \text{ (skipping lots of algebra) } \ldots$$

$$= \frac{(2n+4m-1)(n-m-1)}{6n(n+m)},$$

and when we add these two expressions together, we get

$$\frac{(2m+1)(2m-1)}{6n(m+n)} + \frac{(2n+4m-1)(n-m-1)}{6n(n+m)} = \ldots = \frac{2n-3}{6n} = \frac{1}{3}-\frac{1}{2n},$$

which is *not* equal to 1/3. Drat!

Jasmine: But Emily, for the top side, we still have to add in the area of the triangle of type R, and that has area $1/(2n)$! So when we add that in, it *does* come out to 1/3!

Emily: Wow! This is too good to be true! Let's check the case where n and m are both odd and start in the same corner, which, by symmetry, we can take to be the lower left corner.

Jasmine: Okay. In fact, by symmetry, it suffices to check just the bottom side. On the bottom, there's one triangle of type R. There are m triangles of type H, all with even-horizontal lengths (in units of $1/(nm)$). And there are $(n-m-2)/2$ quadrilaterals of type I corresponding to triangles of type T with even T-lengths ranging from 2 to $n-m-2$. We computed the area of the triangles of type H on the bottom to be $\frac{(m+1)(2m+1)}{3n(m+n)}$. And the quadrilaterals of type I will have total area

$$\left(\frac{n-m-2}{2}\frac{1}{n}\right)-\frac{m^2n}{n^2-m^2}\left(\frac{1}{(nm)^2}\right)\left(\frac{2\left(\frac{n-m-2}{2}\right)\left(\frac{n-m}{2}\right)(n-m-1)}{3}\right)$$

$$= \ldots \text{ (skipping lots of algebra) } \ldots$$

$$= \frac{(n-m-2)(2n+4m+1)}{6n(n+m)}.$$

Emily: So much algebra! When we add these two expressions together we get

$$\frac{(m+1)(2m+1)}{3n(m+n)} + \frac{(n-m-2)(2n+4m+1)}{6n(n+m)} = \ldots = \frac{2n-3}{6n} = \frac{1}{3} - \frac{1}{2n}.$$

Jasmine: Perfect! So when we add in the area of the triangle of type R, which is $1/(2n)$, we again get 1/3!

Emily: I can't believe this is all falling into place! I'll be so disappointed if the last case doesn't work out. So now we have n and m odd, but they start and end in different corners. By symmetry, we again only need to check the bottom edge. There are no triangles of type R, the triangles of type H along the bottom edge correspond to the odd-horizontal lengths (in units of $1/(nm)$) from 1 to $2m - 1$, and there are $(n - m)/2$ quadrilaterals of type I corresponding to triangles of type T with odd T-lenths from 1 to $n - m - 1$.

Jasmine: We compute the sum of the areas of the triangles of type H to be $\frac{(2m+1)(2m-1)}{6n(m+n)}$. The quadrilaterals of type I have total area

$$\left(\frac{n-m}{2}\frac{1}{n}\right) - \frac{m^2 n}{n^2-m^2}\left(\frac{1}{(nm)^2}\right)\left(\frac{\frac{n-m}{2}(n-m+1)(n-m-1)}{3}\right)$$

$$= \ldots \text{ (skipping lots of algebra) } \ldots$$

$$= \frac{(n-m)(2n+4m)+1}{3n(n+m)}.$$

Emily: Phew. My hands are getting tired. Now for the moment of truth:

$$\frac{(2m+1)(2m-1)}{6n(m+n)} + \frac{(n-m)(2n+4m)+1}{3n(n+m)} = \ldots \text{ (skipping lots of algebra) } \ldots = 1/3!$$

It works!

Jasmine: This is so cool! And this result also works for unequal n and m that share common factors, because we can subdivide into copies of the case n/d and m/d, where d is their greatest common factor. And, since in each subdivision the regions above, below, and in between split the corresponding rectangular section into thirds, that will be true for the entire square.

Emily: True! And it doesn't have to be a square; any rectangle will do, just by scaling. This has got to be our sweetest discovery ever!

Mr. ChemCake: Goodness! What's all the excitement?

Jasmine: We just discovered something really cool!

Emily: And proved it, too!

Mr. ChemCake: Really? What'd you find?

Emily and Jasmine excitedly explain their result.

Mr. ChemCake: Are you sure? Does it really work for any *n* and *m*?

Emily: You just can't have *n* equal to *m*. If they're equal, the split will be 1/2 below both, 1/2 above both, and 0 in between, if the two zigzags coincide.

Jasmine: And it'll be 1/4 below both, 1/4 above both, and 1/2 in between if they crisscross.

Mr. Chemcake: Astonishing. I'd say that's a bona fide theorem! How did you discover it?

Emily and Jasmine look at each other.

Emily: We were just playing around computing the areas of the pieces you get when you send two zigzags across a rectangle.

Mr. ChemCake: Are you saying you know the areas of *all* of those crazy pieces?

Jasmine: Yes, actually, we do! That's how we proved the result. The computations are kind of long and messy, but in the end, everything simplifies to 1/3!

Emily: Do you really think it's worthy of theoremhood?

Mr. ChemCake: Well, I'm no mathematician, but, yes, I think so! I had a funny feeling something was up here, so I took a chance and baked you an extra large Brownie Bonanza!

Emily and Jasmine cut a 3-zigzag/7-zigzag pattern across the brownie and apply their theorem to split it three ways to celebrate with Mr. ChemCake. They each grab a piece and clink them together as if they were wine glasses.

Mr. ChemCake: Congratulations!

Feeling encouraged, Emily and Jasmine write in their journals:

> **The ZigZags Theorem** (Emily and Jasmine). *Let* n *and* m *be distinct positive integers. Let a rectangle be crisscrossed by an* n*-zigzag and an* m*-zigzag, each bouncing back and forth between the top and bottom edges. Then the region of the rectangle below both zigzags, the region above both zigzags, and the region between the two zigzags split the rectangle exactly in thirds.*

Jasmine: Say, Emily. This result can't just be some crazy coincidence. There's got to be a deeper reason for it. Is there a more conceptual proof that doesn't involve a ton of algebra, yet miraculously simplifies to 1/3?

To be continued …

Meditate$^{\text{Math}}$

by Lightning Factorial | edited by Amanda Galtman

This installment of Meditate to the Math is a bit different from previous installments. Instead of presenting a mathematical fact and asking you to meditate upon it until you understand why it is true, we present a series of questions and statements that build a conceptual grounding for trigonometry. We hope this provides one answer to the question, “Why trigonometry?”

Please meditate deeply on the blue questions and statements until you’ve “seen the light”!

Similarity

Two things are similar if they are alike in some way. But in geometry, objects are considered **similar** if and only if they are alike in a very specific way: the objects must be scale models of each other. A scale model of the Eiffel Tower *looks* like the Eiffel Tower, but is small enough to pick up and place on your desk. They look alike, just as a picture of your friend looks like your friend, but one is a scaled-down version of the other.

Why do similar objects look alike?

Similar objects look alike because features of each object are in the same relative positions. Our brains are wired to regard similar objects as the same. The reason why our brains work this way can be explained by imagining a friend walking toward you from a distance. As she approaches, she appears to get bigger. Despite this change in apparent size, it would be useful to you to be able to recognize her. That is, it would be valuable for you to be able to recognize your friend at a great distance, when she appears very small, as well as from up close. For this reason, the brain discounts scale in identifying objects and instead focuses on proportions. As your friend nears, her height appears to grow. However, the *ratio* of her height to the length of her arms does *not* change. That is, her proportions do not change. After all, if her proportions changed, she’d need to buy new clothing when she arrived, as her old clothes would no longer fit.

When our brain identifies something, it pays attention to relative proportions, not absolute size.

Angles

How do corresponding angles in similar figures compare?

In similar figures, corresponding angles have equal measure. For example, if the walls and floors of a house meet at right angles, then the walls and floor of a scale model of that house also meet at right angles. (Note that figures with the same angle measures need not be similar. For example, a square and a rectangle both have four right angles, but need not be similar.)

Triangles

Imagine that you’re surrounded by a myriad of triangles, triangles of every shape and size. Some are acute, some are obtuse, some are isosceles, some are equilateral, and so on.

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Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 26 - Meet 1 January 31, 2020	Mentors:	Adeline Hillier, Shradha Kaushal, Tina Lu, Aileen Ma, Elise McCormack, Laura Pierson, Nehar Poddar, Gisela Redondo, Melissa Sherman-Bennett, Rebecca Vessenes, Emma Wang, Hanna Yang, Josephine Yu

The cube and the octahedron are **dual** to each other. This means if you replace the faces of the cube with vertices and the vertices of the cube with faces, you create an octahedron, and vice versa. Find other polyhedra that are dual to each other. Does every polyhedron have a dual? Is any polyhedron its own dual?

Session 26 - Meet 2 February 6, 2020	Mentors:	Katie Gravel, Adeline Hillier, Jenny Kaufman, Shradha Kaushal, Tina Lu, Elise McCormack, Kate Pearce, Nehar Poddar, Gisela Redondo, Melissa Sherman-Bennett, Rebecca Vessenes, Emma Wang, Karissa Wenger, Hanna Yang, Josephine Yu

Let S_n be the symmetric group on n elements. Can you characterize elements of S_n that are the square of a cycle?

Session 26 - Meet 3 February 13, 2020	Mentors:	Katie Gravel, Adeline Hillier, Jenny Kaufman, Tina Lu, Elise McCormack, Valerie Muldoon, Rebecca Nelson, Kate Pearce, Laura Pierson, Nehar Poddar, Melissa Sherman-Bennett, Karissa Wenger, Jasmine Zou

Imagine a party with 10 people who have never met. Each person meets all the other people and greets them with a handshake. How many handshakes occur? Can you come up with a general formula for the number of handshakes if n people meet and each shakes everyone else's hand? Suppose $n > 1$ people meet and each person shakes some hands but not others. Can you show that there will always be two who have shaken hands the same number of times?

Session 26 - Meet 4 February 27, 2020	Mentors:	Adeline Hillier, Shradha Kaushal, Rebecca Nelson, Kate Pearce, Laura Pierson, Nehar Poddar, Gisela Redondo, Emma Wang, Karissa Wenger, Josephine Yu

We tend to express numbers in base 10. That means we use the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 and our place values are powers of ten, $1 = 10^0$, $10 = 10^1$, $100 = 10^2$, etc. There are other number representation systems that use different bases. For example, binary is a base 2 system. In this system you use the 2 digits 0 and 1 and the place values correspond to powers of 2. For example, in binary, 1 corresponds to 2^0, 10 corresponds to 2^1, 100 corresponds to 2^2, etc. Can you train yourself to count in binary using your fingers? If you do so, how high would you be able to count on your fingers?

Calendar

Session 25: (all dates in 2019)

September	12	Start of the twenty-fifth session!
	19	
	26	
October	3	
	10	
	17	
	24	Sally Seaver, Biotechnology Consultant
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	

Session 26: (all dates in 2020)

January	30	Start of the twenty-sixth session!
February	6	
	13	
	20	No meet
	27	
March	5	
	12	
	19	
	26	No meet
April	2	
	9	
	16	
	23	No meet
	30	
May	7	

SUMIT 2020 is scheduled for April 4 and 5, 2020. Registration is now open and conducted on a first-come-first-served basis. We're especially interested in including more high schoolers. Visit http://girlsangle.org/page/SUMIT/SUMIT.html for updates.

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- □ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets) ☐ I will pay on a per meet basis at $20/meet.	☐ I will pay on a per meet basis at $30/meet. ☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____