Girls' Angle Bulletin

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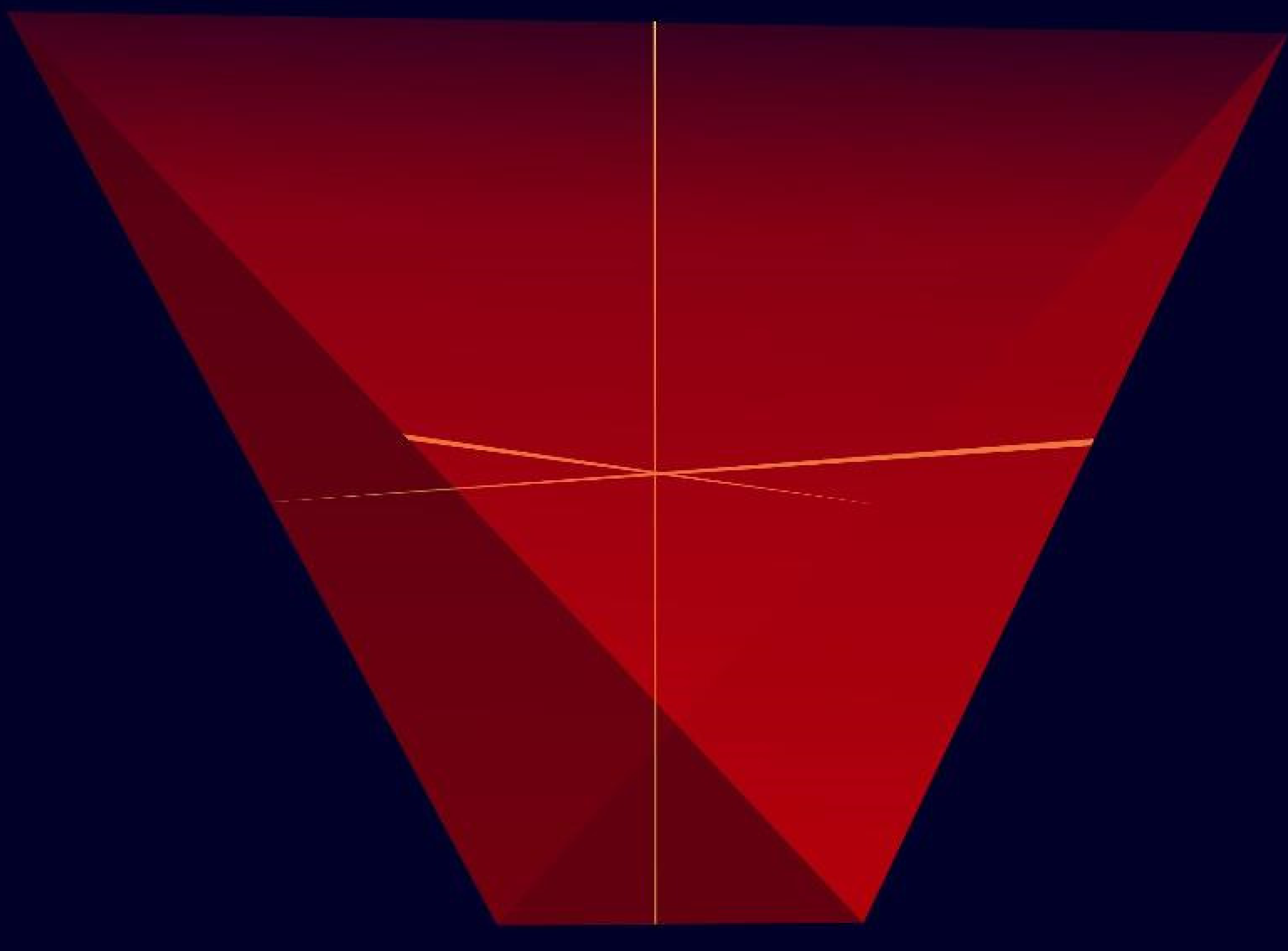
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Though in-person remains better, improvements in technology are making online education an effective option. We're looking forward to working with all our members and mentors again soon, even if it is in cyberspace! - Ken Fan, President and Founder

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Girls' Angle welcomes submissions that pertain to mathematics.

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Editors: Amanda Galtman, Jennifer Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *A scalene triangle can still make many right angles*, by C. Kenneth Fan.

An Interview with Aisha Arroyo

Aisha Arroyo is an Associate Professor of Mathematics at Middlesex Community College. Aisha received her bachelor's degree from Georgia Tech and her Master's degree in mathematics from the University of Illinois at Urbana-Champaign. She is an active member of the American Mathematical Association of Two-Year Colleges (AMATYC) as well as its New England affiliate, the New England Mathematical Association of Two-Year Colleges (NEMATYC).

Grace: I always enjoy hearing about a person's mathematical journey and how it led them to where they currently are. Can you describe a little bit about your journey?

Aisha: I've loved math since I learned to count. My parents fostered this love with books, puzzles, games, and so much more. While I enjoyed school in general, my extracurriculars were fairly math focused from math clubs and competitions to learning to program (in BASIC!). In high school I started tutoring and continued into college where I was a teaching assistant for several years. I entered graduate school with the intention of becoming a research mathematician but found that my true passion was teaching. When I finished my Master's degree I decided to pursue a career in mathematics education instead.

Grace: You participated in REUs and had exposure to research mathematics before entering grad school. How well did you feel this prepared you for the actual experience?

Aisha: The REU's I participated varied from essentially being independent study courses to actual research projects. The research

> *It's important to remember that even solving one question is something to be proud of.*

focused ones were very similar to my experience in graduate school. While graduate-level courses certainly open up more avenues for research, there is a lot of mathematics to explore at the undergraduate level as well.

Grace: When, and how, did it become clear to you that you wanted to pursue a career in mathematics education?

Aisha: I've been interested in helping others learn math since before I was old enough to be considering potential careers. Just like you would read a book to a younger sibling, I would teach my cousins arithmetic and basic algebra. Yet I don't think I really considered a career in education until I was in college. I was always planning on studying mathematics but was going to double major in physics or engineering or computer science (fields my parents considered more "useful"). My sophomore year of college I was a Teaching Assistant for an instructor at my university, someone whose job was teaching-focused instead of research-focused. That position is what solidified that what I really enjoyed doing most was helping others learn.

Grace: Can you describe your current position and the college environment at which you teach?

Aisha: While most of what I do is teach courses, I also help advise students and lead extracurricular activities. Because community colleges accept students with a

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Aisha Arroyo. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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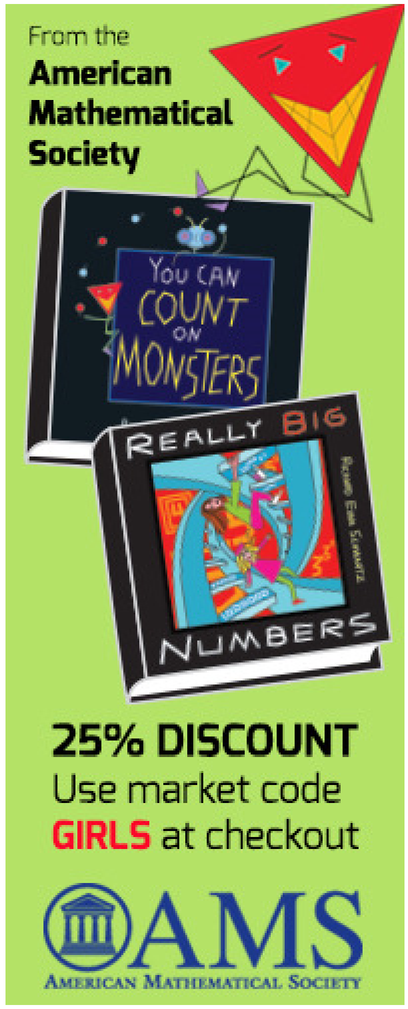

The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

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Meditate$^{\text{Math}}$

by Lightning Factorial | edited by Jennifer Sidney Silva

Let A and B be two points on a number line. Their midpoint can be computed with the formula $(A + B)/2$, also written $(1/2)A + (1/2)B$. But suppose we want to specify a point between A and B that isn't precisely halfway. How can we do that? One way would be to imagine traveling from A to B, but going only a fraction f of the way. This interpretation is expressed by the formula $A + f(B - A)$. As f varies from 0 to 1, the expression yields points that move from A to B.

1. The expression $A + f(B - A)$ makes a and b look different. An expression that accomplishes the same thing, but treats A and B equally, is $xA + yB$, where $x + y = 1$. What values of x and y correspond to $xA + yB$ being between A and B? What values of x and y correspond to being at A or at B? Show that y corresponds to f and x corresponds to $1 - f$.

2. Now suppose that A and B are points in a two- (or more-) dimensional coordinate plane. The same expression $xA + yB$, where x and y are nonnegative real numbers that sum to 1 and addition and multiplication are done for each coordinate separately, can be used as is to specify points on the line segment connecting A and B.

3. Now consider three non-collinear points A, B, and C in 3D coordinate space (or a hyperspace of any dimension). Consider the expression $xA + yB + zC$ where x, y, and z are nonnegative real numbers that sum to 1. Convince yourself that each such triple (x, y, z) corresponds uniquely to a point in or on the boundary of the triangle ABC. Which triples correspond to being at a vertex? Which to the interior of a side? Which to the interior of triangle ABC?

The numbers x, y, and z are called the **barycentric coordinates** of $xA + yB + zC$.

4. When you fix z and let x and y vary (subject to $x + y = 1 - z$), the expression $xA + yB + zC$ yields points along a line segment that is parallel to side AB. As a result, the triangles whose vertices are A, B, and $xA + yB + zC$ (for fixed z) all have the same area. Notice that when $x = 1 - z$ and $y = 0$, the point $xA + yB + zC = (1 - z)A + zC$ reverts to the two-point version of the expression. Deduce that z is the ratio of the area of the triangle with vertices A, B, and $xA + yB + zC$ to the area of triangle ABC.

5. The centroid of triangle ABC is the point $(A + B + C)/3$. Show that the line segments connecting the centroid to the vertices of a triangle split the triangle into three triangles of equal area.

6. Determine the barycentric coordinates of as many special points in triangles as you can.

7. Let's generalize this to four points A, B, C, and D that are not coplanar. We consider four nonnegative real numbers x, y, z, and w that sum to 1 and look at $xA + yB + zC + wD$, which corresponds to a point inside or on the boundary of tetrahedron $ABCD$. What constraints on x, y, z, and w correspond to being at a vertex of tetrahedron $ABCD$? On an edge? On a face? In the interior?

8. Convince yourself that w is the ratio of the volume of the tetrahedron with vertices A, B, C, and $xA + yB + zC + wD$ to the volume of tetrahedron $ABCD$.

Summer Fun!

In the previous issue, we presented the 2020 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Summer Fun!

Cannonballs and Combinatorics

by Girls' Angle Staff

1. There are 45 ways to select 2 apples from 10 apples. There are $n(n-1)/2$ ways to select 2 apples from n apples.

2. We have ${}_nC_k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1} = \frac{n!}{k!(n-k)!}$. (Also, ${}_nC_k \equiv 0$ if $k < 0$ or $k > n$.)

3. (By convention, $0! = 1$.) Pascal's triangle consists of the numbers ${}_nC_k$. The nth row consists of the numbers ${}_nC_0$, ${}_nC_1$, ${}_nC_2$, ${}_nC_3$, …, ${}_nC_n$. We prove this by showing that the numbers ${}_nC_k$ satisfy the defining properties of Pascal's triangle. First note that ${}_nC_0 = {}_nC_n = 1$ for all $n \geq 0$. Next we show that ${}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$ for all $n > 0$, $0 < k \leq n$:

$$\begin{aligned}
{}_nC_{k-1} + {}_nC_k &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{k \cdot n!}{k!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k+1)!} \\
&= \frac{k \cdot n! + (n-k+1)n!}{k!(n-k+1)!} \\
&= \frac{(n+1)n!}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= {}_{n+1}C_k.
\end{aligned}$$

4. For an algebraic proof, see the solution to Problem 3 above. For a bijective proof, let ${}_nS_k$ be the set of subsets of $\{1, 2, 3, \ldots, n\}$ with k elements. For any element S in ${}_{n+1}S_k$, if S does not contain $n + 1$, pair it with itself, considered as a subset of $\{1, 2, 3, \ldots, n\}$. If $n + 1$ is in S, pair it with the subset of $\{1, 2, 3, \ldots, n\}$ which is obtained by removing the element $n + 1$ from S. We must show that no two subsets in ${}_{n+1}S_k$ are paired with the same set, and that every subset in ${}_nS_k$ or ${}_nS_{k-1}$ is matched with some element of ${}_{n+1}S_k$. For the first part, suppose S and T are in ${}_{n+1}S_k$ and are paired with the same set. Since sets that do not contain $n + 1$ are paired with sets of size k, whereas sets that contain $n + 1$ are paired with sets of size $k - 1$, the sets S and T either both contain $n + 1$ or both do not. If neither contains $n + 1$, then they are both paired with themselves, and hence, are equal. If both contain $n + 1$, then they must agree on all their other elements, and since they also both contain $n + 1$, they, again, must be equal. For the second part, note that every element of ${}_nS_k$ is matched with itself, thought of as an element of ${}_{n+1}S_k$, and every element of ${}_nS_{k-1}$ is matched with the element of ${}_{n+1}S_k$ obtained by adding $n + 1$ to the set.

5. There are many patterns in Pascal's triangle, and new ones are still being discovered!

Summer Fun!

6. The nth triangular number is $1 + 2 + 3 + \ldots + n$, which equals $n(n + 1)/2 = {}_{n+1}C_2$.

7a. We seek triangular numbers that are also perfect squares. In other words, we seek integers n and m such that $n(n - 1)/2 = m^2$. If we multiply both sides of this equation by 8, we obtain the equation $4n(n - 1) = 8m^2$, which is equivalent to $(2n - 1)^2 - 1 = 2(2m)^2$. Thus, the problem is equivalent to finding a solution to the Pell equation $x^2 - 2y^2 = 1$, where $x = 2n - 1$ and $y = 2m$.

7b. Let's assume that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}$$

This is an example of a **continued fraction**. What this means is that the sequence of fractions obtained by truncating the continued fraction before each plus sign in the expression converges to the (positive) square root of 2. If we write the fraction obtained by truncating before the nth plus sign as a ratio x_n/y_n of two relatively prime integers x_n and y_n, we find

n	1	2	3	4	5	6	7	8
x_n	1	3	7	17	41	99	239	577
y_n	1	2	5	12	29	70	169	408

Notice that $x_{k+1}/y_{k+1} = 1 + 1/(1 + x_k/y_k) = (x_k + 2y_k)/(x_k + y_k)$ for $k > 0$. We claim that $x_k + 2y_k$ and $x_k + y_k$ are relatively prime, so that, in fact, $x_{k+1} = x_k + 2y_k$ and $y_{k+1} = x_k + y_k$. Suppose that d divides both $x_k + 2y_k$ and $x_k + y_k$. Then d must divide their difference, which is y_k. But if d divides both y_k and $x_k + y_k$, it must also divide x_k. Thus, if x_k and y_k are relatively prime, then d must be 1. Hence, by induction, $x_{k+1} = x_k + 2y_k$ and $y_{k+1} = x_k + y_k$.

Note that $x_{k+1}^2 - 2y_{k+1}^2 = (x_k + 2y_k)^2 - 2(x_k + y_k)^2 = -(x_k^2 - 2y_k^2)$. Since $x_1^2 - 2y_1^2 = -1$, this computation shows that $x_k^2 - 2y_k^2 = (-1)^k$. Thus, x_{2k} and y_{2k} are integer solutions to the Pell equation $x^2 - 2y^2 = 1$.

Furthermore, if x_k is odd, then $x_{k+1} = x_k + 2y_k$ must also be odd, and since $x_1 = 1$ is odd, all the x_k are odd. Since $y_{k+1} = x_k + y_k$, the y_k alternate odd, even, odd, even. Therefore, x_{2k} and y_{2k} are odd and even, respectively. We conclude that $(y_{2k}/2)^2$ is a perfect square which is also a triangular number.

To be complete, we should show that *every* perfect square that is a triangular number will be equal to $(y_{2k}/2)^2$ for some k. To do that, we can solve for x_k and y_k in terms of x_{k+1} and y_{k+1}: We find that $x_k = 2y_{k+1} - x_{k+1}$ and $y_k = x_{k+1} - y_{k+1}$. We can then use these formulas to get smaller solutions to the equations $x^2 - 2y^2 = \pm 1$, eventually reaching the solution $(x, y) = (1, 1)$. For details, see the solution to Problem 14 of *Two Whole Squares* on page 28.

Summer Fun!

7c. Check that when x_k and y_k are so defined, they satisfy the recursion obtained by applying the recursion formulas in the answer to Problem 7b twice.

Correction to the Summer Fun Problem Set: We said that one in from the border of Pascal's triangle, every integer greater than 1 appears twice, but actually 2 only appears once.

8. Let $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$ and $T_n = {}_{n+2}C_3 + {}_{n+1}C_3$. Check that $T_n = n(n + 1)(2n + 1)/6$. Notice that S_n is the unique sequence such that $S_1 = 1$ and $S_{n+1} = S_n + (n + 1)^2$, for $n > 0$. We will show that $S_n = T_n$ by induction on n.

First, $T_1 = 1(1 + 1)(2(1) + 1)/6 = 1$.

Next, for $n > 0$, we compute

$$\begin{aligned} T_n + (n+1)^2 &= n(n+1)(2n+1)/6 + (n+1)^2 \\ &= (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right) \\ &= (n+1)\left(\frac{n(2n+1)+6(n+1)}{6}\right) \\ &= (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\ &= (n+1)\left(\frac{(n+2)(2n+3)}{6}\right) \\ &= T_{n+1}. \end{aligned}$$

9. Multiplying both sides of the equation $\frac{n(n+1)(2n+1)}{6} = m^2$ by 4, we obtain the equation $\frac{4n(n+1)(2n+1)}{6} = 4m^2$, which is equivalent to $\frac{2n(2n+2)(2n+1)}{6} = (2m)^2$. If we let $a = 2n + 2$ and $b = 2m$, this is equivalent to ${}_aC_3 = b^2$.

10. The only perfect square 3 in from the border of Pascal's triangle is … 19,600 = 140^2, which is also equal to ${}_{50}C_3$. In terms of cannonballs, 4900 = 70^2 cannonballs can be arranged in a square pyramid arrangement with 24 layers.

Note that ${}_4C_3 = 4$ is also a perfect square, but ${}_4C_3$ is only 1 in from the border of Pascal's triangle.

11. We refer the reader to "The Square Pyramid Puzzle," by W. S. Anglin on pages 120-124 of *The American Mathematical Monthly*, Volume 97, 1990.

12. We refer the reader to *Proofs from THE BOOK*, by Aigner and Ziegler, pages 15-18.

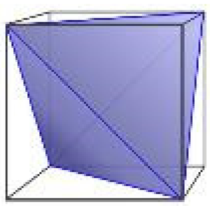
Tetrahedra with Congruent Faces

by Girls' Angle Staff

There are many ways to solve these problems and we urge you to find your own. These solutions are by no means definitive.

1. All tetrahedrons have 4 vertices, 6 edges, and 4 faces.

2. The four faces of a regular tetrahedron of side s are equilateral triangles with side length s. The altitudes of an equilateral triangle with side length s all have length $\sqrt{3}s/2$, which can be found by applying the Pythagorean theorem. Therefore, the area of an equilateral triangle with side length s is $\sqrt{3}s^2/4$. Since a regular tetrahedron has 4 faces, the total surface area is $\sqrt{3}s^2$.

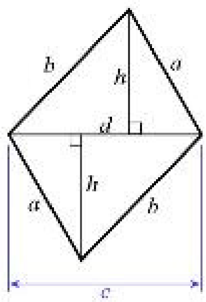


One way to compute the volume of a regular tetrahedron is to use the fact that a regular tetrahedron can be carved out of a cube. If you have a cube of side length L and then paint each vertex of the cube one of two colors in such a way that vertices that are joined by an edge are painted different colors, then you'll find that vertices of the same color form the vertices of a regular tetrahedron of side length $\sqrt{2}L$. To carve one of these tetrahedrons out of the cube, we have to lop off 4 pieces which are all congruent to a right triangular pyramid with height L and an isosceles right triangular base with leg lengths L. The volume of these right triangular pyramids are each $L^3/6$, so combined, they have volume $2L^3/3$. That means that the resulting regular tetrahedron has volume $L^3 - 2L^3/3 = L^3/3$. To make the side length of the regular tetrahedron be s, we must take L to be $s/\sqrt{2}$. In this way, we can deduce that a regular tetrahedron of side length s has volume $\sqrt{2}s^3/12$.

3. Suppose that the face is not acute. Then the side of length c is opposite the angle that has measure 90° or more. (Recall that $a \le b \le c$.) Consider two faces of the tetrahedron that share a side of length c. Let C and C' be the vertices of these two triangles not on the side of length c. Note that C and C' are joined by an edge of the tetrahedron of length c. Let θ be the measure of the dihedral angle between these two faces. Let h be the length of the altitude of these two triangles from C and C', and let d be the distance along the side of length c between the feet of these altitudes. By the Pythagorean theorem, the distance between C and C' is also given by $\sqrt{d^2 + 4h^2 \sin^2(\theta/2)}$. Since $0° < \theta < 180°$, this expression is less than $\sqrt{d^2 + 4h^2}$, which is the value of the expression when θ is a straight angle, which corresponds to the two faces lying in the same plane. When so flattened, both C and C' are inside or on the border of the circle with diameter the common side of length c because the angles at C and C' measure 90° or more. Thus, the maximum distance separating C and C' in the tetrahedron is c, and this can only occur when C and C' are right angles and A is a straight angle, which is impossible since our tetrahedron is not flat. Therefore, the faces must be acute triangles.

5. The area of a triangle in terms of its sides a, b, and c is $\sqrt{s(s-a)(s-b)(s-c)}$, where s is half the perimeter. This is known as Heron's formula.

6. When you derive a formula, it's generally a good idea to check that your formula agrees with known cases. So it's good to check that 4 times the formula found in Problem 5 is consistent with the surface area found in Problem 2 when $a = b = c$.



7. Imagine attaching two triangles with side lengths a, b, and c along their sides of length c to form a parallelogram (not a kite!). Think of the common side of length c as a hinge. The isosceles tetrahedron T can be formed by folding along this hinge until the two vertices opposite the side of length c are, themselves, a distance c apart from each other. Let θ denote the dihedral angle between the two triangles when the isosceles tetrahedron is formed. Let h denote the height of the triangle with side lengths a, b, and c with respect to the side of length c as base. In our parallelogram, the altitudes corresponding to h meet the common side of length c at two points separated by a distance we will denote by d. (See the figure.) Let A, B, and C be the measures of the angles opposite sides a, b, and c, respectively, in the triangle with side lengths a, b, and c. Finally, let K be the area of the triangle with side lengths a, b, and c.

By the Pythagorean theorem, in our tetrahedron T, we have

$$c^2 = d^2 + (2h \sin(\theta/2))^2.$$

Using the half-angle formula for sine and solving for $\cos\theta$, we find

$$\cos\theta = 1 - \frac{c^2 - d^2}{2h^2}.$$

Note that $d = b \cos A - a \cos B$. (Recall that $a \le b \le c$.) Thus,

$$\begin{aligned}
\frac{c^2-d^2}{2h^2} &= \frac{c^2-(b\cos A - a\cos B)^2}{2h^2} \\
&= \frac{a^2+b^2-2ab\cos C-(b\cos A-a\cos B)^2}{2h^2} \quad \text{(using the law of cosines)} \\
&= \frac{a^2\sin^2 B + b^2\sin^2 A + 2ab(\cos A\cos B - \cos C)}{2h^2} \\
&= \frac{2h^2 + 2ab(\cos A\cos B - \cos C)}{2h^2} \quad \text{(because } h = a\sin B = b\sin A\text{)} \\
&= 1 - \frac{ab(\cos A\cos B - \cos C)}{h^2}.
\end{aligned}$$

Hence, $\cos\theta = \dfrac{ab(\cos A\cos B - \cos C)}{h^2}$.

Summer Fun!

Using the formula "1/3 area of base times height," the volume V of T is $Kh(\sin\theta)/3$. Note that $h = 2K/c$ and $K = ab(\sin C)/2$. We compute

$$\begin{aligned}
V &= \frac{Kh}{3}\sqrt{1-\frac{a^2b^2(\cos A\cos B-\cos C)^2}{h^4}} \\
&= \frac{K}{3h}\sqrt{h^4-a^2b^2(\cos A\cos B-\cos C)^2} \\
&= \frac{c}{6}\sqrt{\frac{a^4b^4\sin^4 C}{c^4}-a^2b^2(\cos A\cos B-\cos C)^2} \\
&= \frac{abc}{6}\sqrt{\frac{a^2b^2\sin^4 C}{c^4}-(\cos A\cos B-\cos C)^2} \\
&= \frac{abc}{6}\sqrt{\sin^2 A\sin^2 B-(\cos A\cos B-\cos C)^2} \quad \text{(applying the law of sines).}
\end{aligned}$$

We take a break from our computation to observe that $\cos C = -\cos(A + B)$, since the angles in a triangle sum to 180°. That is, $\cos C = \cos(180° - (A + B)) = -\cos(A + B)$. Also, recall the angle sum formula for cosine, which says that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Continuing our computation, we find

$$\begin{aligned}
V &= \frac{abc}{6}\sqrt{\sin^2 A\sin^2 B-(\cos A\cos B-\cos C)^2} \\
&= \frac{abc}{6}\sqrt{\sin^2 A\sin^2 B-(\cos A\cos B+\cos(A+B))^2} \\
&= \frac{abc}{6}\sqrt{\sin^2 A\sin^2 B-(2\cos A\cos B-\sin A\sin B)^2} \\
&= \frac{abc}{6}\sqrt{4\cos A\cos B\sin A\sin B-4\cos^2 A\cos^2 B} \\
&= \frac{abc}{3}\sqrt{\cos A\cos B(\sin A\sin B-\cos A\cos B)} \\
&= \frac{abc}{3}\sqrt{\cos A\cos B\cos C}\ .
\end{aligned}$$

Substituting expressions for the cosines of the angles of a triangle obtained from the law of cosines yields the formula in the problem statement.

9. See the solution to Problem 7.

10-13. Let's think about the inscribed and circumscribed spheres of T. Let O be the center of the circumscribed sphere and let I be the center of the inscribed sphere.

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The point O is equidistant from the 4 vertices of T. Let H be the foot of the perpendicular dropped from O to one of the faces of T. Call this face F. The 3 right triangles formed by O, H, and each of the 3 vertices of F are congruent because they all share the common leg of length OH and a hypotenuse of length the radius of the circumscribed sphere. Therefore, H is the center of the circumcircle of F. Since all the faces of T are congruent, they all have the same circumradius, and so the distance of O to each face is the same. Hence, $O = I$. (This answers Problem 12.) Since all 4 faces of T have the same area, the barycentric coordinates of $O = I$ must be (1/4, 1/4, 1/4, 1/4), and hence, $O = I$ is also the centroid of T. (This answers Problem 13. Also, see point 8 of *Meditate to the Math* on page 7.)

Let $r = OH$ be the radius of the inscribed sphere. We can split the tetrahedron into 4 congruent triangular pyramids where each face becomes a base of one of these pyramids and they all have apex O and height r. All 4 of these pyramids have the same volume $Kr/3$. Therefore, $V = Sr/3$. Solving for r, we find $r = 3V/S$. (This answers Problem 11.)

Let R be the radius of the circumscribed sphere. By the Pythagorean theorem, $R^2 = r^2 + \rho^2$, where ρ is the radius of the circumcircle of any face. We can express r in terms of a, b, and c by using the answers to Problems 5 and 7, and the formula $r = 3V/S$ from the previous paragraph. We can use the formula $4\rho = abc/K$ and the answer to Problem 5 to express ρ in terms of a, b, and c. One can then verify that the resulting expression for R^2 simplifies to the expression inside the radical of the formula in the statement of Problem 10 to solve Problem 10.

14. Observe that m_c is the altitude to the side of length c in an isosceles triangle with side lengths c, m, and m, where m is the length of the median to side c in any face of T. By Stewart's theorem, $4m^2 = 2a^2 + 2b^2 - c^2$. By the Pythagorean theorem,

$$m_c^2 = m^2 - (c/2)^2 = \frac{a^2 + b^2 - c^2}{2},$$

as desired. The formulas for m_a and m_b follow by symmetry.

15. Return your attention to the figure used in the solution to Problem 7. You are looking straight down upon a parallelogram with one diagonal of length c. Imagine the two triangles rotating about the common side of length c, lifting off the page at the same rate, until T is formed. Note that the side of T opposite the hinge is parallel to the plane of the parallelogram (because the two traveling vertices of the two rotating triangles end up at the same height above the plane of the parallelogram). Also, by symmetry, the midpoint of the side opposite the hinge sits directly above the midpoint of the hinge. That is, m_c is perpendicular to the plane of the parallelogram. The other two segments m_a and m_b connect the midpoints of opposite sides, and these midpoints are all contained in the plane exactly halfway between the plane of the parallelogram and the plane containing the side of T opposite the hinge and parallel to the plane of the parallelogram. Hence, they are both perpendicular to m_c. By a symmetric argument, we conclude that all three segments m_a, m_b, and m_c must be mutually perpendicular to each other. (Also, see this issue's cover image.)

Summer Fun!

Bernoulli Numbers

by Matthew de Courcy-Ireland

For the reader's convenience, we repeat the definition of the sequence B^n here. Let $B^0 = 1$, and for $n \neq 1$, define recursively $(B - 1)^n = B^n$, where this is to be interpreted by expanding $(B - 1)^n$, but then interpreting the exponents as indices in the sequence. For example, the definition entails $B^1 = 1/2$ because the recursion relation for $n = 2$ says that $(B - 1)^2 = B^2$. The expansion of $(B - 1)^2$ is $B^2 - 2B^1 + B^0$, so $B^2 - 2B^1 + B^0 = B^2$. Interpreting the exponents as indices, this simplifies to $B^1 = B^0/2$, and we already know that $B^0 = 1$. Thus, $B^1 = 1/2$.

Remember, in this Summer Fun problem set B^2 and B^3 are not "*B* squared" and "*B* cubed"! Instead, they are the second and third terms of a special sequence of numbers. A superscript that directly decorates a capital *B* stands for an index and not an exponent!

Did you know that what is often cited as the first computer program ever published was an algorithm for computing Bernoulli numbers? It was written by Ada Lovelace and published in 1843. It was written for Charles Babbage's "Analytical Engine," a plan for a computer which was never built.

1. Taking $n = 3$ in the definition gives $B^3 - 3B^2 + 3B^1 - B^0 = B^3$. We know $B^0 = 1$ and $B^1 = 1/2$, while B^3 cancels on both sides. Solving for B^2 gives

$$B^2 = (3B^1 - B^0)/3 = 1/6.$$

When we reason similarly for $n = 4$, everything cancels out. By definition,

$$B^4 - 4B^3 + 6B^2 - 4B^1 + B^0 = B^4.$$

We solve for B^3 using the known values $B^0 = 1$, $B^1 = 1/2$, and $B^2 = 1/6$ and find

$$B^3 = \frac{6B^2 - 4B^1 + B^0}{4} = \frac{6(1/6) - 4(1/2) + 1}{4} = 0.$$

To find B^4, we take $n = 5$ in the definition:

$$B^5 - 5B^4 + 10B^3 - 10B^2 + 5B^1 - B^0 = B^5.$$

Solving for B^4, we find

$$B^4 = \frac{10B^3 - 10B^2 + 5B^1 - B^0}{5} = \frac{0 - 10(1/6) + 5(1/2) - 1}{5} = -\frac{1}{30}.$$

You can continue as far as you like, depending on your patience for binomial coefficients. The first few values might suggest that the Bernoulli numbers are all between -1 and 1, but this is misleading. You will find the pattern broken if you go as far as B^{14}. From that point on, it seems the even terms in the sequence grow very quickly, while the odd terms are 0.

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n	1	3	5	7	9	11	13	15	17	19
B^n	1/2	0	0	0	0	0	0	0	0	0

n	2	4	6	8	10	12	14	16	18	20
B^n	1/6	-1/30	1/42	-1/30	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

2. The general case is similar to the case $p = 1$, but we use the binomial expansion for exponent $p + 1$ instead of second powers. To start, write $B + n - 1 = B - 1 + n$ and then expand:

$$(B + n - 1)^{p+1} = (B - 1 + n)^{p+1} = \sum_{k=0}^{p+1} \binom{p+1}{k} (B-1)^k n^{p+1-k}.$$

The definition implies that $(B - 1)^k = B^k$, except for $k = 1$, so we had better give that term special treatment. Separating the term $k = 1$ from the rest of the sum gives

$$(B + n - 1)^{p+1} = (p + 1)(B - 1)n^p - (p + 1)Bn^p + \sum_{k=0}^{p+1} \binom{p+1}{k} B^k n^{p+1-k}.$$

We recognize this last sum as the binomial expansion of $(B + n)^{p+1}$. Also,

$$(p + 1)(B - 1)n^p - (p + 1)Bn^p = -(p + 1)n^p.$$

Therefore,

$$(B + n - 1)^{p+1} = -(p + 1)n^p + (B + n)^{p+1}.$$

Solving for $(B + n)^{p+1}$ completes our mission.

3. We make repeated use of the previous identity: $(B + n)^{p+1} = (B + n - 1)^{p+1} + (p + 1)n^p$.

Applying it again with $n - 1$ in place of n gives

$$(B + n)^{p+1} = (B + n - 2)^{p+1} + (p + 1)(n - 1)^p + (p + 1)n^p.$$

And once more, with $n - 2$ in place of n, gives

$$(B + n)^{p+1} = (B + n - 3)^{p+1} + (p + 1)((n - 2)^p + (n - 1)^p + n^p).$$

After n applications, for each of $n, n - 1, n - 2, \ldots, 1$, we obtain

$$(B + n)^{p+1} = (B + n - n)^{p+1} + (p + 1)(1^p + \ldots + (n - 2)^p + (n - 1)^p + n^p).$$

Finally, we solve for the sum and note that $(B + n - n)^{p+1}$ is just B^{p+1}. Therefore,

$$1^p + \ldots + (n-2)^p + (n-1)^p + n^p = \frac{(B+n)^{p+1} - B^{p+1}}{p+1}.$$

Using the binomial expansion, we can restate this formula in terms of the sequence B^0, B^1, B^2, … without mention of the indeterminate B itself. The term $B^{p+1}n^0$ cancels with B^{p+1}, leaving only

$$1^p + \ldots + (n-2)^p + (n-1)^p + n^p = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k}B^k n^{p+1-k}.$$

4. We substitute the known values of B^k in the formula obtain in Problem 3.

For $p = 1$, the sum involves only $B^0 = 1$ and $B^1 = 1/2$. The binomial coefficients are 1 for $k = 0$ and 2 for $k = 1$. It follows that

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}\left(\binom{1+1}{0}B^0 n^2 + \binom{1+1}{1}B^1 n\right) = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}.$$

For $p = 2$ we also need to know $B^2 = 1/6$. The binomial coefficients in this case are 1, 3, and 3. The general formula for $1^p + \ldots + n^p$ implies

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}(n^3 + 3\times\frac{1}{2}n^2 + 3\times\frac{1}{6}n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}.$$

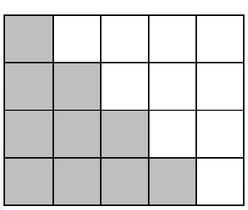
(Note that the fraction on the right must be an integer for all integer values of n, because the left is clearly an integer. This can also be seen directly: either n or $n + 1$ is even, and one of n, $n + 1$, or $2n + 1$ is a multiple of 3, so their product is divisible by 6.)

For $p = 3$, we substitute $B^3 = 0$ and the binomial coefficients are 1, 4, 6, and 4. The result is

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}(n^4 + 4\times\frac{1}{2}n^3 + 6\times\frac{1}{6}n^2 + 4\times 0 n^1) = \frac{n^4 + 2n^3 + n^2}{4}.$$

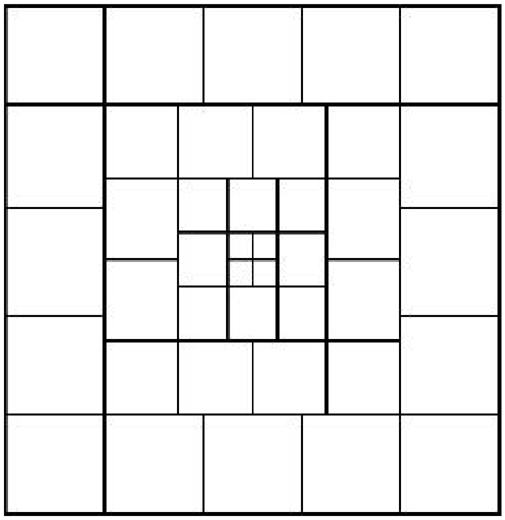
We can recognize this as a square, in fact, the square of $1 + 2 + 3 + \ldots + n$!

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$



5. Here is a picture for $n = 4$. The sum $1 + 2 + 3 + \ldots + n$ gives the area of a staircase built with one extra square at each level. Two such staircases make a rectangle of area $n \times (n + 1)$, so each has area $n(n + 1)/2$. The geometric meaning of the sum formula is that the same area is expressed in two different ways.

To see how this sum is related to $1^3 + 2^3 + 3^3 + \ldots + n^3$, examine this figure:



The third-power sum is the square of the first-power sum. This result is called Nicomachus's Theorem, and an online search for "Nicomachus's Theorem" will lead you to a variety of proofs and geometric interpretations of the identity. The one in the hint goes as follows. In the middle, we have 4 squares of side 1. These are surrounded by 8 squares of side 2, then 12 of side 3, and 16 of side 4. The picture can be continued with $4n$ squares of side n at the nth layer. The areas of all these squares together can be computed in two ways. Layer by layer, we obtain

$$4 \times 1^2 + (4 \times 2) \times 2^2 + \ldots + (4 \times n) \times n^2,$$

which is 4 times the third-power sum. On the other hand, the big square formed from all of these has area $(n(n + 1))^2$ because each side consists of $n + 1$ squares of side n. Comparing these areas and dividing by 4 completes the proof.

6. There are two trivial solutions and another very interesting one. The sum of the first $n = 0$ squares is empty, and an empty sum should always be interpreted as 0 (so that we can add it to other sums without changing their value). Of course $0 = 0^2$ counts as a perfect square. Only slightly less vacuous is the case $n = 1$, where $1^2 = 1^2$.

The interesting case is $n = 24$. Using your formula for the sum of the first n squares,

$$1^2 + 2^2 + 3^2 + \ldots + 24^2 = 24(24 + 1)(2(24) + 1)/6 = 70^2.$$

It turns out that there are no other solutions. The problem asks for integer solutions to the equation $y^2 = x(x + 1)(2x + 1)/6$, which is an example of an "elliptic curve." The well-developed theory of equations of this type makes it possible to show that this one has no other integer solutions. There are also more elementary proofs, as discussed in W. S. Anglin on pages 120-124 of *The American Mathematical Monthly*, Volume 97, 1990.

7. The expressions are $\cos(Bx) = \frac{x}{2}\cot\frac{x}{2}$ and $\sin(Bx) = \frac{x}{2}$. This implies that all odd-indexed Bernoulli numbers are 0, except for B^1; just compare both sides of

$$\frac{x}{2} = \sin(Bx) = B^1 x - \frac{B^3}{3!}x^3 + \frac{B^5}{5!}x^5 - \frac{B^7}{7!}x^7 + \frac{B^9}{9!}x^9 - \dots.$$

The key property used to find these expressions is that for any power series $f(x)$, we have

$$f(Bx) - f((B-1)x)$$

is the linear part of $f(x)$. What we mean by the "linear part" of a series $f(x) = \sum a_n x^n$ is $a_1 x$. The sum is finite if $f(x)$ is a polynomial, but could equally well extend over all $n \geq 0$. We have

$$f(Bx) - f((B-1)x) = \sum a_n B^n x^n - \sum a_n (B-1)^n x^n.$$

By definition, $(B-1)^n = B^n$ except when $n = 1$. Therefore, $f(Bx) - f((B-1)x) = a_1 x$.

For example, consider the power series $f(x) = \sin x = x - x^3/6 + x^5/120 - \dots$. Its linear part is x, so

$$\sin(Bx) - \sin((B-1)x) = x.$$

For $f(x) = \cos x = 1 - x^2/2 + x^4/24 - \dots$, the linear part disappears. In this case

$$\cos(Bx) - \cos((B-1)x) = 0.$$

We apply the angle-sum formulas for sine and cosine to these equations, namely

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Let us do the calculation formally before discussing why these identities remain valid in the presence of B. We have first, using the identity for $\cos(u+v)$, with $u = Bx$ and $v = -x$,

$$0 = \cos(Bx) - \cos((B-1)x) = \cos(Bx) - \cos(Bx)\cos(x) - \sin(x)\sin(Bx).$$

Second, using the identity for $\sin(u+v)$, we find

$$x = \sin(Bx) - \sin((B-1)x) = \sin(Bx) - \sin(Bx)\cos(x) + \sin(x)\cos(Bx).$$

We now have two linear equations for $\sin(Bx)$ and $\cos(Bx)$. Write the unknowns as $X = \cos(Bx)$ and $Y = \sin(Bx)$, and to abbreviate the coefficients of the equations, let $c = \cos x$ and $s = \sin x$:

$$(1-c)X - sY = 0$$
$$sX + (1-c)Y = x$$

The solution is $(X, Y) = (\frac{1}{2}\frac{s}{1-c}x, \frac{1}{2}x)$.

This can be simplified using some classical trigonometry (no B involved), in particular the double angle formulas $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 1 - 2\sin^2(x)$. We find

$$\frac{s}{1-c} = \frac{\sin x}{1-\cos x} = \frac{2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)} = \cot(x/2).$$

Thus, $\cos(Bx) = \frac{x}{2}\cot\frac{x}{2}$ and $\sin(Bx) = \frac{x}{2}$.

To find an expression for $\exp(Bx)$, we can take what we already know for $\cos(Bx)$ and $\sin(Bx)$ and feed it into Euler's formula $\exp(u) = \cos(u/i) + i\sin(u/i)$, where $i^2 = -1$. We find

$$\exp(Bx) = \cos(Bx/i) + i\sin(Bx/i) = \frac{x/i}{2}\cot\left(\frac{x/i}{2}\right) + \frac{ix/i}{2}.$$

The cotangent function for imaginary arguments can be handled using Euler's formula again. Recall that

$$\cot u = \frac{\cos u}{\sin u} = \frac{(\exp(iu)+\exp(-iu))/2}{(\exp(iu)-\exp(-iu))/(2i)} = i\frac{\exp(iu)+\exp(-iu)}{\exp(iu)-\exp(-iu)}.$$

Taking $u = (x/i)/2$ gives

$$\cot(\frac{x/i}{2}) = i\frac{\exp(i\frac{x/i}{2})+\exp(-i\frac{x/i}{2})}{\exp(i\frac{x/i}{2})-\exp(-i\frac{x/i}{2})} = i\frac{\exp(x)+1}{\exp(x)-1}.$$

Substituting this into $\exp(Bx)$, we obtain

$$\exp(Bx) = \frac{x/i}{2}i\frac{\exp(x)+1}{\exp(x)-1} + \frac{x}{2} = x\frac{\exp(x)}{\exp(x)-1} = \frac{x}{1-\exp(-x)}.$$

Finally, we comment on why the trigonometric identities remain valid. This is because these identities can be proved by manipulating the power series of sine and cosine, and substituting the relation $(B-1)^n = B^n$. What would not be valid is multiplying terms:

$$B^n B^m \neq B^{n+m}.$$

For example, $B^1B^1 = (1/2)(1/2) = 1/4$, whereas $B^2 = 1/6$. For example, one trigonometric identity that does not work is $\sin^2(Bx) + \cos^2(Bx) \neq 1$.

Summer Fun!

8. Computing $1^{10} + 2^{10} + 3^{10} + \ldots + 1000^{10}$ was an example Bernoulli gave to illustrate the power of the formula, writing that he could compute the sum "intra semiquadrantum horum," that is, within a half of a quarter of an hour. To use the formula

$$1^p + \ldots + (n-2)^p + (n-1)^p + n^p = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k}B^k n^{p+1-k},$$

we need the binomial coefficients

1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11

To give ourselves a head start, assume these have been computed in advance. Likewise, we refer to our table of Bernoulli numbers instead of computing them again. We only need them as far as B^{10}. Thus, before the clock starts ticking, we know

$$1^{10} + 2^{10} + 3^{10} + \ldots + n^{10} = \frac{1}{11}(n^{11} + \frac{11}{2}n^{10} + \frac{55}{6}n^9 - \frac{330}{30}n^7 + \frac{462}{42}n^5 - \frac{165}{30}n^3 + 11\times\frac{5}{66}n).$$

Many of the fractions simplify, even more so than in the seemingly easier case $1^9 + 2^9 + \ldots + n^9$:

$$1^{10} + 2^{10} + 3^{10} + \ldots + n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n.$$

Now we let $n = 1000$ and write the fractions as repeating decimals. Then the powers of n will simply shift the decimal place. So the thousand-term sum is equal to

$$(0.09\ldots)10^{33} + (0.5)10^{30} + (0.833\ldots)10^{27} - 10^{21} + 10^{15} - (0.5)10^9 + (0.0757575\ldots)10^3.$$

The positive contributions are

+90909090909090909090909090909090.909090909090…
+00500000000000000000000000000000.000000000000…
+00000833333333333333333333333333.333333333333…
+00000000000000001000000000000000.000000000000…
+00000000000000000000000000000075.757575757575…

The negative ones are

-00000000001000000000000000000000.000000000000…
-00000000000000000000000500000000.000000000000…

The result is

91,409,924,241,424,243,424,241,924,242,500.

Matrix Expedition

by Jasmine Zou

Consider the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Notice that $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Also, notice that M is a **linear transformation**, that is, $M(av + bw) = aMv + bMw$, where a and b are scalars and v and w are vectors.

1. The reflection in the vertical axis sends the position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to the position vector $\begin{pmatrix} -x \\ y \end{pmatrix}$. In particular, it sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Applying the comment at the top of this page, we conclude that the matrix that represents reflection in the vertical axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. As in our solution to Problem 1, we note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are swapped by the reflection in the line $y = x$, hence the answer is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. The 180° rotation about the origin sends the position vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and the position vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Thus, the answer is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

4. The matrix $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ corresponds to a counterclockwise rotation about the origin by the angle t. To see this, note that its first and second columns correspond to the counterclockwise rotation of the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by the angle t about the origin.

5. The matrix corresponding to **1** sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We compute

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a(1)+\mathrm{b}(0) & a(0)+b(1) \\ c(1)+d(0) & c(0)+d(1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A similar computation shows that $\mathbf{1}M = M$ for any matrix M.

Summer Fun!

6. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. This means that $2a + 3b = 3$ and $2c + 3d = 2$. This is a system of 2 linear equations in 4 unknowns. Rearranging, we find that $b = 1 - 2a/3$ and $d = 2(1 - c)/3$. Thus, the most general matrix that sends $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is $\begin{pmatrix} a & (3-2a)/3 \\ c & (2-2c)/3 \end{pmatrix}$.

7. Let M be a desired matrix. The position vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ points to a point on the line $y = 2$. Hence, M must send it to a position vector that points to a point on the line $y = 1 - x$, so let's declare that M sends $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} a \\ 1-a \end{pmatrix}$, for some a. Every vector is a linear combination of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, so if we also specify where M maps $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, the matrix M will be determined. Now M must also send $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ to a position vector that points to a point on the line $y = 1 - x$, so let's send it to the vector $\begin{pmatrix} b \\ 1-b \end{pmatrix}$, for some b. If $M\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 1-a \end{pmatrix}$, then $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a/2 \\ (1-a)/2 \end{pmatrix}$, by linearity. Also, since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right)$, it must be that $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} b \\ 1-b \end{pmatrix} - \begin{pmatrix} a \\ 1-a \end{pmatrix}\right) = \begin{pmatrix} (b-a)/2 \\ (a-b)/2 \end{pmatrix}$, by linearity. Therefore, $M = \begin{pmatrix} (b-a)/2 & a/2 \\ (a-b)/2 & (1-a)/2 \end{pmatrix}$. If we set $y = a/2$ and $x = (b - a)/2$, this can be expressed in the form $\begin{pmatrix} x & y \\ -x & 1/2 - y \end{pmatrix}$.

8. While the commutative property of multiplication holds for all numbers (i.e. $ab = ba$ for all real numbers a and b), this property does not hold for matrices. For example

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \text{ whereas } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

9. A 2 by 2 matrix M that satisfies $M^2 = \mathbf{1}$ must be $\mathbf{1}$, $\mathbf{-1}$, or a matrix of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $a^2 + bc = 1$.

10. In fact, $E = \mathbf{1}$ works. Another example is $E = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$.

Summer Fun!

Girls!

Learn Mathematics!

Make new Friends!

Meet
Professional Women
who use math in their work!

Improve how you
Think and Dream!

Girls' Angle

A math club for ALL girls, grades 5-12.

girlsangle@gmail.com
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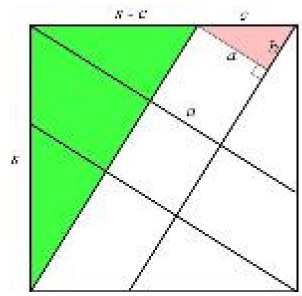
Two Whole Squares

by Girls' Angle Staff

2. We have $a = \dfrac{sc}{\sqrt{2s^2 - 2sc + c^2}}$.

3. When $c = 0$, the value of a is 0. When $c = s$, the value of a is s. As c goes from 0 to s, the value of a varies continuously from 0 to s and, hence, passes through every whole number less than or equal to s.

4. See the solutions to Problems 5 through 14.



Recall that we are restricting c to be strictly between 0 and s.

5. The length c is the hypotenuse of a triangle with one leg of length a, hence $c > a$.

6. The green and red triangles in the figure at right are similar right triangles. By similarity, $b/a = (s - c)/s$.

7. Since we are assuming that s, c, and a are all whole numbers, the formula in Problem 6 shows that b must be a rational number. Also $b^2 = c^2 - a^2$ is an integer. If the square root of an integer is rational, the integer must be a perfect square because the square root of a non-square integer is irrational.

8. If you were unable to show that a, b, and c are a primitive Pythagorean triple, congratulations, because you are correct! Apologies for our mistake!

Solving the equation in the solution to Problem 6 for s, we find that $s = ac/(a - b)$. Now consider any primitive Pythagorean triple a, b, c with $c > a > b$. We then construct the square of side length $s = ac/(a - b)$. This side length may not be an integer. However, we can scale the figure by the factor $a - b$. We would then obtain the triple $S = ac$, $C = c(a - b)$, and $A = a(a - b)$, which we claim is a primitive solution to our problem.

To see this, note that $a - b$ is relatively prime to a, for if not, then we would have a common prime factor p of $a - b$ and a, and p must divide the difference $a - (a - b) = b$. But if p divides both a and b, then p must divide c, since $c^2 = a^2 + b^2$, contradicting our assumption that a, b, c is a primitive Pythagorean triple. Similarly, $a - b$ is relatively prime to b. We also claim that $a - b$ is relatively prime to c. For suppose we have a prime factor p of both $a - b$ and c. Note that p cannot be 2 since primitive Pythagorean triples all have odd hypotenuse. Now observe that p must divide $(a - b)^2 = a^2 + b^2 - 2ab = c^2 - 2ab$, and hence p divides ab. This implies p divides a or b. But if p divides two sides of an integer-sided right triangle, it must also divide the third, contradicting our assumption that a, b, and c are a primitive Pythagorean triple.

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Now suppose there is a prime number p which divides S, C, and A. Since a and c are relatively prime, p must divide either a or c, but not both. If p divides a, then the fact that p divides C shows that p divides $a - b$. If p divides c, then the fact that p divides A shows that p divides $a - b$. Either way, we get a contradiction. Therefore, S, C, A is primitive and all primitive solutions must be obtainable in this manner (why?).

In summary, for each primitive Pythagorean triple, we get a primitive solution to our Diophantine problem, but the corresponding primitive solution may involve a non-primitive Pythagorean triple a, b, and c. For fun, let's find those primitive solutions s, c, a for which the corresponding triple c, a, b *is* a primitive Pythagorean triple.

9-11. When will c, a, b be a primitive Pythagorean triple? In our correction to Problem 8, we saw that it will be primitive precisely when $a - b = 1$. So we seek primitive Pythagorean triples whose leg lengths are consecutive whole numbers.

13. We show this by induction on k. Note that $x_0 = 1$ and $y_0 = 0$, so $x_0^2 - 2y_0^2 = 1$. Also,

$$x_{k+1} + y_{k+1}\sqrt{2} = (1 + \sqrt{2})^{k+1} = (1 + \sqrt{2})(x_k + y_k\sqrt{2}) = (x_k + 2y_k) + (x_k + y_k)\sqrt{2}.$$

Thus, $x_{k+1} = x_k + 2y_k$ and $y_{k+1} = x_k + y_k$. We compute

$$x_{k+1}^2 - 2y_{k+1}^2 = (x_k + 2y_k)^2 - 2(x_k + y_k)^2 = -x_k^2 + 2y_k^2 = -(-1)^k = (-1)^{k+1}.$$

The identity follows by induction.

14. (Modified to find primitive solutions that induce a primitive Pythagorean triple.) We claim that all integer solutions to $x^2 - 2y^2 = \pm 1$ in positive integers x and y correspond to some pair of terms x_k and y_k from the sequence defined in Problem 13. To see this, suppose that we have positive integers m and n that satisfy $m^2 - 2n^2 = \pm 1$, with $n > 0$. By reversing the map that sends (x_k, y_k) to (x_{k+1}, y_{k+1}) in the solution to Problem 13, we can get another solution to the equation $x^2 - 2y^2 = \pm 1$. That is, $(2n - m, m - n)$ is also a solution to $x^2 - 2y^2 = \pm 1$. Furthermore, we claim that $0 \le m - n < m$. Since $n > 0$, we know $m - n < m$. Also, $m^2 = \pm 1 + 2n^2$. Since $n > 0$, we know that $n \ge 1$, so $n^2 \ge 1$ and $2n^2 = n^2 + n^2 \ge n^2 + 1$. Hence, $2n^2 - 1 \ge n^2$. Thus, $m^2 \ge n^2$, which means that $m \ge n$, so $m - n \ge 0$. This means that our new solution has a smaller, nonnegative value for y. By continuing this procedure, we eventually arrive at a solution where $y = 0$, and the only such solution has $x = 1$, that is, when $x = x_0$ and $y = y_0$. We can then undo the steps taken to get to the solution (1, 0) to see that our original solution (m, n) is equal to (x_k, y_k) for some positive integer k.

For positive even k, we let $n = y_k$ and $m = x_k + y_k$, and our Pythagorean triple is $c = m^2 + n^2$, $a = m^2 - n^2$, and $b = 2mn$. And for positive odd k, we let $n = y_k$ and $m = x_k + y_k$, and our Pythagorean triple is $c = m^2 + n^2$, $a = 2mn$, and $b = m^2 - n^2$. In both cases, the square will have side length $ca = m^4 - n^4$.

Summer Fun!

Calendar

Session 27: (all dates in 2020)

September	10	Start of the twenty-seventh session!
	17	
	24	
October	1	
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls’ Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls’ Angle Members receive a significant discount on club attendance fees.

Who are the Girls’ Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls’ Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls’ Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members’ efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of “catching up with the group” doesn’t apply.

Where is Girls’ Angle located? Girls’ Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls’ Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls’ Angle activities are tailored to each girl’s specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls’ Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________
(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: __________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____