Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

An Interview with Petronela Radu
Order In Disorder 1
Multiplex Juggling and its Connection to Kostant's Partition Function, Part 2

Summer Fun Problem Sets:
Wythoff's Game, Parting Ways,
All About Randomness
Notes from the Club

From the Founder

What is the essence of math? Is it the legacy of theorems left by mathematicians? Or, is it the process by which those theorems were created? Whatever the answer, it is my hope that more time is spent on the latter, which has been neglected. - Ken Fan, President and Founder

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Just a Crumple?* by C. Kenneth Fan. For more on crumpled paper, see *Order in Disorder 1* on page 8.

An Interview with Petronela Radu

I was always surrounded by students and colleagues that seemed smarter and quicker than me, but I really enjoyed math and did not want (or know) how to give up.

Petronela Radu is Olson Professor of Mathematics at the University of Nebraska-Lincoln and is the Undergraduate Chair of her department. She earned her doctoral degree in mathematics from Carnegie Mellon University under the supervision of Luc Charles Tartar. She was raised in Romania.

Ken: I'd like to begin with math. You're an expert on partial differential equations and integral equations. To me, those topics have always seemed unapproachably difficult because it's easy to write down simple equations that seem impossible to solve. So there are the equations that are taught in a course, such as the wave equation or Schroedinger's equation for a hydrogen atom, but after that, it seems like you are suddenly confronted with scaling a cliff. Would you be able to say something to sort of "bridge the gap" between a student's understanding of such equations and the professional understanding? Would you describe how your thinking about such equations evolved as you went from student to professional?

Petronela: The fields of mathematics in which I work are connected to laws of motion, physical processes. Let's think about Newton's second law that simply says that the force acting on a body is equal to its mass times the acceleration. In the simplest case, these are all constant, but what if the force changes from point to point, or from one second to the next? Then we introduce functions of multiple variables, with space and time as inputs. For these functions we measure the rates of change (i.e. their derivatives, as we learn in Calculus). Partial differential equations arise when we know the derivatives, or a relationship among them, and we need to find the function. Complex relationships between derivatives could lead to very difficult problems. A famous example is one of the Clay Institute Millennium problems (for which there is a million dollar prize!) which asks to prove (or disprove) if a "nice" solution exists for the Navier-Stokes system of equations which models the dynamics of fluids. Solving such equations does not mean that you will be able to give an exact formula, like

$$u(x, y) = \sin(x + y),$$

but rather you want to show that a solution exists. It is similar to having the equation

$$x^3 + 2x = 0,$$

and you are not interested in knowing the exact values, but rather you want to know if there exist such values (and how many). Or more interestingly, take the equation

$$\cos x = x,$$

which cannot be solved with elementary functions, so we can only show that such a value x exists, but cannot write it explicitly. For algebraic equations, Calculus is a very powerful tool for showing existence of solutions; for partial differential equations a

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For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Petronela Radu and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
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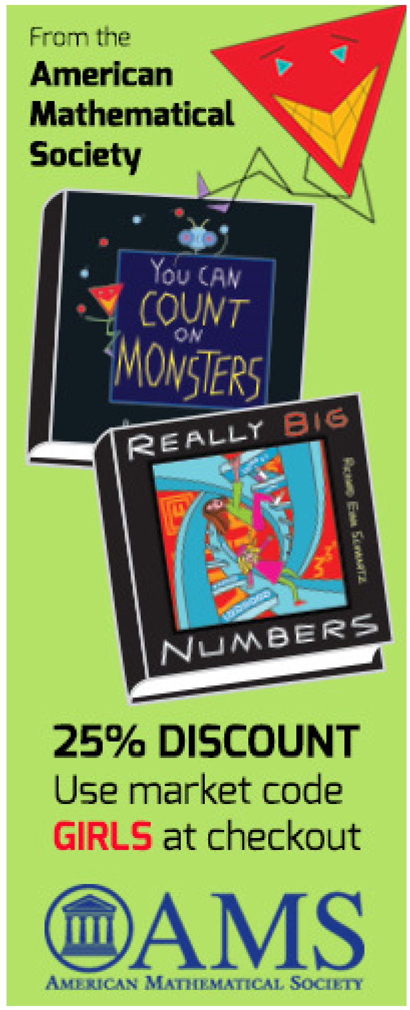

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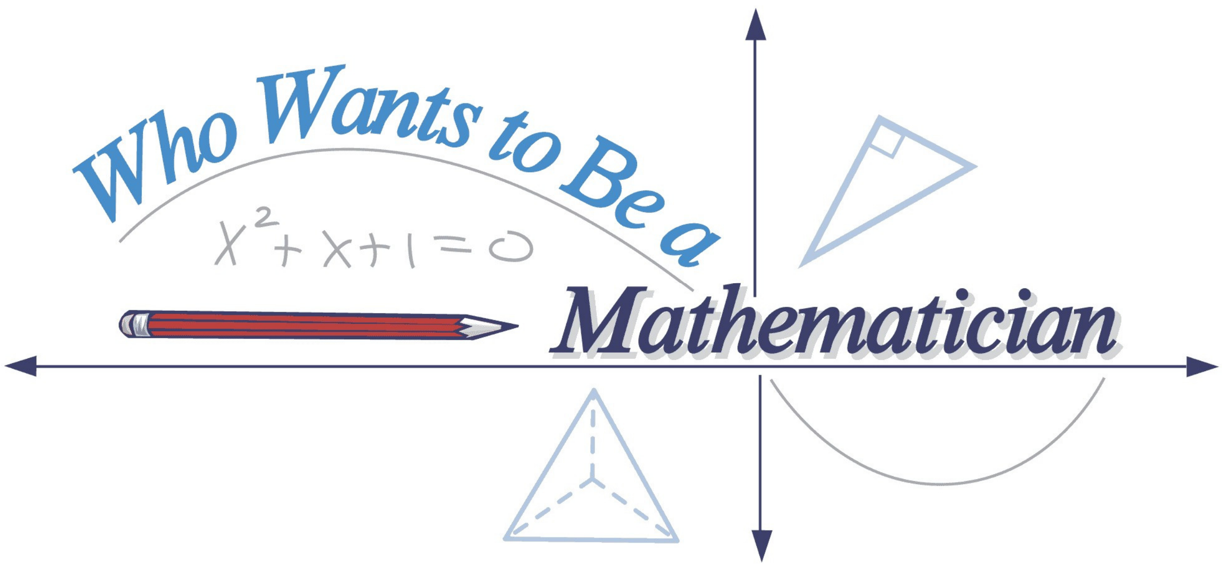

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Order In Disorder 1:

Uncovering the Mathematical Rules of Crumpling

by Jovana Andrejevic[1] | edited by Amanda Galtman

Take a sheet of paper and crumple it into a ball. Then, carefully unfold and flatten it. Inscribed in the sheet is a complex network of creases where stresses sharply focused during compaction, introducing permanent, plastic deformation. Now, crumple your sheet of paper again and unfold it. New creases have developed, but you might expect that the length of added creases is not as great as in the first iteration, since the sheet could exploit some of its existing folds during re-crumpling.

The way in which materials wear and accumulate damage under repeated loading is a common question of interest for a variety of materials: How do the fibers in our clothing relax over many cycles of stretching? How do the blades of a wind turbine wear out due to sustained vibration from the wind? A thin sheet – whether it's paper, aluminum, or plastic – likewise incurs incremental damage as it is crumpled over and over again. It leaves scientists to wonder: Is there a rule for how such damage accumulates?

Surprising mathematical order to crumpling

The pattern of creases that develops as a sheet is crumpled appears quite disordered and chaotic at first glance. However, in 2018, researchers revealed that there is indeed some mathematical order to crumpling [1]. The experimental research group of Prof. Shmuel Rubinstein conducted studies of systematic crumpling using Mylar, a thin polyester film that crumples similarly to paper. The experimenters placed a rolled sheet of Mylar into a cylindrical container and compressed the sheet several times with a piston to a fixed compaction ratio. The compaction ratio, denoted $\tilde{\Delta}$, is the ratio of the final to initial height of the sheet. Between successive compressions, the sheet was removed from the container, gently flattened, and scanned with a **profilometer**, an instrument that measures the height profile of the sheet. Individual creases were visualized by calculating the curvature of the sheet from the height map and coloring the creases blue or red, depending on whether they pucker outward (mountain creases) or inward (valley creases), respectively. The experimental procedure is outlined in Figure 1.

Reminiscent of busy roadmaps, the crease patterns after each crumple were analyzed using image analysis techniques to measure the total crease length, or *mileage*: the distance traversed by tracing all the creases in the sheet. The researchers found that the mileage, l, of a single sheet followed a logarithm in n, the number of times crumpled, according to the following formula:

$$l(n) = c_1(1-\tilde{\Delta})\log(1+c_2\frac{n}{\tilde{\Delta}}) \qquad (1)$$

In this equation, c_1 and c_2 are fitting parameters, and log refers to the natural logarithm. (The base of the logarithm is not critical, however, as it can be changed to base 10 with an adjustment factor of $1/\log_{10} e$ that can be absorbed in the parameter c_1.) This result was

[1] This work is supported in part by a grant from MathWorks.

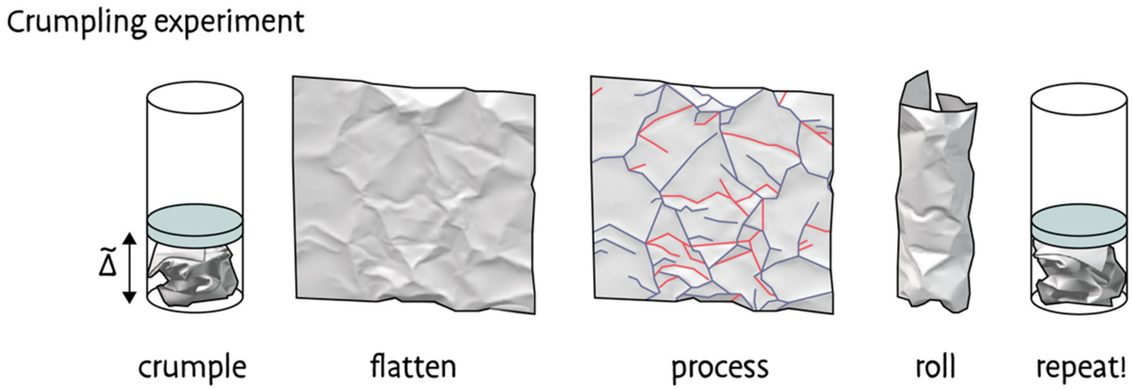


Figure 1: **Systematic crumpling experiment.** A sheet of Mylar is placed inside a cylindrical container and compressed to a target compaction ratio Δ, then carefully flattened and scanned with a profilometer for image processing. The sheet is returned to the container and the process repeated a number n times.

highly robust over several different sheets and compaction ratios, despite the high variation in the spatial arrangement of creases from sample to sample.

Logarithms

Logarithms emerge in many real-world applications. Often, they are used to map measured quantities that vary widely in orders of magnitude to a more practical range. For example, the intensity I of a sound source, which can vary from 10^{-10} W/m^2 (a whisper) to 10^{-1} W/m^2 (a rock concert) can be more conveniently expressed in decibels [2]. The decibel level is given by $10 \log_{10}(I/I_0)$, with I_0 the lowest threshold of human hearing. The acidity of aqueous solutions can be characterized by the pH, given by the negative logarithm of the Hydrogen ion concentration [3]: pH = $-\log_{10}$ [H^+]. The key idea is that large changes in the input of the logarithm have diminishing effect on the output. Computer scientists are pleased when the execution time of their algorithm scales as $\log N$, where N might be the length of a vector or the size of a matrix, because it means that scaling up the system size will incur smaller penalties in the required time. Since the input and output of a logarithmic function vary on widely different scales, plots often use logarithmic axes for the input variable, to resolve the small changes that happen at large input values, as shown in Figure 2.

To understand the significance of a logarithm in the context of crumpling, the researchers probed their model to extract the predicted *change* in mileage, δl, over one crumple, $\delta n = 1$. To estimate δl, they calculated the derivative of l in Equation 1 with respect to n:

$$\delta l \approx \frac{\partial l}{\partial n} = \frac{c_1 c_2 (1-\tilde{\Delta})}{\tilde{\Delta}(1+c_2 \frac{n}{\tilde{\Delta}})}.$$

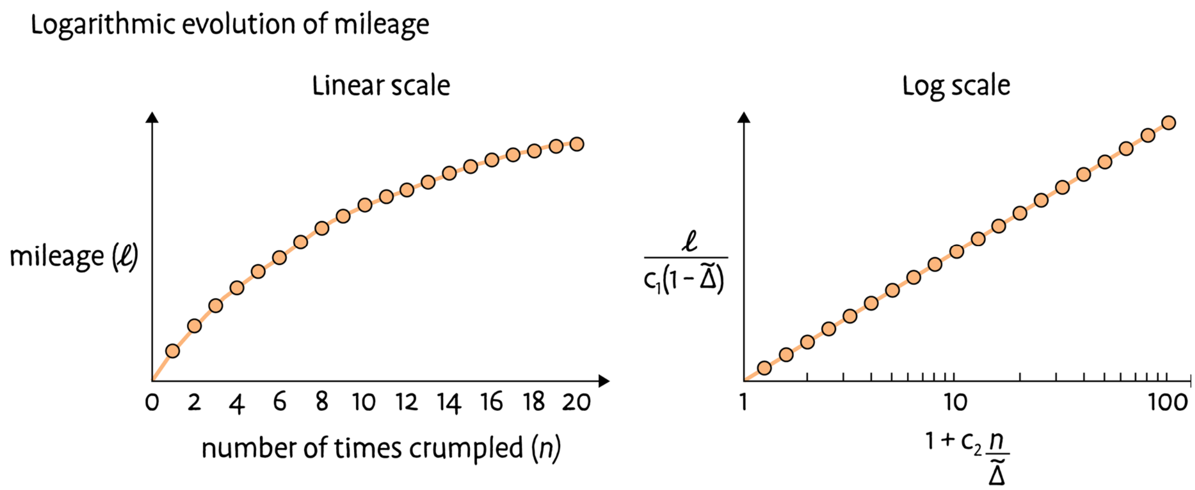


Figure 2: **Logarithmic evolution of mileage.** On the left, a sample curve modeled by Equation 1 is plotted on a linear scale in the number of crumples, n. On the right, the equation is rearranged to the form $y = \log x$ and plotted using a log scale on the x-axis. The relationship between input and output now appears as a straight line.

The right-hand side expression may be slightly reworked by recognizing that

$$\frac{1}{1+c_2\frac{n}{\tilde{\Delta}}} = e^{-l/c_1(1-\tilde{\Delta})},$$

which modifies the equation to

$$\delta l \approx c_1 c_2 \frac{1-\tilde{\Delta}}{\tilde{\Delta}} e^{-\frac{l}{c_1(1-\tilde{\Delta})}}. \tag{2}$$

An interesting consequence of this result is its prediction that damage accumulation is **memoryless**: The added crease length over one crumple depends only on the current total crease length, l, and the compaction ratio used during this iteration, $\tilde{\Delta}$, not on the prior history of crumples performed on the sheet. To test this prediction, the researchers performed some additional experiments, as follows.

Suppose you crumple a single sheet 24 times to a compaction ratio $\tilde{\Delta} = 0.63$. Your friend instead crumples a sheet just two times, but with a stronger compaction $\tilde{\Delta} = 0.18$, to a smaller final height. Upon visual inspection, the crease patterns likely appear quite different, but surprisingly have comparable mileage, as shown in Figure 3. Now, suppose another friend prepares their sheet entirely differently by pre-folding an ordered pattern that happens to have the same mileage as your crumpled sheet. You now have three distinct crease patterns with different crumpling histories, but roughly the same mileage. The equation for δl predicts that if each sheet is now crumpled once with the same compaction ratio, say $\tilde{\Delta} = 0.45$, the added damage to each will be the same, as all three sheets have the same mileage. Indeed, the researchers' experiments confirmed this surprising outcome. In other words, mileage accumulation seems to be independent of the spatial details of crease networks!

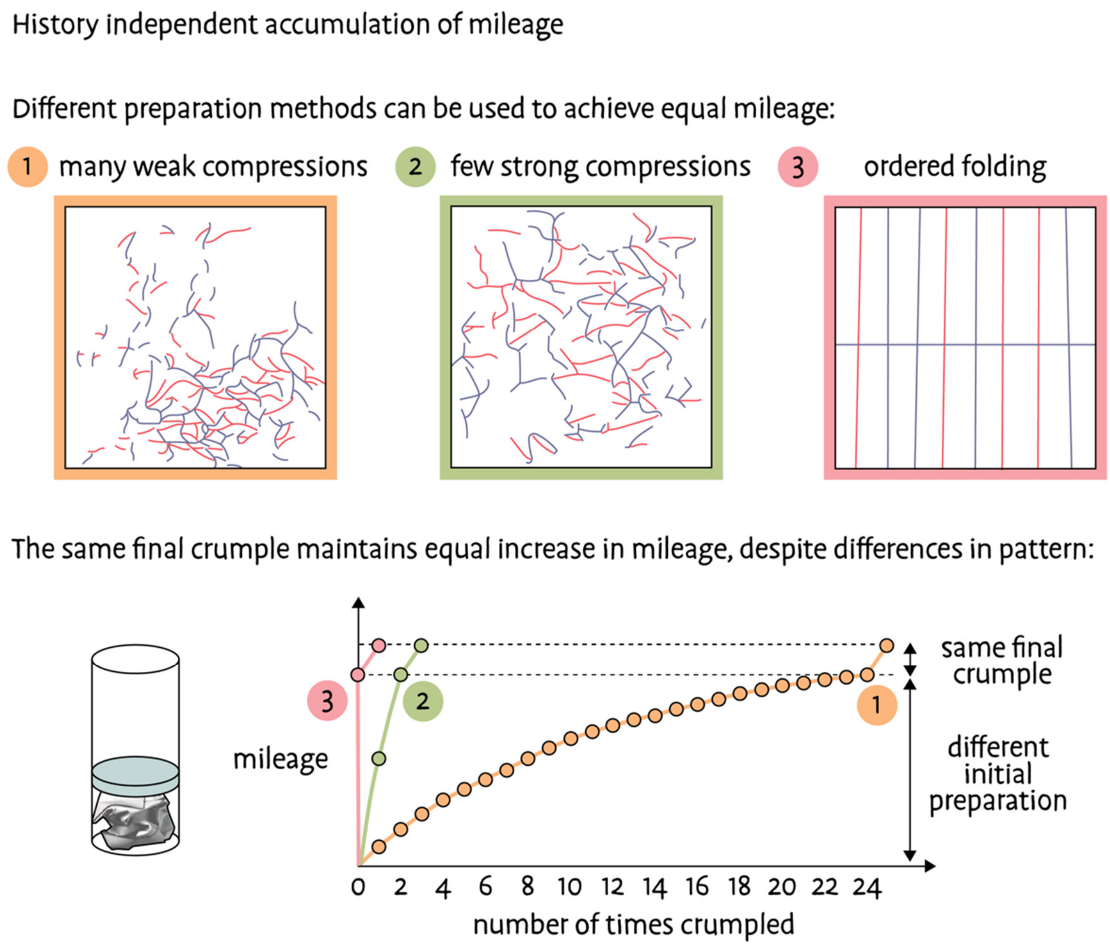


Figure 3: **Damage accumulation is history independent.** Different preparation methods, such as crumpling weakly many times, crumpling strongly a few times, or folding in an ordered manner, can produce visually distinct crease patterns with comparable mileage. When these sheets are crumpled once in the same way, they will accumulate the same damage, regardless of the differences in the spatial arrangement of creases across the samples.

References

[1] Gottesman, O., Andrejevic, J., Rycroft, C. H., and Rubinstein, S. M. (2018). A state variable for crumpled thin sheets. *Communications Physics*, 1(1), 1-7.

[2] Intensity and the decibel scale. Available at https://www.physicsclassroom.com/class/sound/Lesson-2/Intensity-and-the-Decibel-Scale Accessed 05/06/2021.

[3] pH, pOH, and the pH scale. Available at https://www.khanacademy.org/science/ap-chemistry/acids-and-bases-ap/acids-bases-and-ph-ap/a/ph-poh-and-the-ph-scale Accessed 05/06/2021.

Multiplex Juggling and its Connection to Kostant's Partition Function, Part 2

by Pamela E. Harris and Maria Rodriguez Hertz[2] | edited by Jennifer Sidney Silva

In Part 1, we explained mathematical juggling and Kostant's partition function. We left you with a tantalizing connection between the two, which we repeat here for your convenience:

Theorem 1 (Corollary 3.8 of [10]). For any n and r nonnegative integers

$$\text{mjs}(\langle n\rangle, \langle n\rangle, r, n) = \wp(n(\alpha_1 + \alpha_2 + \cdots + \alpha_r)). \tag{1}$$

Here, we explain the connection.

The key insight in establishing Theorem 1 is keeping track of how each ball is thrown in the juggling sequence that encodes the parts we use in the partition. That is, whenever we throw a ball at time i to height j, this corresponds to using the coin $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j-1}$. On the other hand, if the partition uses the part $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j-1}$, this corresponds to a throw of a ball at time i to height j in the juggling sequence.

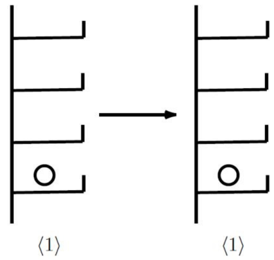
To simplify things, we explain Equation 1 in the case that $r = 3$ while n is a nonnegative integer. This means that $\wp(n\alpha_1 + n\alpha_2 + n\alpha_3))$ is the number of ways we can add multiples of α_1, α_2, α_3, $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, and $\alpha_1 + \alpha_2 + \alpha_3$ to get $n\alpha_1 + n\alpha_2 + n\alpha_3$. Let's simplify further and consider the case where $n = 1$. From Example 3 we know that there are four partitions of $\alpha_1 + \alpha_2 + \alpha_3$, one of which is given by $1(\alpha_1) + 1(\alpha_2) + 1(\alpha_3)$. We now decode these parts, as they have encoded the throws in a juggling sequence as follows: the part α_1 is a throw of a ball at time 1 to height 1, the part α_2 is a throw of a ball at time 2 to height 1, and the part α_3 is a throw of a ball at time 3 to height 1. This juggling sequence is illustrated in Figure 1c; note that based on these throws we do end up creating a juggling sequence in the set MJS($\langle 1\rangle$, $\langle 1\rangle$, 3, 1).

The other three partitions from Example 3 correspond to the following juggling sequences in MJS($\langle 1\rangle$, $\langle 1\rangle$, 3, 1):

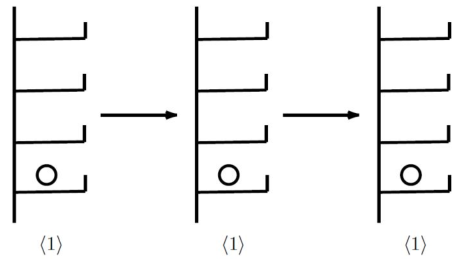
- $1(\alpha_1 + \alpha_2) + 1(\alpha_3)$ corresponds to the juggling sequence in MJS($\langle 1\rangle$, $\langle 1\rangle$, 3, 1) where we throw a ball at time 1 to height 2 and we throw a ball at time 3 to height 1,
- $1(\alpha_1) + 1(\alpha_2 + \alpha_3)$ corresponds to the juggling sequence in MJS($\langle 1\rangle$, $\langle 1\rangle$, 3, 1) where we throw a ball at time 1 to height 1 and we throw a ball at time 2 to height 2, and
- $1(\alpha_1 + \alpha_2 + \alpha_3)$ corresponds to the juggling sequence in MJS($\langle 1\rangle$, $\langle 1\rangle$, 3, 1) where we throw a ball at time 1 to height 3.

These juggling sequences are illustrated in Figure 2.

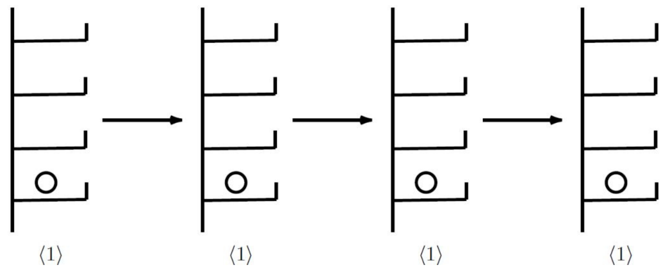
[2] This work is supported in part by a grant from MathWorks.



(a) An element of MJS(⟨1⟩, ⟨1⟩, 3, 1) at time 1

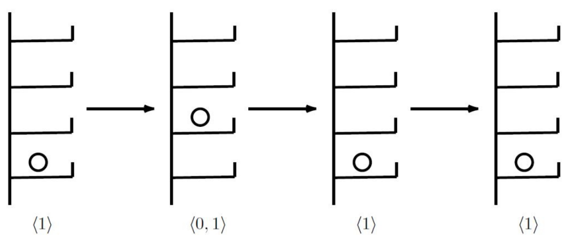


(b) An element of MJS(⟨1⟩, ⟨1⟩, 3, 1) at time 2

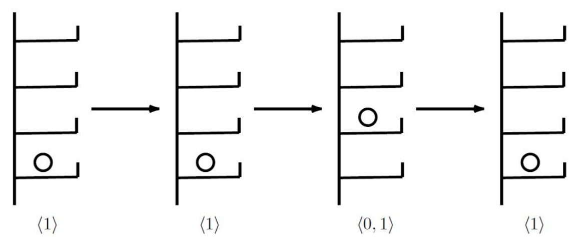


(c) The element of MJS(⟨1⟩, ⟨1⟩, 3, 1) that corresponds with $1(\alpha_1) + 1(\alpha_2) + 1(\alpha_3)$

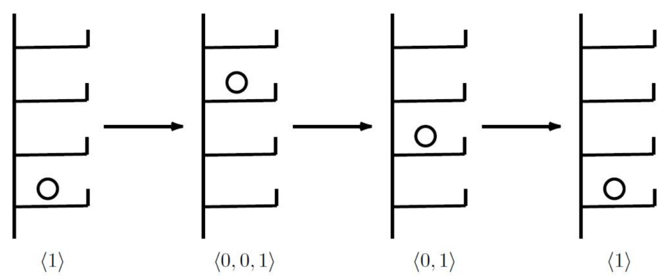
Figure 1. Constructing the element of MJS(⟨1⟩, ⟨1⟩, 3, 1) that corresponds with $1(\alpha_1) + 1(\alpha_2) + 1(\alpha_3)$.



(a) The element of MJS(⟨1⟩, ⟨1⟩, 3, 1) that corresponds with $1(\alpha_1 + \alpha_2) + 1(\alpha_3)$



(b) The element of MJS(⟨1⟩, ⟨1⟩, 3, 1) that corresponds with $1(\alpha_1) + 1(\alpha_2 + \alpha_3)$



(c) The element of MJS(⟨1⟩, ⟨1⟩, 3, 1) that corresponds with $1(\alpha_1 + \alpha_2 + \alpha_3)$

Figure 2. The rest of the elements of MJS(⟨1⟩, ⟨1⟩, 3, 1).

Let's also do a slightly different example to further illustrate the key insight in the result. Now consider the case in which $n = r = 2$ where we are computing partitions of $2\alpha_1 + 2\alpha_2$ using the parts α_1, α_2, and $\alpha_1 + \alpha_2$. In this case, we'd have the starting and ending states of our juggling sequences as ⟨2⟩, the time is set to 2, and the hand capacity is 2 as well. This time, Theorem 2 asserts that

$$\mathrm{mjs}(\langle 2\rangle, \langle 2\rangle, 2, 2) = \wp(2\alpha_1 + 2\alpha_2). \tag{2}$$

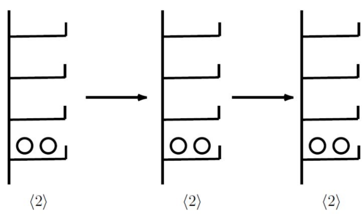
Here there are three ways to write $2\alpha_1 + 2\alpha_2$, which are:

$$2(\alpha_1) + 2(\alpha_2),$$
$$1(\alpha_1) + 1(\alpha_2) + 1(\alpha_1 + \alpha_2), \text{ and}$$
$$2(\alpha_2 + \alpha_3).$$

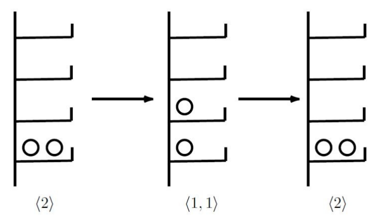
We now convert each of these partitions into juggling sequences as we did above. We find that

- $2(\alpha_1) + 2(\alpha_2)$ corresponds to the juggling sequence in MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2) where we throw two balls at time 1 to height 1 and we throw two balls at time 2 to height 1,
- $1(\alpha_1) + 1(\alpha_2) + 1(\alpha_1 + \alpha_2)$ corresponds to the juggling sequence in MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2) where we throw a ball at time 1 to height 1, we throw a ball at time 2 to height 1, and we throw a ball at time 1 to height 2, and
- $2(\alpha_2 + \alpha_3)$ corresponds to the juggling sequence in MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2) where we throw two balls at time 1 to height 2.

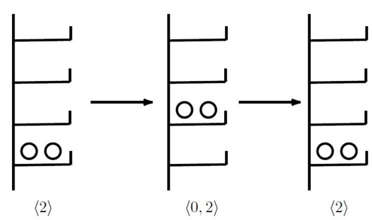
These juggling sequences are illustrated in Figure 3.



(a) The element of MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2) that corresponds with $2(\alpha_1) + 2(\alpha_2)$



(b) The element of MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2) that corresponds with $1(\alpha_1) + 1(\alpha_2) + 1(\alpha_1 + \alpha_2)$



(c) The element of MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2) that corresponds with $2(\alpha_1 + \alpha_2)$

Figure 3. All the elements of MJS($\langle 2\rangle$, $\langle 2\rangle$, 2, 2).

In order to verify that the two sides of Equation 1 are equal, though, we also have to check that if we start with the juggling sequences, we can get back all of the partitions. We want to make sure that we don't have two of the juggling sequences sending us back to the same partition. So we can go back to our original example, the one in Figures 8 and 9. However, this time, we are going to start with the juggling sequences and create the partitions.

To do this, recall that a throw at time i to height j corresponds to a part $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j-1}$. Whenever we throw a ball at time i to height $j = 1$, it corresponds to a part in the partition of the form α_i, since $i = i + 1 - 1 = i + j - 1$. First, let's start with the juggling sequence in Figure 1c. Notice that at time 1 we throw a ball to height 1. This gives us a part of the form α_1. You'll notice that at time 2 we also throw a ball up to height 1. Then, again, this gives us α_2. Lastly, we throw a ball to height 1 at time 3, giving us α_3. We can put it all together and get $(\alpha_1) + (\alpha_2) + (\alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$, which is what we started with.

We can keep doing this with the juggling sequence in Figure 2a. In this sequence, we throw the ball to height 2 at time 1, and then to height 1 at time 3. This gives the parts $\alpha_1 + \alpha_2$ and α_3, respectively. Note that the sum of these parts is again $\alpha_1 + \alpha_2 + \alpha_3$ as claimed. Then, in Figure 2b, we throw the ball to height 1 at time 1, and to height 2 at time 2. This give us α_1 and $\alpha_2 + \alpha_3$, respectively, and again their sum is $\alpha_1 + \alpha_2 + \alpha_3$. Finally, we can look at Figure 2c, where we throw the ball up to height 3 at time 1. This gives the part $\alpha_1 + \alpha_2 + \alpha_3$, which is also $\alpha_1 + \alpha_2 + \alpha_3$. So, for this example, we have shown that

$$\mathrm{mjs}(\langle 1\rangle, \langle 1\rangle, 3, 1) = \wp(\alpha_1 + \alpha_2 + \alpha_3).$$

We encourage you to follow what we did for this example and consider other values of n and r to check that Equation 1 holds as well. For the proof of the general case, see [10]. Happy juggling… or partitioning… of vectors!

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[10] C. Benedetti, C. R. H. Hanusa, P. E. Harris, A. Morales, and A. Simpson, Kostant's partition function and magic multiplex juggling sequences, *Annals of Combinatorics* **24** (2020), no. 3, 439 – 473. Preprint arXiv:2001.03219.

Appendix: Solutions to exercises.

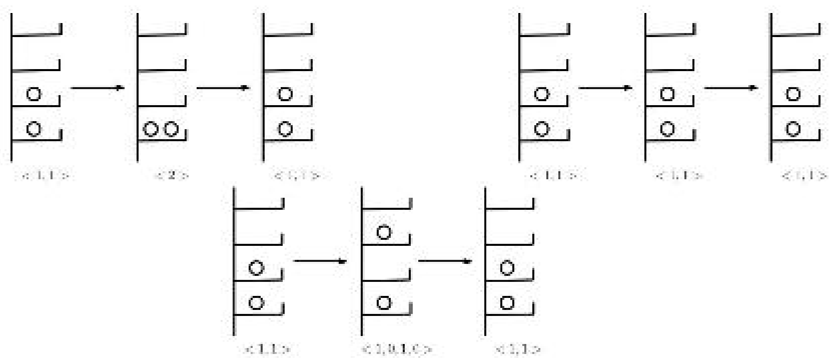


Figure 4. The elements of MJS($\langle 1, 1\rangle$, $\langle 1, 1\rangle$, 2, 2). (Solution to the question at the end of Example 2 of Part 1.)

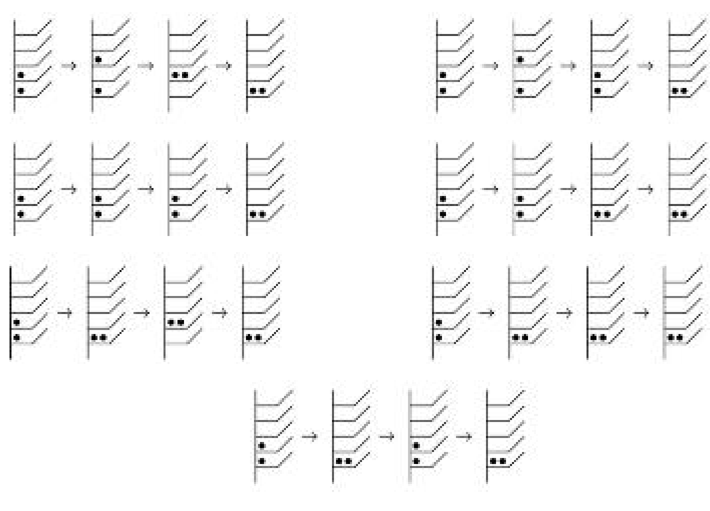


Figure 5. The elements of MJS($\langle 1, 1\rangle$, $\langle 2\rangle$, 3, 2). (Solution to Exercise 1 from Part 1.)

Summer Fun!

The best way to learn math is to do math. Here are the 2021 Summer Fun problem sets.

We invite all members and subscribers to send any questions and solutions to us at girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems on your own.

Some problems are very challenging and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

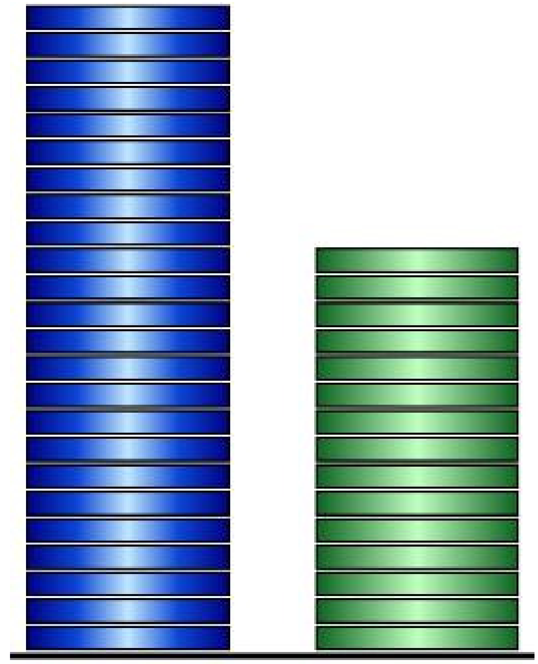
If you're used to solving problems fast, it can be frustrating to work on problems that take weeks to solve. Try to enjoy the journey and don't be afraid to follow detours. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Wythoff's Game

by Laura Pierson



(24, 15)

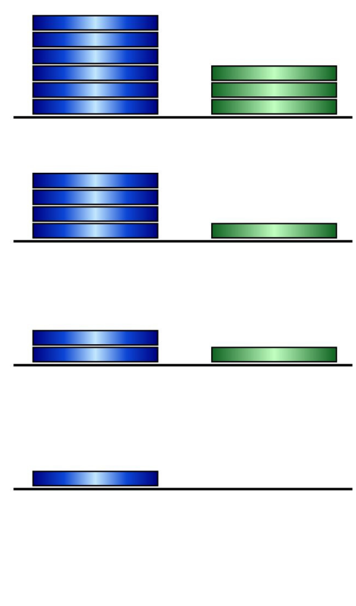
The Rules

In this game, there are two players who take turns making moves and two piles of objects. We will write the pile sizes using an ordered pair, e.g., (3, 5) means the first pile has size 3 and the second pile has size 5. On each turn, a player has two choices:

1. Take some number of items from **one pile**.
2. Take the **same number** from **both piles**.

Passing is not allowed. The player who makes the last move wins.

For instance, a possible game starting from (6, 3) could be:



1. The first player takes 2 from each pile, leaving (4, 1).
2. The second player takes 2 from the pile with 4, leaving (2, 1).
3. The first player takes 1 from each pile, leaving (1, 0)
4. The second player takes the remaining 1 object and wins.

Winning and Losing Positions

We will call a pair of pile sizes a **winning position** if the first player has a winning strategy from that position, and a **losing position** if the second player has a winning strategy. A **winning strategy** means that player can win regardless of what their opponent does.

1. Play some sample games by yourself or with a friend to get a feel for how it works. Convince yourself that every position is either a winning position or a losing position, and (a, b) is a winning position if and only if (b, a) is.

2. Show that (n, n) and $(0, n)$ are winning positions for every n.

3. Show that (1, 2) is a losing position.

4. Show that $(1, n)$ is a winning position for $n > 2$, and $(2, n)$ and $(n, n + 1)$ are winning positions for $n > 1$.

5. Show that (3, 5) is a losing position.

6. Show that $(3, n)$ is a winning position for $n > 5$, $(5, n)$ is a winning position for $n > 3$, and $(n, n + 2)$ is a winning position for $n > 3$.

7. Show that if there is **any way** to get to a losing position from (m, n), then (m, n) is a winning position.

8. Show that if there is **no way** to get to a losing position from (m, n), then (m, n) is a losing position.

9. Show that (4, 7) and (6, 10) are losing positions. Use the ideas in Problem 4 and Problem 6 to list some other winning positions.

10. Show that among the losing position (m, n) with $m < n$, each difference $n – m$ shows up exactly once, and each positive integer shows up exactly once as either an m or n.

11. Use the results of the previous problem to list the next few losing positions.

The Golden Ratio

The **golden ratio** φ (the Greek letter "phi") is a famous mathematical constant that you may have heard of before. It is connected to the Fibonacci sequence and regular pentagons, and even comes up in nature!

Summer Fun!

One way to define it is that if you take a $1 \times \varphi$ rectangle and cut out a 1×1 square, the remaining $\varphi - 1 \times 1$ rectangle is similar to the big one, that is, $\varphi - 1 : 1 = 1 : \varphi$.

12. Show that the golden ratio satisfies the quadratic equation $\varphi^2 - \varphi - 1 = 0$.

13. Use the quadratic formula to find a formula for φ and show that $\varphi \approx 1.618\ldots$.

Beatty Sequences

The **floor** of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x. For example, $\lfloor 2 \rfloor = 2$, $\lfloor 4.5 \rfloor = 4$, and $\lfloor \pi \rfloor = 3$.

The **Beatty sequence** for a positive irrational number r is the sequence $\lfloor r \rfloor$, $\lfloor 2r \rfloor$, $\lfloor 3r \rfloor$, $\lfloor 4r \rfloor$, …. For instance, if $r = \sqrt{2}$, the first few terms are $\lfloor \sqrt{2} \rfloor = \lfloor 1.41\ldots \rfloor = 1$, $\lfloor 2\sqrt{2} \rfloor = \lfloor 2.82\ldots \rfloor = 2$, $\lfloor 3\sqrt{2} \rfloor = \lfloor 4.24\ldots \rfloor = 4$.

If r and s are positive irrational numbers with $1/r + 1/s = 1$, the sequences $\lfloor r \rfloor$, $\lfloor 2r \rfloor$, $\lfloor 3r \rfloor$, $\lfloor 4r \rfloor$, … and $\lfloor s \rfloor$, $\lfloor 2s \rfloor$, $\lfloor 3s \rfloor$, $\lfloor 4s \rfloor$, … are called **complementary Beatty sequences**.

14. Suppose $\lfloor r \rfloor$, $\lfloor 2r \rfloor$, $\lfloor 3r \rfloor$, $\lfloor 4r \rfloor$, … and $\lfloor s \rfloor$, $\lfloor 2s \rfloor$, $\lfloor 3s \rfloor$, $\lfloor 4s \rfloor$, … are complementary Beatty sequences. Show that every positive integer shows up in **exactly one** of the two sequences.

15. Show that $1/\varphi + 1/\varphi^2 = 1$, so that the sequences $\lfloor \varphi \rfloor$, $\lfloor 2\varphi \rfloor$, $\lfloor 3\varphi \rfloor$, $\lfloor 4\varphi \rfloor$, … and $\lfloor \varphi^2 \rfloor$, $\lfloor 2\varphi^2 \rfloor$, $\lfloor 3\varphi^2 \rfloor$, $\lfloor 4\varphi^2 \rfloor$, … are complementary Beatty sequences.

16. Compute the first few terms of the Beatty sequences $\lfloor \varphi \rfloor$, $\lfloor 2\varphi \rfloor$, $\lfloor 3\varphi \rfloor$, $\lfloor 4\varphi \rfloor$, … and $\lfloor \varphi^2 \rfloor$, $\lfloor 2\varphi^2 \rfloor$, $\lfloor 3\varphi^2 \rfloor$, $\lfloor 4\varphi^2 \rfloor$, …. How do they correspond to the entries of the losing positions in Wythoff's game?

17. Show that $\lfloor n\varphi^2 \rfloor - \lfloor n\varphi \rfloor = n$.

18. Use Problem 10 and Problems 14-17 to show that the losing positions of Wythoff's game are precisely the pairs $(\lfloor n\varphi \rfloor, \lfloor n\varphi^2 \rfloor)$ and $(\lfloor m\varphi^2 \rfloor, \lfloor m\varphi \rfloor)$, for positive integers n and m.

Summer Fun!

Parting Ways

An Exploration of Fibonacci Partitions
by AnaMaria Perez and Josh Josephy-Zack

1. You can express positive integers as the sum of other positive integers in many ways. For example, we can express 4 as the sum of positive integers in 5 different ways, listed below:

$1 + 1 + 1 + 1$ $\quad$ $1 + 1 + 2$ $\quad$ $1 + 3$ $\quad$ $2 + 2$ $\quad$ 4

These sums are called **partitions**. Let us define a function $p(n)$ that takes a positive integer n as its input, and returns the number of partitions as its output. The first few values of $p(n)$ are calculated in the table below.

n	**Unique partitions of n**	$p(n)$
1	1	1
2	1 + 1, 2	2
3	1 + 1 + 1, 1 + 2, 3	3
4	1 + 1 + 1 + 1, 1+ 1+ 2, 1+ 3, 2 + 2, 4	5

Calculate $p(n)$ for $n = 5$, 6, 7, and 8.

2. Recall that the Fibonacci sequence is the sequence of numbers that begins with two 1's, then each successive term is obtained by adding the previous two terms. Let f_n be the Fibonacci sequence so that $f_1 = f_2 = 1$ and, for $n > 1$, we have $f_{n+1} = f_n + f_{n-1}$.

A. Calculate the first 12 terms of the Fibonacci sequence.

B. Notice that $f_{n-1} + f_{n-2}$ is a partition of f_n, for $n > 2$. How else could you partition a Fibonacci number using only Fibonacci numbers without repeats? We'll call such a partition a **Fibonacci partition**. (Note: Both f_1 and f_2 are 1, so 1 can appear up to two times in a Fibonacci partition.)

Example. The Fibonacci number 5 can be written 1 + 1 + 3, 2 + 3, or 5. Thus, 5 has 3 Fibonacci partitions. Note that we exclude 1 + 1 + 1 + 1 + 1 and 1 + 2 + 2 since these include repeated numbers, but 1 + 1 + 3 is fine since we're allowed to use 1 up to two times. Further, we exclude partitions such as 1 + 4 since 4 is not a Fibonacci number.

Write the Fibonacci partitions for the first 12 Fibonacci numbers.

C. What patterns do you notice? Write down your best guess as to what the pattern might be. Try to write the pattern as a recursive formula; we will return to this pattern later in the problem set.

Summer Fun!

3. As we've seen so far, Fibonacci partitions can be made by applying the definition of a Fibonacci number to generate a sum. Can we write other numbers as a Fibonacci partition? Sure! For example, we can write 7 as 5 + 2.

A. Write the numbers 1-12 as a Fibonacci partition. Can you do it for all of them?

B. Suppose you can write all the numbers less than a specific Fibonacci number f_n as a Fibonacci partition. Can you show that you can then write all the numbers less than f_{n+1} as a Fibonacci partition? Why?

C. What does this mean in general? Can all numbers be written as a Fibonacci partition? Prove that you can using induction.

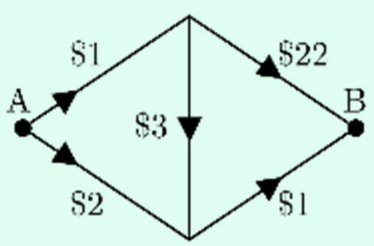
Proof by Induction

Suppose you want to prove a sequence of statements P_n, $n = 1, 2, 3, \ldots$. One way to do so is to use a basic induction argument, which proceeds as follows:

1. First, prove that P_1 is true. This is called the **base case**.
2. Next, assume that the statement P_n is true (or, that the statements P_k are true for all $k \leq n$). This is called the **inductive hypothesis**.
3. Finally, show that under the inductive hypothesis, P_{n+1} is true. This is called the **inductive step**.

For more, see *Mathematical Induction* on pages 7-10 of Volume 8, Number 5 of this Bulletin.

Greedy Algorithms



A greedy algorithm is an algorithm that chooses optimally at each step, but they do not always produce the best solution. For example, suppose you wish to go from point A to point B as cheaply as possible. If you use the greedy algorithm, you'd choose to pay the $1 toll, then the $3 toll, then the $1 toll again, costing a total of $5 instead of the $3 you would've spent had you taken the $2 path.

4. Let's consider Fibonacci partitions of positive integers.

A. Write out the Fibonacci partitions for a few (non-Fibonacci) numbers of your choice.

B. What method are you using to find the partitions?

C. Come up with an algorithm that could be used to find the shortest Fibonacci partition of a number (i.e., for 6, the Fibonacci partition 5 + 1 is a shorter than 3 + 2 + 1).

D. Does there always exist a partition that does not include consecutive Fibonacci numbers? If so, prove it. If not, give a contradiction.

5. Let's return to the idea alluded to in Problem 2C.

A. Prove the lemma given below.

> **Lemma 1**. For any Fibonacci number f_n, the following relation holds:
>
> $$f_n = 1 + \sum_{k=1}^{n-2} f_k .$$

B. Using your proof of Lemma 1 in Problem 5A, prove the recursive pattern you found back in Problem 2C.

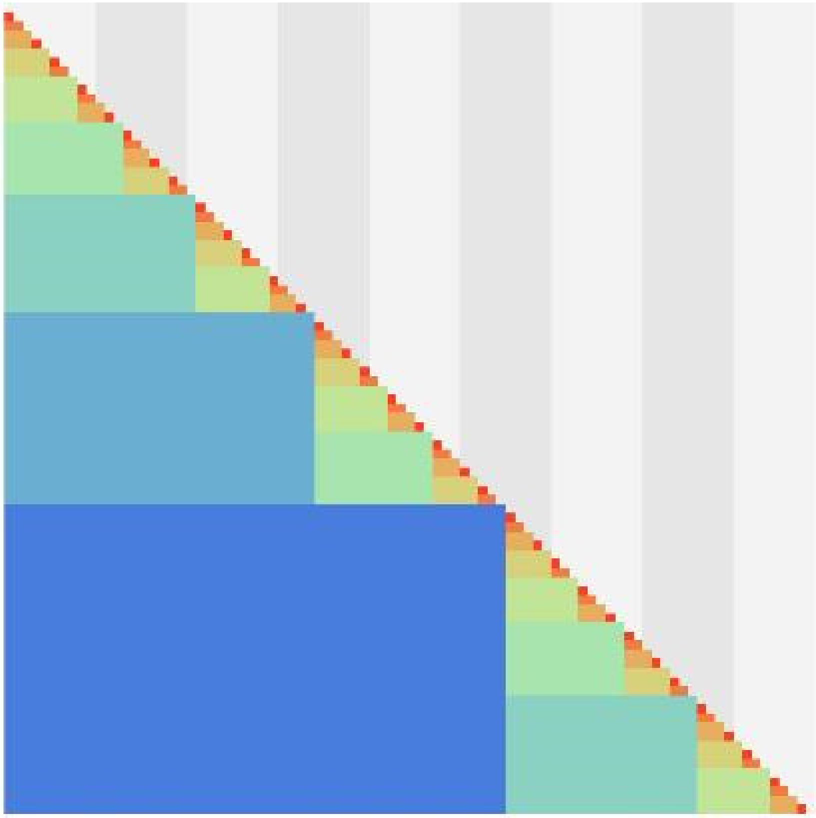


Image created by Nomen4Omen courtesy of commons.wikimedia.org. See https://commons.wikimedia.org/wiki/File:Zeckendorf_representations_89px.svg

Figure 1. Fibonacci integer staircase. This is a visual representation of Zeckendorf's theorem! Zeckendorf's theorem is about the Fibonacci partitions of the positive integers and how they can be built uniquely without consecutive Fibonacci numbers.

All About Randomness

by Fan Wei | edited by Jennifer Sidney Silva

Preparation

A random variable denotes the *numerical* outcome of a random event. Since the event is random, the outcomes are associated with some likelihood, which we call a probability distribution. For example, if we flip a coin, we can define a random variable X, where $X = 1$ if heads comes up and $X = 0$ if tails comes up. If the coin is fair, then the two outcomes $X = 1$ and $X = 0$ are equally likely, thus both have probability 0.5. This probability distribution is summarized in the table below.

X	0 (tails)	1 (heads)
probability	½	½

In general, for a random variable X with finitely many outcomes, there is a positive integer n such that the probability distribution of X looks like the following table:

X	c_1	c_2	…	c_{n-1}	c_n
probability	p_1	p_2	…	p_{n-1}	p_n

where the outcomes c_i occur with probability p_i. The probabilities p_i are all real numbers between 0 and 1, inclusive, and must add up to 1. Such a random variable is an example of a **discrete random variable** with finitely many outcomes.

1. Let D be the random variable that returns the outcome of the toss of a standard fair die. Write down the probability distribution of D.

Let's say we toss a coin twice, and we want to count the total number of heads. We could again define a random variable X which can take values 0, 1, or 2, denoting the number of heads in the two tosses, and we could write down the probability distribution. Alternatively, we can define two random variables X_1 and X_2, where $X_1 = 1$ denotes heads coming up on the first toss, $X_1 = 0$ denotes tails coming up on the first toss, and X_2 is similarly defined for the outcomes of the second toss. With these two random variables, we can create the random variable for the number of heads in two tosses by writing $X_1 + X_2$.

2. Write down the distribution of $X_1 + X_2$.

3. You have two standard six-sided fair dice. Both are tossed. Let D_1 be the number showing on top of the first die, and let D_2 be the number showing on top of the second die.
Write down the distribution of $D_1 + D_2$.

Summer Fun!

4. Let X be the number of cakes Karen eats every day and let Y be the number of cakes Susan eats every day. The following table shows the probability distribution for various outcomes of (X, Y). From this information, figure out the probability distribution of X.

(X, Y)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
probability	0.1	0.2	0	0.25	0.4	0.05

Continuous Random Variables

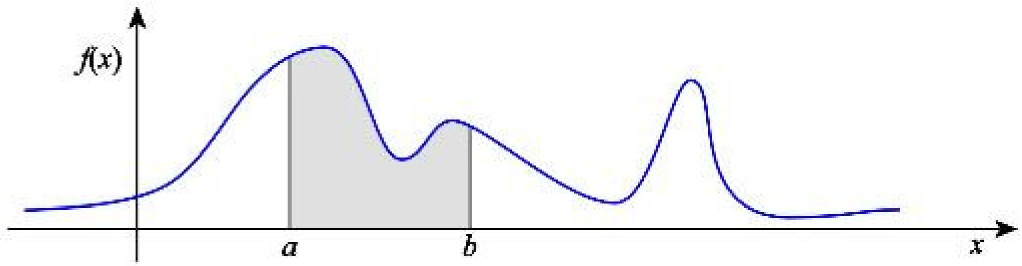


Figure 1. Graph of a probability density function for a continuous random variable X. The probability that $a < X < b$ is the area of the shaded region.

The outcomes of a continuous random variable X, in contrast to a discrete random variable, can be a continuum. Such a variable is associated with a probability density function $f(x)$. The probability that $a < X < b$ is given by the integral $\int_a^b f(x)dx$, as illustrated in Figure 1 above. The probability density $f(x)$ must satisfy $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. For example, generating a real number uniformly at random from the interval [0, 1] corresponds to a continuous random variable U with the probability density $f(x) = 1$ for $0 \leq x \leq 1$, and $f(x) = 0$ otherwise.

5. What is the probability that $1/2 < U < 2/3$? What is the probability that $U = 1/2$?

Expectation and Variance

Let X be a discrete random variable with outcomes c_i, for $i = 1, 2, 3, \ldots, n$ such that the probability of outcome c_i is p_i. The **expectation** $\boldsymbol{E[X]}$ (the expected value of X) is defined by the formula $E[X] = \sum_{i=1}^{n} p_i c_i$. In other words, the expectation of X is a weighted sum of the possible outcomes of X, where each outcome is weighted by its associated probability of occurring. The **variance** of X is defined as $\mathrm{Var}[X] = E[X^2] - E[X]^2$. (Here, X^2 is the random variable that has the outcome c_i^2 with probability p_i.) The variance gives a sense of how far X will be from $E[X]$. We'll explore these notions further in the problems below.

6. Find an expression for $\mathrm{Var}[X]$ in terms of the p_i and c_i.

7. Show that $\mathrm{Var}[X] = E[(X - E[X])^2]$.

8. The table below is the grade distribution of the midterm exam in Karen's math class.

Grade	$A = 5$	$B = 4$	$C = 3$	$D = 2$	$F = 0$
Percentage of Students	40%	45%	10%	0%	5%

Let X be the random variable denoting the grade of a random student we pick in Karen's class, with each student equally likely to be chosen. Compute $E[X]$ and Var$[X]$.

In Problem 8, the expectation of X is the same as the "average math midterm grade" among the students in Karen's class.

9. Let Y be the random variable denoting the grade of a random student we pick in Karen's class *if we only pick from among those students who got an A or B grade*, with each such student equally likely to be chosen. What is the distribution of Y? What is $E[Y]$ and Var$[Y]$?

10. For any exam, can you show that there must always be a student whose grade is at most the expected grade? In what situation can it be that all the students' grades are at least the expected grade?

For Problems 11-15, the discrete random variable X denotes how many carrots FluffyFur's rabbit friend Buffy will eat tomorrow.

11. Suppose the probability that Buffy will eat 2 carrots is 0.5, the probability that she will eat 3 carrots is 0.3, and the probability that she will eat 5 carrots is 0.2. What are $E[X]$ and Var$[X]$?

12. Suppose instead that the probability Buffy will eat 5 carrots is 1. What are $E[X]$ and Var$[X]$?

13. Suppose that the probability Buffy will eat 4 carrots is 0.5 and the probability of her eating 6 carrots is 0.5. What are $E[X]$ and Var$[X]$?

14. Suppose instead that the probability that Buffy will eat 2 carrots is 0.5 and the probability of her eating 8 carrots is 0.5. What are $E[X]$ and Var$[X]$?

15. Compare your answers to Problems 12, 13, and 14. What does this suggest to you about the meaning of Var$[X]$?

16. What can you say about a random variable X if Var$[X] = 0$?

17. A useful property of the expectation of random variables is its **linearity**, i.e., if X_1 and X_2 are random variables, then $E[X_1 + X_2] = E[X_1] + E[X_2]$. Prove this. (Can you prove that for n random variables $X_1, X_2, X_3, \ldots, X_n$, we have

$$E[X_1 + X_2 + X_3 + \ldots + X_n] = E[X_1] + E[X_2] + E[X_3] + \ldots + E[X_n]?)$$

For Problems 18 and 19, there are six students with unique names, and six envelopes each containing the name of a different student. The six envelopes cannot be distinguished without opening them. Prizes are given to those students who open an envelope that contains their name.

18. Suppose each student in turn picks one of the six envelopes (each with equal probability), opens and sees the name inside, and then returns the envelope as if it had not been opened before. What is the probability that exactly 3 students win a prize? What is the expected number of students to win a prize?

19. Suppose instead that the six envelopes are distributed to the six students uniformly at random. Now what's the probability that there are exactly 3 students who win prizes, and what's the expected total number of students who win the prizes?

20. How would you define $E[X]$ and $\mathrm{Var}[X]$ for a continuous random variable?

For a continuous random variable, we can define $E[X]$ to be $\int_{-\infty}^{\infty} xf(x)dx$.

21. Let X be a continuous random variable with a density function $f(x)$ which is 0 outside the interval [0, 1]. Let $g(x)$ be the probability that $X \geq x$. Show that $E[X] = \int_0^1 g(x)dx$.

Independent Random Variables

Roughly speaking, we say that two discrete random variables X_1 and X_2 are **independent** if their outcomes have no bearing on each other. Formally, X_1 and X_2 are independent if and only if the probability that $X_1 = a$ and $X_2 = b$ is equal to the product of the probabilities that $X_1 = a$ and $X_2 = b$, for all a and b. We say that three random variables X_1, X_2, and X_3 are **independent** if any two of them are independent *and* the probability that $X_1 = a$, $X_2 = b$, and $X_3 = c$ is equal to the product of the separate probabilities that $X_1 = a$, $X_2 = b$, and $X_3 = c$.

22. Suppose we know that X_1, X_2, and X_3 are independent. Show that they are pairwise independent, that is, any two are independent.

23. Suppose we know that X_1, X_2, and X_3 are pairwise independent. Can we conclude that X_1, X_2, and X_3 are independent? In general, does pairwise independence imply independence?

Here's the setup for Problems 24-31: There are n candies labeled 1 through n and k baskets labeled 1 through k. Each candy is placed into a random basket with each basket being an equally likely placement. Let B_i be the random variable that tells which basket candy i ends up in. Let N_i be the random variable which tells the number of candies in basket i.

24. What's the probability that there are exactly k candies in basket 1?

Summer Fun!

25. What is the expected number of candies in basket 1? Can you compute this expectation without using your answer to Problem 24?

26. Are B_1 and B_2 independent random variables? Are B_1, B_2, and B_3 mutually independent?

27. Are N_1 and N_2 independent random variables?

28. If $N_1 = n$, how will the probability distribution of N_2 change?

29. If $N_1 = 0$, how will the probability distribution of N_2 change?

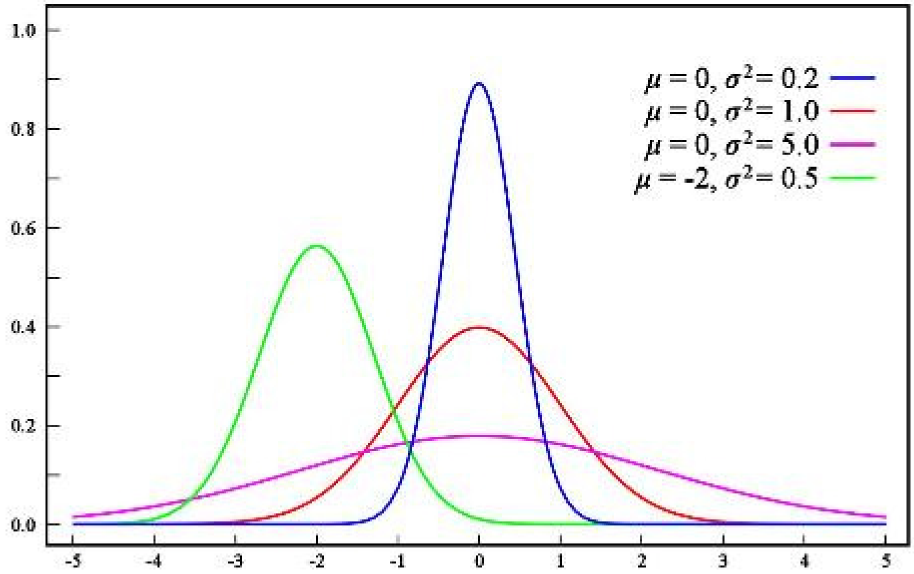
30. Suppose $k = 2$ and n is an even number. If you know that $N_1 = n/2$, how will that knowledge affect the probability distribution of N_2?

31. Let's focus on the probability distribution of N_1. As the number of candies n gets larger and larger, what does the distribution of N_1 begin to look like?

An important continuous random variable $X_{\mu,\sigma}$ has the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $E[X_{\mu,\sigma}] = \mu$ and $\mathrm{Var}(X_{\mu,\sigma}) = \sigma^2$. This is called the **normal distribution** with mean μ and variance σ^2. Below is a graph of $f(x)$ for various values of μ and σ.



32. Show that $X_{\mu_1+\mu_2, \sqrt{\sigma_1^2+\sigma_2^2}} = X_{\mu_1,\sigma_1} + X_{\mu_2,\sigma_2}$.

Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 28 - Meet 12 May 6, 2021	Mentors:	Cecilia Esterman, Rebecca Nelson, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Savannah Tynan, Jane Wang, Karissa Wenger, Rebecca Whitman, Angelina Zhang, Rachel Zheng

After skipping our traditional end-of-session Math Collaboration last fall due to the pandemic, we resumed them with a fun filled math extravaganza designed by mentors Vievie Romanelli, Rachel Zheng, and Head Mentor Grace Work. The theme: *A Trip Down Memory Lane*. Try your hand at solving this sampling of problems from the event:

For each number d from 1 to 20, inclusive, determine the smallest n such that the sum $1 + 2 + 3 + \ldots + n$ is divisible by d.

If two standard, fair 6-sided dice are rolled, find the probability that the product of the numbers rolled is a perfect square.

Come up with an equation whose graph is a smiley face.

From day one, we dreamed of hiring a woman with a doctoral degree in mathematics to serve as our Head Mentor. This dream became a reality on July 1, 2019, with the hiring of Grace Work as Girls' Angle first full-time Head Mentor, thanks to a major grant from the Mathenaeum Foundation.

The pandemic created a number of challenges, including the transition to a virtual club. Grace played a critical and large role in effecting this transition, helping to keep the mathematical momentum built up over the years alive.

Unfortunately, due to the pandemic and the more difficult fundraising environment, we are losing Grace who will be joining the mathematics department of the University of Wisconsin-Madison this fall.

We thank Grace for her 2-years of service to Girls' Angle's mission. We wish her success and joy in all her future endeavors, as well as a return to regular games of Ultimate Frisbee and Disc Golf!

Let $N > 1$. There are N light bulbs labeled 1 through N arranged in clockwise order around a circle. All the bulbs are initially on. You start at bulb 1. If it is on, you change the state of the next bulb (in clockwise order, which is bulb 2), and if it is off, you do nothing. You then move clockwise one bulb to bulb 2. If bulb 2 is on, you change the state of the next bulb (in clockwise order), otherwise you do nothing. (The next bulb after bulb N is bulb 1.) You continue around the bulbs in this manner, switching the state of the next bulb (in clockwise order) if the bulb you're at is on, or leaving the switches as they are if it is off. If you continue round and round in this manner, prove that there will be another moment when all the bulbs are on.

Calendar

Session 28: (all dates in 2021)

January	28	Start of the twenty-eighth session!
February	4	
	11	
	18	No meet
	25	
March	4	Daina Taimina, Cornell University
	11	
	18	
	25	No meet
April	1	
	8	
	15	
	22	No meet
	29	
May	6	

Session 29: (all dates in 2021)

September	9	Start of the twenty-ninth session!
	16	
	23	
	30	
October	7	
	14	
	21	
	28	
November	4	
	11	
	18	
	25	Thanksgiving - No meet
December	2	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ______________________________________